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SHALLOW ARCHES WITH WEAK AND STRONG DAMPING

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Abstract. The paper develops a rigorous mathematical framework for the behavior of arch and membrane like structures. Our main goal is to incorporate moving point loads. Both the weak and the strong damping cases are considered. First, we prove the existence and the uniqueness of the solutions. Then it is shown that the solution in the weak damping case is the limit of the strong damping solutions, as the strong damping vanishes. The theory is applied to a car moving on a bridge.

1. Introduction

Building long span arch roofs and bridges has been an important practical problem that has occupied structural engineers for many years. The motion of such structures has been studied by engineers and mathematicians since at least 1930s, see [9]. A general mathematical model for the motion of arches and membrane like structures can be found in [4], and it is given by the following non-local integro-differential equation

$$
\begin{aligned}
&y_{tt} + \alpha\Delta^2 y - \left(\beta + \gamma \int_\Omega |\nabla y|^2\, dx\right)\Delta y + \xi y + \kappa y_t - \lambda \Delta y_t + \mu \Delta^2 y_t \\
&- \left(\delta \left| \int_\Omega \nabla y \cdot \nabla y_t\, dx \right|^{q-2} \int_\Omega \nabla y \cdot \nabla y_t\, dx\right)\Delta y = f.
\end{aligned}
\tag{1.1}
$$

The function $y = y(x,t)$ describing the membrane's deflection is defined on $\Omega \times (0,T)$, where $\Omega \in \mathbb{R}^d$. In the one-dimensional case $d = 1$, function $y(x,t)$ describes the deflection of an arch, which is positioned over the interval $\Omega = (0,l)$ of the x axis.

The physical meaning of the parameters in (1.1) can be found in [4]. We just mention that the parameters α and γ are always positive, parameters δ, λ, μ are non-negative, and the others are arbitrary real numbers. The unsigned parameter β accounts for the axial force acting in the reference configuration. Values $\beta > 0$ appear when the beam is stretched, and $\beta < 0$ when the beam

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is compressed. Both hinged ends and clamped (or built-in) ends boundary conditions are considered. Mathematical and engineering issues of interest related to (1.1) are local and global existence and uniqueness, stability and bifurcation of solutions, as well as their chaotic and snap-through behavior.

These and other results on the weak solutions for (1.1) have been established by us in a more general framework in [7] assuming $f \in L^2(0,T;L^2(\Omega))$. This paper also contains some additional background information.

There has been significant recent activity in the engineering research on shallow arches with the emphasis on concentrated, distributed and moving loads. Assuming the initial arch shape to be half-sine, circular or parabolic, experimental and computer simulated behavior of arches has been studied, [3], [8]. The questions are the static and dynamic buckling of the arch.

To simulate the concentrated load, the acting forces have been modeled with delta functions as $f(x,t) = P\delta(x)S(t)$, where the constant P is the load magnitude, and $S(t)$ is a step function, see [5]. In [2], the Fourier expansion of the solution has been used to study the snap-through buckling of a shallow arch under a moving point load $f = Q\delta(x-ct)$. However, in such a setting we can no longer assume that $f \in L^2(0,T;L^2(\Omega))$, since the load f should have the values in a larger space of distributions. Thus the previously mentioned theoretical results are no longer valid.

The main purpose of this paper is to establish the existence and uniqueness and the stability of solutions of (1.1) under such more general assumptions on the load f. The new elements of our study are considerations of the weak ($\mu = 0$), and the strong ($\mu > 0$) damping cases, and the relations between them as $\mu \to 0$. These results form a theoretical foundation for subsequent studies of the stability of shallow arches with constant and moving loads.

The problem is set on a bounded domain $\Omega \subset \mathbb{R}^d$ with a sufficiently smooth boundary $\partial\Omega$. The governing equation is

$$y_{tt} + \alpha\Delta^2 y - \left[\beta + \gamma \int_\Omega |\nabla y|^2\, dx\right]\Delta y + \mu\Delta^2 y_t + \kappa y_t = f \tag{1.2}$$

on $\Omega \times (0,T)$. The initial conditions are

$$y(x,0) = u_0(x), \quad y_t(x,0) = v_0(x), \quad x \in \Omega. \tag{1.3}$$

The boundary conditions are either of the hinged type

$$y = \Delta y = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.4}$$

or the clamped (built-in) type

$$y = \frac{\partial y}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{1.5}$$

Here $\alpha, \gamma \in \mathbb{R}^+ = (0,\infty)$, $\mu \geq 0$, $\beta, \kappa \in \mathbb{R}$, and the load $f \in L^2(0,T;V')$. The space V' is defined in the next section, where we consider the weak formulation of the problem. Note that the parameter δ in (1.1) is set to zero.

However this assumption does not restrict the generality of our approach. For the extension to the case $\delta > 0$ see [1].

The abstract framework for the problems is given in Section 2. The uniqueness and the existence of solutions in the strong damping case $\mu > 0$ is considered in Section 3. The weak damping case $\mu = 0$ is examined in Section 4. The regularization, i.e., the continuous dependence of the solutions on the damping parameter μ, as $\mu \to 0$, is considered in Section 5. The developed theory is applied to a car moving across a long bridge in Section 6.

2. Setup

Let the Hilbert space $H = L^2(\Omega)$ have the norm $|u|$, and the inner product (u, v). For the hinged boundary conditions we choose $V = H_0^1(\Omega) \cap H^2(\Omega)$, and for the clamped boundary conditions let $V = H_0^2(\Omega)$. In both cases V is a Hilbert space with the inner product $((u, v)) = (\Delta u, \Delta v)$, and the norm $\|u\| = |\Delta u|,\ u, v \in V$. This norm is equivalent to the standard norm in $H^2(\Omega)$, see [6].

Since $C_0^\infty(\Omega)$ is dense in H, it follows that V is densely embedded in H. In fact, the embedding is compact and continuous. Identifying H with its dual gives a Gelfand triple $V \subset H \subset V'$, where the duality pairing $\langle \cdot, \cdot \rangle$ between V and its dual V' with the norm $\| \cdot \|_{V'}$ is consistent with the inner product in H. Given a function f defined on $[0, T]$ with values in a Banach space X, we denote by $\dot{f}$ its derivative with respect to t in an appropriate sense.

To set up the weak formulation of the problem (1.2)-(1.5), we introduce operators A, B, and G, corresponding to the terms of the equation (1.2), and state their properties in the following lemmas, proved in [7].

Lemma 2.1. (i) *Define operator A by*

$$\langle Au, v \rangle = \int_\Omega \Delta u \Delta v \, dx, \quad u, v \in V. \tag{2.1}$$

Then A is a linear continuous operator from V into V'.

(ii) *Define operator B by*

$$Bu = \Delta u, \quad u \in V. \tag{2.2}$$

Then B is a linear continuous operator from V into H.

(iii) *Define operator G by*

$$Gu = |\nabla u|^2 \Delta u, \quad u \in V. \tag{2.3}$$

Then G is a nonlinear continuous operator from V into H, which is Lipschitz continuous on bounded subsets of V, that is,

$$|Gu - Gv| \le c(\|u\|^2 + \|v\|^2)\|u - v\|,\ u, v \in V. \tag{2.4}$$

Considered as an operator from V into V', operator G is Lipschitz continuous on bounded subsets of V, that is,

$$\|Gu - Gv\|_{V'} \le c(\|u\|^2 + \|v\|^2)|\nabla u - \nabla v|,\ u, v \in V. \tag{2.5}$$

Furthermore, operators B and G map weakly convergent sequences in V into strongly convergent sequences in V'.

We also need the following technical result established in [7].

Lemma 2.2. *Let $h \in L^1(0,T)$. Suppose that $y_k \in L^\infty(0,T;V)$, $\dot{y}_k \in L^\infty(0,T;H)$, with $\|y_k(t)\| \le 1$, and $|\dot{y}_k(t)| \le h(t)$ a.e. on $[0,T]$ for any natural number $k \in \mathbb{N}$. Suppose that $y_k \rightharpoonup y$, weakly in $L^2(0,T;V)$ as $k \to \infty$. Then, after a modification on a set of measure zero in $[0,T]$,*

(i) *$y_k \to y$ in $C([0,T];H)$ as $k \to \infty$,*
(ii) *$|\nabla y_k - \nabla y| \to 0$ in $L^2(0,T)$ as $k \to \infty$.*

Let X be a Banach space, and $H^1(0,T;X) = \{f : f, \dot{f} \in L^2(0,T;X)\}$. We will need the following integration by parts formula.

Lemma 2.3. *If $f \in H^1(0,T;V')$, and $g \in H^1(0,T;V)$, then*

$$\int_0^t \langle f, \dot{g}\rangle\, ds = \langle f(t), g(t)\rangle - \langle f(0), g(0)\rangle - \int_0^t \langle \dot{f}, g\rangle\, ds. \tag{2.6}$$

Proof. Formula (2.6) is valid for any $f \in C^1([0,T],V')$, and $g \in C^1([0,T],V)$. Fix $g \in C^1([0,T],V)$, and approximate given f by smooth functions $f_n \in C^1([0,T],V')$ in such a way that $f_n \to f$ and $\dot{f}_n \to \dot{f}$ in $L^2(0,T;V')$ as $n \to \infty$. Then $f_n \to f$ in $C([0,T],V')$. Use (2.6) with f replaced by f_n, and pass to the limit as $n \to \infty$ to obtain (2.6) valid for any $f \in H^1(0,T;V')$, and $g \in C^1([0,T],V)$. Now approximate g by g_n in $H^1(0,T;V)$. Passing to the limit in (2.6) for such a sequence gives the desired result. $\square$

Let the operator A be defined by equation (2.1). Its eigenfunctions $\{w_k\}_{k=1}^\infty \subset D(A)$ form an orthonormal basis in H. Let $Aw_k = \mu_k w_k$, $k \in \mathbb{N}$. Then system $\{w_k/\sqrt{\mu_k}\}_{k=1}^\infty$ is an orthonormal basis in V, see Section 2.2.1 of [11]. We have the following results needed for the approximation of the solutions, see [7].

Lemma 2.4. *Let $m \in \mathbb{N}$, and the operator $P_m : H \to H$ and $P_m^* : V' \to V'$ be defined by*

$$P_m h = \sum_{k=1}^m (h, w_k) w_k, \quad h \in H, \tag{2.7}$$

and

$$\langle P_m^* g, v\rangle = \langle g, P_m v\rangle, \quad g \in V',\ v \in V. \tag{2.8}$$

Then

(i) *Operator P_m is an orthogonal projection in H, with $|P_m h| \le |h|$ for any $h \in H$. Also, $|P_m h - h| \to 0$ as $m \to \infty$.*
(ii) *Operator P_m is an orthogonal projection in V, with $\|P_m v\| \le \|v\|$ for any $v \in V$. Also, $\|P_m v - v\| \to 0$ as $m \to \infty$.*

(iii) *Operator* P_m^* *is the adjoint operator of* P_m *in* V'*, with* $\|P_m^* g\|_{V'} \leq \|g\|_{V'}$ *for any* $g \in V'$*. Also* $P_m^* g \rightharpoonup g$ *weakly in* V' *as* $m \to \infty$*.*

3. Existence of solutions in the strong damping case $\mu > 0$

First, we consider the theory for the strong damping case $\mu > 0$, and postpone the weak damping case $\mu = 0$ to Section 4. Here and in the sequel c denotes various constants that do not depend on the solution y or its approximations y_m. Keeping in mind that one of our goals is to consider what happens to the solutions when the strong damping vanishes, i.e., $\mu \to 0$, we will keep an explicit dependency on the damping parameter μ. Let

$$W_r[0,T] = \{y \ : \ y \in L^2(0,T;V), \quad \dot{y} \in L^2(0,T;V), \quad \ddot{y} \in L^2(0,T;V')\}.$$

Here $\dot{y}$, $\ddot{y}$ denote the derivatives with respect to t. They are understood in the sense of distributions with the values in V and V', see [10]. Space $W_r[0,T]$ becomes a Hilbert space when its inner product is set to be the sum of the inner products in the constituent spaces.

Definition 3.1. Let $u_0 \in V$, $v_0 \in H$, $T > 0$, and $f \in L^2(0,T;V')$. Function $y \in W_r[0,T]$ is called a *weak solution* of the problem (1.2)-(1.5), if $y \in L^\infty(0,T;V)$, $\dot{y} \in L^\infty(0,T;H)$, equation

$$\ddot{y} + \mu A\dot{y} + \kappa\dot{y} + \alpha Ay - \beta By - \gamma Gy = f \tag{3.1}$$

is satisfied in V' a.e. on $[0,T]$, and the initial conditions

$$y(0) = u_0, \quad \dot{y}(0) = v_0, \tag{3.2}$$

are satisfied in $V$and H respectively. We write $y = y(u_0, v_0, f)$ to emphasize the dependence of y on the initial conditions and f. The initial conditions make sense, since it follows from Lemma 3.3, that y is continuous in V, and $\dot{y}$ is continuous in H.

Definition 3.2. Let $m \in \mathbb{N}$. Function y_m is called an approximate solution of the abstract problem (3.1)-(3.2), if y_m, $\dot{y}_m \in W_r[0,T] \cap L^\infty(0,T;V)$, equation

$$\ddot{y}_m + \mu A\dot{y}_m + \kappa\dot{y}_m + \alpha Ay_m - \beta By_m - \gamma Gy_m = P_m^* f \tag{3.3}$$

is satisfied in V' a.e. on $[0,T]$, and the initial conditions

$$y_m(0) = P_m u_0, \quad \dot{y}_m(0) = P_m v_0, \tag{3.4}$$

are satisfied in V. The solution y_m will be denoted by $y_m(u_0, v_0, f)$, when it will be necessary to indicate its dependence on the initial conditions and f.

The following lemma is central to our method.

Lemma 3.3. *Let* $w \in W_r[0,T]$*. Then, after a modification on the set of measure zero,* $w \in C([0,T];V)$*,* $\dot{w} \in C([0,T];H)$ *and, in the sense of distributions on* $(0,T)$*, one has*

$$\frac{d}{dt}||w||^2 = 2\langle Aw, \dot{w}\rangle, \quad \text{and} \quad \frac{d}{dt}|\dot{w}|^2 = 2\langle \ddot{w}, \dot{w}\rangle, \tag{3.5}$$

where the linear operator A is defined in (2.1).

Proof. According to Lemma 2.3.2 in [11], if $u \in L^2(0,T;V)$ and its derivative $\dot{u} \in L^2(0,T;V')$, then u is continuous from $[0,T]$ into H after a modification on a set of measure zero, and it satisfies $d/dt|u|^2 = 2\langle \dot{u}, u\rangle$. Letting $u = \dot{w}$ we get $\dot{w} \in C([0,T];H)$, and the second equality in (3.5). For the first equality in (3.5) we can use Lemma 2.3.2 in [11] with $V = H = V'$, or just notice that the mapping $x \to \|x\|^2$ is Fréchet differentiable. $\square$

First, we give a priori estimates for the weak solutions y. They are also valid for the approximate solutions y_m with the same c, since (3.3) is the same as (3.1) with f replaced by $P_m^* f$, and $\|P_m^* f\|_{V'} \le \|f\|_{V'}$.

Lemma 3.4. *Let y be a solution of* (3.1)-(3.2). *Then $y \in C([0,T];V)$, $\dot{y} \in C([0,T];H)$. Furthermore, there exists c, independent of μ, such that*

(i) *If $f \in L^2(0,T;H)$, then*

$$|\dot{y}(t)|^2 + \|y(t)\|^2 \le c\left(|v_0|^2 + \|u_0\|^2 + \|u_0\|^4 + \int_0^t |f|^2\,ds\right), \tag{3.6}$$

(ii) *If $f \in L^2(0,T;V')$, then*

$$\begin{aligned} &|\dot{y}(t)|^2 + \|y(t)\|^2 + \int_0^t \|\dot{y}(s)\|^2\,ds \\ &\le \frac{c}{\mu^2}\left(|v_0|^2 + \|u_0\|^2 + \|u_0\|^4 + \int_0^t \|f\|_{V'}^2\,ds\right), \end{aligned} \tag{3.7}$$

(iii) *If $f \in H^1(0,T;V')$, then*

$$|\dot{y}(t)|^2 + \|y(t)\|^2 \le c\left(|v_0|^2 + \|u_0\|^2 + \|u_0\|^4 + \|f(0)\|_{V'}^2 + \int_0^t \|\dot{f}\|_{V'}^2\,ds\right) \tag{3.8}$$

for any $t \in [0,T]$.

Proof. Take the inner product of (3.1) with $\dot{y}$, and use Lemma 3.3 to get

$$\frac{1}{2}\frac{d}{dt}\{|\dot{y}|^2 + \alpha\|y\|^2 + \frac{\gamma}{2}|\nabla y|^4\} + \mu\|\dot{y}(t)\|^2 = \langle f, \dot{y}\rangle - \kappa|\dot{y}|^2 + \beta(\Delta y, \dot{y}). \tag{3.9}$$

Integrate (3.9) from 0 to t to get

$$\begin{aligned} &|\dot{y}|^2 + \alpha\|y\|^2 + \frac{\gamma}{2}|\nabla y|^4 + 2\mu\int_0^t \|\dot{y}\|^2\,ds - |v_0|^2 - \alpha\|u_0\|^2 - \frac{\gamma}{2}|\nabla u_0|^4 \\ &= -2\kappa\int_0^t |\dot{y}|^2\,ds + 2\beta\int_0^t (\Delta y, \dot{y})\,ds + 2\int_0^t \langle f, \dot{y}\rangle\,ds. \end{aligned} \tag{3.10}$$

The estimate of the last term depends on the condition imposed on f.

If $f \in L^2(0,T;H)$, then

$$2\left|\int_0^t \langle f, \dot{y}\rangle\,ds\right| \le \int_0^t |f|^2\,ds + \int_0^t |\dot{y}|^2\,ds.$$

Therefore, using $|\nabla u_0| \leq c\|u_0\|$,

$$|\dot{y}|^2 + \alpha\|y\|^2 \leq |v_0|^2 + \alpha\|u_0\|^2 + c\|u_0\|^4 + 2|\kappa| \int_0^t |\dot{y}|^2\, ds + |\beta| \int_0^t (\|y\|^2 + |\dot{y}|^2)\, ds + \int_0^t |f|^2\, ds + \int_0^t |\dot{y}|^2\, ds,$$

and (3.6) follows from the Gronwall's inequality.

If $f \in L^2(0,T;V')$, then

$$2\left|\int_0^t \langle f, \dot{y}\rangle\, ds\right| \leq \frac{1}{\mu}\int_0^t \|f\|_{V'}^2\, ds + \mu \int_0^t \|\dot{y}\|^2\, ds,$$

and

$$|\dot{y}|^2 + \alpha\|y\|^2 + \mu \int_0^t \|\dot{y}\|^2 \;\; - (|v_0|^2 + \alpha\|u_0\|^2 + c\|u_0\|^4), ds \leq |\beta| \int_0^t (\|y\|^2 + |\dot{y}|^2)\, ds + 2|\kappa| \int_0^t |\dot{y}|^2\, ds + \frac{1}{\mu}\int_0^t \|f\|_{V'}^2\, ds.$$

Whence we have

$$|\dot{y}|^2 + \|y\|^2 + \int_0^t \|\dot{y}\|^2 ds \leq \frac{c}{\mu^2}\left(|v_0|^2 + \|u_0\|^2 + |\nabla u_0|^4 + \int_0^t \|f\|_{V'}^2 ds + \int_0^t (\|y\|^2 + |\dot{y}|^2)\, ds\right),$$

and Gronwall's inequality gives (3.7).

To obtain inequality (3.8), we use the integration by parts formula (2.6) to estimate the f term in the above derivation. First, notice that the term $\|f(t)\|_{V'}^2$ can be estimated as follows

$$\|f(t)\|_{V'} \leq \|f(0)\|_{V'} + \int_0^t \|\dot{f}\|_{V'}\, ds \leq \|f(0)\|_{V'} + \sqrt{t}\left(\int_0^t \|\dot{f}\|_{V'}^2 ds\right)^{1/2},$$

which implies

$$\|f(t)\|_{V'}^2 \leq 2\|f(0)\|_{V'}^2 + 2t \int_0^t \|\dot{f}\|_{V'}^2\, ds.$$

Then

(3.11)
$$\begin{aligned} 2\left|\int_0^t \langle f, \dot{y}\rangle\, ds\right| &= 2\left|\langle f(t), y(t)\rangle - \langle f(0), u_0\rangle - \int_0^t \langle \dot{f}, y\rangle\, ds\right| \\ &\leq \frac{\alpha}{2}\|y(t)\|^2 + \frac{2}{\alpha}\|f(t)\|_{V'}^2 + \|f(0)\|_{V'}^2 + \|u_0\|^2 + \int_0^t \|\dot{f}\|_{V'}^2\, ds + \int_0^t \|y\|^2\, ds \\ &\leq \frac{\alpha}{2}\|y(t)\|^2 + \left(\frac{4}{\alpha} + 1\right)\|f(0)\|_{V'}^2 + \left(\frac{4}{\alpha}\, t + 1\right)\int_0^t \|\dot{f}\|_{V'}^2 ds + \int_0^t \|y\|^2 ds. \end{aligned}$$

Therefore

$$|\dot y|^2+\alpha\|y\|^2+\frac{\gamma}{2}|\nabla y|^4+2\mu\int_0^t\|\dot y\|^2\,ds-|v_0|^2-\alpha\|u_0\|^2-\frac{\gamma}{2}|\nabla u_0|^4$$
$$=-2\kappa\int_0^t|\dot y|^2\,ds+2\beta\int_0^t(\Delta y,\dot y)\,ds+2\int_0^t\langle f,\dot y\rangle\,ds$$
$$\le 2|\kappa|\int_0^t|\dot y|^2\,ds+|\beta|\int_0^t(\|y\|^2+|\dot y|^2)\,ds+\frac{\alpha}{2}\|y(t)\|^2+\left(\frac{4}{\alpha}+1\right)\|f(0)\|^2_{V'}$$
$$+\left(\frac{4}{\alpha}t+1\right)\int_0^t\|\dot f\|^2_{V'}ds+\int_0^t\|y\|^2ds.$$

Whence we have

$$|\dot y|^2+\|y\|^2\le c(|v_0|^2+\|u_0\|^2+|\nabla u_0|^4+\|f(0)\|^2_{V'})$$
$$+c\left(\int_0^t\|\dot f\|^2_{V'}ds+\int_0^t(\|y\|^2+|\dot y|^2)\,ds\right),$$

and Gronwall's inequality gives (3.8).

The conclusions $y\in C([0,T];V)$, $\dot y\in C([0,T];H)$ follow from Lemma 3.3. □

Next, we show the continuous dependence of the solutions $y(u_0,v_0,f)$ on the initial conditions and f. Just like in Lemma 3.4, there are three estimates, depending on the conditions imposed on f.

Lemma 3.5. *Let $u_0,\bar u_0\in V$, $v_0,\bar v_0\in H$, $f,\bar f\in L^2(0,T;V')$, and $y=y(u_0,v_0,f)$, $\bar y(\bar u_0,\bar v_0,\bar f)$ be the corresponding solutions of* (3.1)–(3.2). *Then there exists a constant c, independent of μ, such that*

(i) *If $f,\bar f\in L^2(0,T;H)$, then*

$$\begin{aligned}&|\dot{\bar y}(t)-\dot y(t)|^2+\|\bar y(t)-y(t)\|^2\\ &\le c\left(|\bar v_0-v_0|^2+\|\bar u_0-u_0\|^2+\|\bar u_0-u_0\|^4+\int_0^t|\bar f-f|^2\,ds\right),\end{aligned}\tag{3.12}$$

(ii) *If $f,\bar f\in L^2(0,T;V')$, then*

$$\begin{aligned}&|\dot{\bar y}(t)-\dot y(t)|^2+\|\bar y(t)-y(t)\|^2+\int_0^t\|\dot{\bar y}-\dot y\|^2\,ds\\ &\le \frac{c}{\mu^2}\left(|\bar v_0-v_0|^2+\|\bar u_0-u_0\|^2+\|\bar u_0-u_0\|^4+\int_0^t\|\bar f-f\|^2_{V'}\,ds\right),\end{aligned}\tag{3.13}$$

(iii) *If $f,\bar f\in H^1(0,T;V')$, then*

$$\begin{aligned}&|\dot{\bar y}(t)-\dot y(t)|^2+\|\bar y(t)-y(t)\|^2\le c\bigg(|\bar v_0-v_0|^2+\|\bar u_0-u_0\|^2\\ &+\|\bar u_0-u_0\|^4+\|\bar f(0)-f(0)\|^2_{V'}+\int_0^t\|\dot{\bar f}-\dot f\|^2_{V'}\,ds\bigg)\end{aligned}\tag{3.14}$$

for any $t \in [0,T]$.

In every case the solution of the problem (3.1)–(3.2) *is unique.*

Proof. Let y, $\bar{y}$ be two solutions of (3.1)–(3.2). Then their difference $z = \bar{y} - y$ satisfies

$$\ddot{z} + \mu A\dot{z} + \kappa\dot{z} + \alpha Az - \beta Bz - \gamma(G\bar{y} - Gy) = \bar{f} - f.$$

By Lemma 2.1, $B, G : V \to H$ are Lipschitz continuous on bounded subsets of V. Also,

$$|G\bar{y}(t) - Gy(t)| \leq C\|\bar{y}(t) - y(t)\|, \quad t \in [0,T].$$

The same inequality is also satisfied when operator G is replaced by B. The constant C depends on the bounds for $\bar{y}$ and y in V. These bounds depend only on the initial conditions, as well as on the functions $\bar{f}$ and f according to Lemma 3.4. Now we can argue as in Lemma 3.4 to obtain the required inequalities. The uniqueness follows from these inequalities when $\bar{u}_0 = u_0$, $\bar{v}_0 = v_0$, and $\bar{f} = f$. □

It remains to show the existence of the weak solutions in the strong damping case.

Theorem 3.6. *Fix* $\mu > 0$. *Let* $u_0 \in V$, $v_0 \in H$, $T > 0$, *and* $f \in L^2(0,T;V')$.

(i) *Then there exists a unique solution* y *of the problem* (3.1)–(3.2). *The solution satisfies* $y \in W_r[0,T] \cap C([0,T];V)$, *and* $\dot{y} \in C([0,T];H)$.

(ii) *If* y_m *is an approximate solution, then* $y_m \to y$ *in* $C([0,T];V)$, *and* $\dot{y}_m \to \dot{y}$ *in* $C([0,T];H)$ *as* $m \to \infty$.

Proof. Fix $m \in \mathbb{N}$. Let us construct an approximate solution x_m with values in $V_m = span\{w_1, w_2, \ldots, w_m\}$ as follows. Let

$$x_m(t) = \sum_{j=1}^{m} g_{j,m}(t)w_j, \tag{3.15}$$

where the expansion is over the eigenfunctions of A, and real-valued functions $g_j := g_{j,m}(t)$, $j = 1, 2, \ldots, m$ are the solutions of the following system of m equations

$$\begin{gathered}\langle \ddot{x}_m + \mu A\dot{x}_m + \kappa\dot{x}_m + \alpha Ax_m - \beta Bx_m - \gamma Gx_m, w_k\rangle = \langle f, w_k\rangle, \\ (x_m(0), w_k) = (P_m u_0, w_k), \quad (\dot{x}_m(0), w_k) = (P_m v_0, w_k),\end{gathered} \tag{3.16}$$

where $k = 1, 2, \ldots, m$. Here we used $\langle P_m^* f, w_k\rangle = \langle f, P_m w_k\rangle = \langle f, w_k\rangle$ for such k.

Since $(\nabla w_k, \nabla w_j) = 0$ for $k \neq j$, and $|\nabla w_k|^2 = \lambda_k := \sqrt{\mu_k}$, we get an explicit expression

$$\ddot{g}_k(t)+\mu\lambda_k^2 g_k(t)+\kappa\dot{g}_k(t)+\alpha\lambda_k^2 g_k(t)+\beta\lambda_k g_k(t)+\gamma\lambda_k \sum_{j=1}^{m} \lambda_j^2 |g_j(t)|^2\, g_k(t)=\langle f, w_k\rangle,$$

$$g_k(0) = (u_0, w_k), \quad \dot{g}_k(0) = (v_0, w_k),$$

where $k = 1, 2, \dots, m$. This initial value problem for the system of m ODEs has a unique solution satisfying $g_k, \dot{g}_k \in C[0,T]$, $\ddot{g}_k \in L^2[0,T]$. Thus $x_m, \dot{x}_m \in C([0,T]; V_m)$, and $\ddot{x}_m \in L^2(0,T;V_m)$.

To conclude that x_m is an approximate solution of the problem (3.1)-(3.2) in the sense of Definition 3.2, it is enough to establish that

$$\langle \ddot{x}_m + \mu A\dot{x}_m + \kappa\dot{x}_m + \alpha A x_m - \beta B x_m - \gamma G x_m, w_k\rangle = \langle P_m^* f, w_k\rangle \tag{3.17}$$

is satisfied for any $k \in \mathbb{N}$. For $1 \le k \le m$, equation (3.17) is the same as (3.16), which is satisfied by the construction of x_m. For $k > m$, the left side of (3.17) becomes 0, and the right side is also 0, because $\langle P_m^* f, w_k\rangle = \langle f, P_m w_k\rangle = 0$ for such k. The uniqueness of the approximate solutions follows from Lemma 3.5. Therefore we conclude that x_m is the only approximate solution of (3.1)-(3.2), i.e., $x_m = y_m$.

By Lemma 3.4, the approximate solutions y_m remain in the same bounded ball in $L^\infty(0,T;V)$, and their derivatives $\dot{y}_m$ remain in a bounded ball of $L^\infty(0,T;H)$ for all $m \in \mathbb{N}$. Furthermore, $\dot{y}_m$ stay within a bounded ball in $L^2(0,T;V)$. Let us show that $\ddot{y}_m$ are also staying within a bounded ball for all m.

By Lemmas 2.1, 2.4, and using the continuous embedding of $V \subset H$, for each $w \in V$ we have

$$\begin{aligned}|\langle \ddot{y}_m, w\rangle| &\le \mu|\langle A\dot{y}_m, w\rangle| + |\kappa|\,|(\dot{y}_m, w)| + \alpha\,|\langle Ay_m, w\rangle| + \beta\,|(By_m, w)| \\ &\quad + \gamma\,|(Gy_m, w)| + |\langle P_m^* f, w\rangle| \\ &\le c(\|\dot{y}_m\| + |\dot{y}_m| + \|y_m\| + \|y_m\| + \|y_m\|^3 + \|f\|_{V'})\,\|w\|.\end{aligned}$$

Thus

$$\|\ddot{y}_m\|_{V'} \le c(\|\dot{y}_m\| + |\dot{y}_m| + \|y_m\| + \|y_m\|^3 + \|f\|_{V'}).$$

Together with already established bounds for y_m and $\dot{y}_m$, it concludes the proof that y_m remains within the same bounded ball in $W_r[0,T]$ for all $m \in \mathbb{N}$.

Since $W_r[0,T]$ is a reflexive space, we can find a subsequence of y_m (still denoted by y_m) such that it and the derivatives $\dot{y}_m$, $\ddot{y}_m$ are weakly convergent in the spaces $L^2(0,T;V)$, $L^2(0,T;H)$, and $L^2(0,T;V')$ correspondingly. Since the derivatives are taken in the distributional sense, it follows that there exists $y \in W_r[0,T]$ such that

$$y_m \rightharpoonup y, \quad \dot{y}_m \rightharpoonup \dot{y}, \quad \ddot{y}_m \rightharpoonup \ddot{y} \tag{3.18}$$

weakly in the corresponding spaces. Since sequence y_m is bounded in $L^\infty(0,T;V)$, and sequence $\dot{y}_m$ is bounded in $L^\infty(0,T;H)$, by Lemma 2.2 the weak convergence (3.18) yields

$$y_m(t) \rightharpoonup y(t) \text{ in } V, \quad \dot{y}_m(t) \rightharpoonup \dot{y}(t) \text{ in } H \tag{3.19}$$

weakly for all $t \in [0,T]$. By the property of the weak convergence the limiting function y is in $W_r[0,T]$.

Now we are going to show that y satisfies the abstract problem (3.1)–(3.2), i.e., it is a weak solution of the problem (1.2)–(1.5). By the definition of the approximate solution

$$\ddot{y}_m + \mu A\dot{y}_m + \kappa\dot{y}_m + \alpha Ay_m - \beta By_m - \gamma Gy_m = P_m^* f \tag{3.20}$$

in V', a.e. on $[0,T]$, and

$$y_m(0) = P_m u_0, \quad \dot{y}(0) = P_m v_0. \tag{3.21}$$

Clearly, we can pass to the limit in V' for $\ddot{y}_m$, $\kappa\dot{y}_m$, By_m, and $P_m^* f$ as $m \to \infty$. For the nonlinear operator G we have estimate (2.5)

$$\|Gy_m - Gy\|_{V'} \le c(\|y_m\|^2 + \|y\|^2)|\nabla y_m - \nabla y|.$$

The norms $\|y_m\|$ and $\|y\|$ are bounded by (3.7). Now we conclude by Lemma 2.2 that $|\nabla y_m - \nabla y| \to 0$ in $L^2(0,T)$ as $m \to \infty$. Thus $Gy_m \to Gy$ in $L^2(0,T;V')$, which implies the validity of the passing to the limit.

As for $\mu A\dot{y}_m$, and Ay_m, the linear operator A is continuous from V into V', therefore it continuous from $L^2(0,T;V)$ into $L^2(0,T;V')$. Thus it is weakly continuous in these spaces, and the passage to the limit as $m \to \infty$ in (3.20) is justified.

Concerning the initial conditions (3.21), it was also argued in Lemma 2.2 that the weak convergence of y_m to y in $L^2(0,T;V)$ implies that $y_m(t) \rightharpoonup y(t)$ weakly in V for any $t \in [0,T]$. Since $y_m(0) = P_m u_0 \to u_0$ in V, we conclude that $y(0) = u_0$. A straightforward modification of Lemma 2.2 shows that $\dot{y}_m \rightharpoonup \dot{y}$ weakly in H for any $t \in [0,T]$. Therefore $\dot{y}(0) = v_0$.

Assume that $f \in L^2(0,T;H)$. For such an f, we have $P_m^* f = P_m f$. Let $y_m, y \in W_r[0,T]$ be the solutions obtained in this theorem. Then we have inequality

$$\begin{aligned} &|\dot{y}_m(t) - \dot{y}(t)|^2 + \|y_m(t) - y(t)\|^2 \\ &\le c\left(|P_m v_0 - v_0|^2 + \|P_m u_0 - u_0\|^2 + \|P_m u_0 - u_0\|^4 + \int_0^t |P_m f(s) - f(s)|^2\, ds\right) \end{aligned} \tag{3.22}$$

for any $t \in [0,T]$, since it is (3.12) with $\bar{u}_0 = P_m u_0$, $\bar{v}_0 = P_m v_0$, and $\bar{f} = P_m f$. It implies the convergence of y_m to y in $C([0,T];V)$, and $\dot{y}_m$ to $\dot{y}$ in $C([0,T];H)$ as $m \to \infty$.

Finally, assume that $f \in L^2(0,T;V')$. Approximate f by a sequence $g_n \in L^2(0,T;H)$, such that $g_n \to f$ in $L^2(0,T;V')$ as $n \to \infty$. We have

$$\begin{aligned} &\|y_m(u_0,v_0,f) - y(u_0,v_0,f)\|_V \\ &\le \|y_m(u_0,v_0,f) - y_m(u_0,v_0,g_n)\|_V + \|y_m(u_0,v_0,g_n) - y(u_0,v_0,g_n)\|_V \\ &\quad + \|y(u_0,v_0,g_n) - y(u_0,v_0,f)\|_V. \end{aligned} \tag{3.23}$$

Let $\epsilon > 0$. The first and the third terms in the right side of (3.23) are less than ϵ, for sufficiently large n, by (3.13). The second term is less than ϵ, for sufficiently large m, by (3.22). The same argument goes for $|\dot{y}_m(u_0, v_0, f) - \dot{y}(u_0, v_0, f)|$. This establishes the second claim of the theorem. $\square$

4. Existence of solutions in the weak damping case $\mu = 0$

An arch is said to have a weak damping, if $\mu = 0$ in the system (1.2)–(1.5). The goal of this section is to prove the existence of weak solutions for this problem under the condition $f \in H^1(0, T; V')$. Since less damping is present in the system, the solutions are less regular. The main tool in the proof is Lemma 4.3. However, it is applicable only if $f \in L^2(0, T; H)$. Therefore we are forced to take a more indirect route. Let

$$W[0,T] = \{y \ : \ y \in L^2(0,T;V), \quad \dot{y} \in L^2(0,T;H), \quad \ddot{y} \in L^2(0,T;V')\}.$$

Definition 4.1. Let $u_0 \in V$, $v_0 \in H$, $T > 0$, and $f \in L^2(0, T; V')$. Function $y \in W[0, T]$ is called a *weak solution* of the problem (1.2)–(1.5) in the weak damping case, if $y \in L^\infty(0, T; V)$, $\dot{y} \in L^\infty(0, T; H)$, equation

$$\ddot{y} + \kappa\dot{y} + \alpha Ay - \beta By - \gamma Gy = f \tag{4.1}$$

is satisfied in V' a.e. on $[0, T]$, and the initial conditions

$$y(0) = u_0, \quad \dot{y}(0) = v_0 \tag{4.2}$$

are satisfied in V and H correspondingly. We write $y = y(u_0, v_0, f)$ to emphasize the dependence of y on the initial conditions and f.

Definition 4.2. Let $m \in \mathbb{N}$. Function y_m is called an approximate solution of the abstract problem (4.1)–(4.2), if $y_m \in W[0,T] \cap L^\infty(0,T;V)$, $\dot{y}_m \in L^\infty(0, T; H)$, equation

$$\ddot{y}_m + \kappa\dot{y}_m + \alpha Ay_m - \beta By_m - \gamma Gy_m = P_m^* f \tag{4.3}$$

is satisfied in V' a.e. on $[0, T]$, and the initial conditions

$$y_m(0) = P_m u_0, \quad \dot{y}_m(0) = P_m v_0 \tag{4.4}$$

are satisfied in V and H correspondingly. The solution y_m will be denoted by $y_m(u_0, v_0, f)$, when it will be necessary to indicate its dependence on the initial conditions and f.

The following crucial result is established in Lemma 2.4.1 of [11]:

Lemma 4.3. *Let $A : V \to V'$ be defined by* (2.1)*. Suppose that $y \in L^2(0, T; V)$, $\dot{y} \in L^2(0, T; H)$, and $\ddot{y} + Ay \in L^2(0, T; H)$. Then, after a modification on a set of measure zero, $y \in C([0, T]; V)$, $\dot{y} \in C([0, T]; H)$ and, in the sense of distributions on $(0, T)$, one has*

$$(\ddot{y} + Ay, \dot{y}) = \frac{1}{2}\frac{d}{dt}\{|\dot{y}|^2 + \|y\|^2\}. \tag{4.5}$$

Suppose that y is a weak solution of (1.2)–(1.5) with $\mu = 0$. Since A, B, G are Lipschitz continuous on bounded subsets of V, and $y \in L^\infty(0,T;V)$, $\dot{y} \in L^2(0,T;H)$, it follows from Lemma 2.1 that $Ay, By, Gy \in L^2(0,T;V')$, so equation (4.1) makes sense. Functions $y : [0,T] \to V$, and $\dot{y} : [0,T] \to H$ are weakly continuous, so conditions (4.2) make sense as well.

The main result of this section is:

Theorem 4.4. *Let $u_0 \in V$, $v_0 \in H$, $T > 0$, and $f \in H^1(0,T;V')$. Then*

(i) *There exists a unique weak solution $y \in W[0,T]$ of the problem (1.2)–(1.5) with $\mu = 0$. Furthermore, $y \in C([0,T];V)$, $\dot{y} \in C([0,T];H)$.*

(ii) *The solution satisfies*

(4.6)
$$|\dot{y}(t)|^2 + \|y(t)\|^2 \le c\left(|v_0|^2 + \|u_0\|^2 + \|u_0\|^4 + \|f(0)\|^2_{V'} + \int_0^t \|\dot{f}(s)\|^2_{V'}\,ds\right)$$

for any $t \in [0,T]$.

(iii) *Let $\bar{y} = \bar{y}(\bar{u}_0, \bar{v}_0, \bar{f})$ be the weak solution for the initial conditions $\bar{u}_0 \in V$, $\bar{v}_0 \in H$, and $\bar{f} \in H^1(0,T;V')$. Then the difference $y - \bar{y}$ satisfies the inequality*

$$\begin{aligned} &\|\dot{y}(t) - \dot{\bar{y}}(t)|^2 + \|y(t) - \bar{y}(t)\|^2 \\ &\le c\Big(|v_0 - \bar{v}_0|^2 + \|u_0 - \bar{u}_0\|^2 + \|u_0 - \bar{u}_0\|^4 + \|f(0) - \bar{f}(0)\|^2_{V'} \\ &\quad + \int_0^t \|\dot{f}(s) - \dot{\bar{f}}(s)\|^2_{V'}\,ds\Big) \end{aligned} \tag{4.7}$$

for any $t \in [0,T]$.

(iv) *If y_m is an approximate solution, then $y_m \to y$ in $C([0,T];V)$, and $\dot{y}_m \to \dot{y}$ in $C([0,T];H)$ as $m \to \infty$.*

The proof of the theorem is provided below. It follows the proof in the strong damping case with some modifications. In particular, Lemma 4.3 is used instead of Lemma 3.3. The existence result was previously obtained by us in [7] for $f \in L^2(0,T;H)$.

Lemma 4.5. *The solution of the problem (4.1)–(4.2) is unique.*

Proof. Let y, $\bar{y}$ be two solutions of (4.1)–(4.2) with the same initial conditions and f. Their difference z satisfies

$$\ddot{z} + \kappa\dot{z} + \alpha A z - \beta B z - \gamma(Gy - G\bar{y}) = 0, \tag{4.8}$$

with $z(0) = 0$, $\dot{z}(0) = 0$. By Lemma 2.1 in conjunction with $z \in L^\infty(0,T;V)$, $\dot{z} \in L^\infty(0,T;H)$ we have

$$\ddot{z} + \alpha A z = -\kappa\dot{z} + \beta B z + \gamma(Gy - G\bar{y}) \in L^2(0,T;H). \tag{4.9}$$

Multiply both sides of (4.9) by $\dot{z}$. Then use $\ddot{z} + \alpha A z \in L^2(0,T;H)$ and Lemma 4.3 to rewrite the result as

$$\frac{1}{2}\frac{d}{dt}\left[|\dot{z}|^2 + \alpha\|z\|^2\right] = -\kappa|\dot{z}|^2 + \beta(Bz, \dot{z}) + \gamma(Gy - G\bar{y}, \dot{z}). \tag{4.10}$$

By the Lipschitz continuity of $G, B : V \to H$ on bounded subsets of V according to Lemma 2.1, we have

$$|(\gamma G y - \gamma G \bar{y}, \dot{z})| \leq c(\|y\|^2 + \|\bar{y}\|^2)\|z\| \, |\dot{z}| \leq c\|z\| \, |\dot{z}|,$$

and

$$|(Bz, \dot{z})| \leq c\|z\| \, |\dot{z}|.$$

The solutions and their derivatives are weakly continuous from $[0, T]$ into V and H correspondingly, so their values are well-defined for any t, see [7]. Therefore we can integrate (4.10) on $[0, t]$, and use the above inequalities to get

$$|\dot{z}(t)|^2 + \|z(t)\|^2 \leq c \left(\int_0^t (|\dot{z}|^2 + \|z\|^2) \, ds \right).$$

Gronwall's inequality implies that $z = 0$, i.e., the uniqueness of the solution. □

The following estimates are provided for completeness, as well as because they are needed in subsequent sections.

Lemma 4.6. *Suppose that $u_0 \in V$, $v_0 \in H$, $T > 0$, and $f \in L^2(0, T; H)$. Let $y = y(u_0, v_0, f)$.*

(i) *Then*

$$|\dot{y}(t)|^2 + \|y(t)\|^2 \leq c \left(|v_0|^2 + \|u_0\|^2 + \|u_0\|^4 + \int_0^t |f|^2 \, ds \right). \tag{4.11}$$

(ii) *If $\bar{u}_0 \in V$, $\bar{v}_0 \in H$, $\bar{f} \in L^2(0, T; H)$, and $\bar{y}(\bar{u}_0, \bar{v}_0, \bar{f})$ is the corresponding solution, then*

$$\begin{aligned} &|\dot{\bar{y}}(t) - y(t)|^2 + \|\bar{y}(t) - y(t)\|^2 \\ &\leq c \left(|\bar{v}_0 - v_0|^2 + \|\bar{u}_0 - u_0\|^2 + \|\bar{u}_0 - u_0\|^4 + \int_0^t |\bar{f} - f|^2 \, ds \right). \end{aligned} \tag{4.12}$$

Proof. We have

$$\ddot{y} + \alpha A y = -\kappa \dot{y} + \beta B y + \gamma G y + f \in L^2(0, T; H).$$

Take the inner product of this equality with $\dot{y}$ in H, and use Lemma 4.3 to rewrite the result as

$$\frac{1}{2} \frac{d}{dt} \left[|\dot{y}|^2 + \alpha \|y\|^2 \right] = -\kappa |\dot{y}|^2 + \beta (By, \dot{y}) + \gamma (Gy, \dot{y}) + (f, \dot{y}).$$

Integrate it from 0 to t, and use $|(f, \dot{y})| \leq |f||\dot{y}|$, as well as other such estimates to get

$$|\dot{y}(t)|^2 + \|y(t)\|^2 \leq c \left(|v_0|^2 + \|u_0\|^2 + \|u_0\|^4 + \int_0^t |f|^2 \, ds + \int_0^t (|\dot{y}|^2 + \|y\|^2) \, ds \right).$$

The Gronwall's inequality gives (4.11).

Estimate (4.12) is proved similarly, by applying this method to the difference $z = \bar{y} - y$, as in the proof of Lemma 4.5. □

Lemma 4.7. *Fix $m \in \mathbb{N}$. Let $V_m = span\{w_k,\ k = 1, 2, \ldots, m\}$. Suppose that $u_0 \in V,\ v_0 \in H,\ T > 0$, and $f \in H^1(0, T; V')$. Then*

(i) *There exists a unique approximate solution y_m of the problem (4.1)–(4.2). This solution satisfies $y_m, \dot{y}_m \in C([0, T]; V_m)$, $\ddot{y}_m \in L^2(0, T; V_m)$. Furthermore, for any $t \in [0, T]$*

$$|\dot{y}_m(t)|^2 + \|y_m(t)\|^2 \le c\left(|v_0|^2 + \|u_0\|^2 + \|u_0\|^4 + \|f(0)\|_{V'}^2 + \int_0^t \|\dot{f}(s)\|_{V'}^2\, ds\right), \tag{4.13}$$

where the constant c is independent of m. Also, there exists $C = C(u_0, v_0, f)$ independent of m, such that

$$\|\ddot{y}_m\|_{L^2(0,T;V')} \le C \tag{4.14}$$

for any $m \in \mathbb{N}$.

(ii) *Let $\bar{y}_m = \bar{y}_m(\bar{u}_0, \bar{v}_0, \bar{f})$ be the approximate solution for the initial conditions $\bar{u}_0 \in V,\ \bar{v}_0 \in H$, and $\bar{f} \in H^1(0, T; V')$. Then the difference $y_m - \bar{y}_m$ satisfies the inequality*

$$\begin{aligned} &|\dot{y}_m(t) - \dot{\bar{y}}_m(t)|^2 + \|y_m(t) - \bar{y}_m(t)\|^2 \\ &\le c\Big(|v_0 - \bar{v}_0|^2 + \|u_0 - \bar{u}_0)\|^2 + \|f(0) - \bar{f}(0)\|_{V'}^2 + \int_0^t \|\dot{f}(s) - \dot{\bar{f}}(s)\|_{V'}^2\, ds\Big). \end{aligned} \tag{4.15}$$

Proof. The uniqueness of the approximate solution follows form Lemma 4.5, since it is applicable to (4.3)–(4.4) with f replaced by $P_m^* f$.

Now we construct an approximate solution x_m with values in V_m as in Theorem 3.6. Let

$$x_m(t) = \sum_{j=1}^m g_{j,m}(t) w_j, \tag{4.16}$$

where the expansion is over the eigenfunctions of A, and real-valued functions $g_j := g_{j,m}(t),\ j = 1, 2, \ldots, m$ are the solutions of the following system of m equations

$$\begin{gathered} \langle \ddot{x}_m + \kappa \dot{x}_m + \alpha A x_m - \beta B x_m - \gamma G x_m, w_k\rangle = \langle f, w_k\rangle, \\ (x_m(0), w_k) = (P_m u_0, w_k), \quad (\dot{x}_m(0), w_k) = (P_m v_0, w_k), \end{gathered} \tag{4.17}$$

where $k = 1, 2, \ldots, m$. Here we used $\langle P_m^* f, w_k\rangle = \langle f, P_m w_k\rangle = \langle f, w_k\rangle$ for such k.

The solution x_m of this system of ODEs satisfies $x_m,\ \dot{x}_m \in C([0, T]; V_m)$, $\ddot{x}_m \in L^2(0, T; V')$. The uniqueness of the approximate solutions implies that $y_m = x_m$.

The next step is to obtain estimate (4.13). Multiply (4.3) by $\dot{y}_m$, and rewrite the result in the form

$$\frac{1}{2}\frac{d}{dt}\left[|\dot{y}_m|^2 + \alpha\|y_m\|^2 + \frac{\gamma}{2}|\nabla y_m|^4\right] = -\kappa|\dot{y}_m|^2 + \beta(\nabla y_m, \dot{y}_m) + \langle f, \dot{y}_m\rangle. \tag{4.18}$$

Integrate (4.18) from 0 to t to get

$$
\begin{aligned}
&|\dot y_m|^2+\alpha\|y_m\|^2+\frac{\gamma}{2}|\nabla y_m|^4=|v_0|^2+\alpha\|u_0\|^2+\frac{\gamma}{2}|\nabla u_0|^4\\
&-2\kappa\int_0^t|\dot y_m|^2\,ds+2\beta\int_0^t(\nabla y_m,\dot y_m)\,ds+2\int_0^t\langle f,\dot y_m\rangle\,ds\\
&\le |v_0|^2+\alpha\|u_0\|^2+\frac{\gamma}{2}|\nabla u_0|^4+2|\kappa|\int_0^t|\dot y_m|^2\,ds+2|\beta|c\int_0^t\|y_m\|\,|\dot y_m|\,ds\\
&+2\left|\int_0^t\langle f,\dot y_m\rangle\,ds\right|.
\end{aligned}
$$

The last integral can be estimated using the integration by parts formula established in Lemma 2.3 like (3.11) with $y=y_m\in C^1([0,T];V_m)$. Hence

$$
\begin{aligned}
|\dot y_m|^2+\|y_m\|^2&\le c(|v_0|^2+\|u_0\|^2+|\nabla u_0|^4+\|f(0)\|_{V'}^2)\\
&\le c\left(\int_0^t\|\dot f\|_{V'}^2ds+\int_0^t(|\dot y_m|^2+\|y_m\|^2)ds\right).
\end{aligned}
$$

Now $|\nabla u_0|\le c\|u_0\|$, and Gronwall's inequality gives (4.13). Here the constant c is independent of m, but is dependent on T.

Let us prove inequality (4.14). By Lemmas 2.1, 2.4, and using the continuous imbedding of $V\subset H$, we have for each $w\in V$

$$
\begin{aligned}
|\langle\ddot y_m,w\rangle|&\le|\kappa|\,|(\dot y_m,w)|+\alpha\,|\langle Ay_m,w\rangle|+\beta\,|(By_m,w)|\\
&\quad+\gamma\,|(Gy_m,w)|+|\langle P_m^*f,w\rangle|\\
&\le c(|\dot y_m|\,\|w\|+\|y_m\|\,\|w\|+\|y_m\|\,\|w\|+\|y_m\|^3\,\|w\|+\|f\|_{V'}\,\|w\|).
\end{aligned}
$$

Thus

$$
\|\ddot y_m\|_{V'}\le c(|\dot y_m|+\|y_m\|+\|y_m\|^3+\|f\|_{V'})
$$

and the result follows.

Finally, let us show (4.15). The difference $z_m=y_m-\bar y_m$ satisfies

$$
\ddot z_m+\kappa\dot z_m+\alpha Az_m-\beta Bz_m-\gamma Gy_m+\gamma G\bar y_m=P_m^*(f-\bar f) \tag{4.19}
$$

in V'. Multiply (4.19) by $\dot z_m\in V$ to get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left[|\dot z_m|^2+\alpha\|z_m\|^2\right]=&-\kappa|\dot z_m|^2+\beta(Bz_m,\dot z_m)\\
&+\gamma(Gy_m-G\bar y_m,\dot z_m)+\langle P_m^*(f-\bar f),\dot z_m\rangle.
\end{aligned} \tag{4.20}
$$

By the Lipschitz continuity of $G,B:V\to H$ on bounded subsets of V, according to Lemma 2.1, and (4.13), we have

$$
|(\gamma Gy_m-\gamma G\bar y_m,\dot z_m)|\le c(\|y_m\|^2+\|\bar y_m\|^2)\|z_m\||\dot z_m|\le c\|z_m\|\;|\dot z_m|
$$

and

$$
|(Bz_m,\dot z_m)|\le c\|z_m\|\;|\dot z_m|,
$$

where c is independent of m. Similarly to the estimate that used the integration by parts formula in the previous part

$$2\left|\int_0^t \langle P_m^*(f-\bar f), \dot z_m\rangle\,ds\right| \le \frac{\alpha}{2}\|z_m(t)\|^2 + \left(\frac{4}{\alpha}+1\right)\|f(0)-\bar f(0)\|_{V'}^2 + \left(\frac{4}{\alpha}t+1\right)\int_0^t \|\dot f-\dot{\bar f}\|_{V'}^2\,ds + \int_0^t \|z_m\|^2\,ds.$$

Integrate (4.20) over $[0,t]$, and apply the above three inequalities to get

$$\begin{aligned}&|\dot z_m|^2+\|z_m\|^2\\ &\le c(|v_0-\bar v_0|^2+\|v_0-\bar v_0\|^2)\\ &\quad + c\left(\|f(0)-\bar f(0)\|_{V'}^2+\int_0^t\|\dot f-\dot{\bar f}\|_{V'}^2\,ds+\int_0^t(|\dot z_m|^2+\|z_m\|^2)\,ds\right).\end{aligned}$$

Then Gronwall's inequality gives (4.15). □

Proof Theorem 4.4. The uniqueness of the weak solutions has already been proved in Lemma 4.5. For the existence, let $y_m,\ m\in\mathbb{N}$ be the sequence of the approximate solutions. By Lemma 4.7 this sequence is bounded in $W[0,T]$. Since $W[0,T]$ is a reflexive space, we can find a subsequence of y_m (still denoted by y_m) such that it and the derivatives $\dot y_m$, $\ddot y_m$ are weakly convergent in the spaces $L^2(0,T;V)$, $L^2(0,T;H)$, and $L^2(0,T;V')$ correspondingly. Since the derivatives are taken in the distributional sense, it follows that there exists $y\in W[0,T]$ such that

$$y_m \rightharpoonup y,\quad \dot y_m\rightharpoonup \dot y,\quad \ddot y_m\rightharpoonup\ddot y \tag{4.21}$$

weakly in the corresponding spaces. Since sequence y_m is bounded in $L^\infty(0,T;V)$, and sequence $\dot y_m$ is bounded in $L^\infty(0,T;H)$, by Lemma 2.2 the weak convergence (4.21) yields

$$y_m(t)\rightharpoonup y(t)\ \text{ in }\ V,\quad \dot y_m(t)\rightharpoonup\dot y(t)\ \text{ in }\ H \tag{4.22}$$

weakly for all $t\in[0,T]$. By the property of the weak convergence

$$\|y(t)\|\le\liminf_{m\to\infty}\|y_m(t)\|,\quad |\dot y(t)|\le\liminf_{m\to\infty}|\dot y_m(t)|. \tag{4.23}$$

Therefore inequality (4.6) follows from (4.13) by taking the limit as $m\to\infty$.

Now we are going to show that y satisfies the abstract problem (4.1)–(4.2), i.e., it is a weak solution of the problem (1.2)–(1.5). By the definition of the approximate solution

$$\ddot y_m+\kappa\dot y_m+\alpha Ay_m-\beta By_m-\gamma Gy_m=P_m^*f \tag{4.24}$$

in V', a.e. on $[0,T]$, and

$$y_m(0)=P_mu_0,\quad \dot y(0)=P_mv_0. \tag{4.25}$$

Clearly, we can pass to the limit in V' for $\ddot{y}_m, \kappa\dot{y}_m, Ay_m, By_m$, and $P_m^* f$ as $m \to \infty$. For the nonlinear operator G we have estimate (2.5)

$$\|Gy_m - Gy\|_{V'} \le c(\|y_m\|^2 + \|y\|^2)|\nabla y_m - \nabla y|.$$

The norms $\|y_m\|$ and $\|y\|$ are bounded by estimates (4.13) and (4.6). Since y_m and y satisfy all the conditions of Lemma 2.2, we conclude that $|\nabla y_m - \nabla y| \to 0$ in $L^2(0,T)$ as $m \to \infty$. Thus $Gy_m \to Gy$ in $L^2(0,T;V')$, and the passage to the limit as $m \to \infty$ in (4.24) is justified.

Concerning the initial conditions (4.25), it was also argued in Lemma 2.2 that the weak convergence of y_m to y in $L^2(0,T;V)$ implies that $y_m(t) \rightharpoonup y(t)$ weakly in V for any $t \in [0,T]$. Since $y_m(0) = P_m u_0 \to u_0$ in V, we conclude that $y(0) = u_0$. A straightforward modification of Lemma 2.2 shows that $\dot{y}_m \rightharpoonup \dot{y}$ weakly in H for any $t \in [0,T]$. Therefore $\dot{y}(0) = v_0$.

Inequality (4.7) follows from (4.15) by passing to the limit as $m \to \infty$, like in (4.22). Also note that since the weak solution y is unique, then the entire sequence $\{y_m\}_{m=1}^{\infty}$ of the approximate solutions is weakly convergent to y, and not just its subsequence.

The only parts of the Theorem that still need a proof are the assertions about the convergence of the approximate solutions, and the continuity of the solutions. We proceed as at the end of the proof for the strong damping case.

Approximate f by a sequence $g_n \in L^2(0,T;H)$, such that $g_n \to f$ in $L^2(0,T;V')$, and $\dot{g}_n \to \dot{f}$ in $L^2(0,T;V')$ as $n \to \infty$. We have

(4.26)
$$\begin{aligned}
&\|y_m(u_0,v_0,f) - y(u_0,v_0,f)\|_V \\
&\le \|y_m(u_0,v_0,f) - y_m(u_0,v_0,g_n)\|_V \\
&\quad + \|y_m(u_0,v_0,g_n) - y(u_0,v_0,g_n)\|_V + \|y(u_0,v_0,g_n) - y(u_0,v_0,f)\|_V.
\end{aligned}$$

Let $\epsilon > 0$. The first and the third terms in the right side of (4.26) are less than ϵ, for sufficiently large n, by (4.7). The second term is less than ϵ, for sufficiently large m, by Lemma 4.8. The same argument goes for $|\dot{y}_m(u_0,v_0,f) - \dot{y}(u_0,v_0,f)|$. Since the $y_m \in C([0,T];V)$, and $\dot{y}_m \in C([0,T];H)$, it implies that $y \in C([0,T];V)$, and $\dot{y} \in C([0,T];H)$. □

Lemma 4.8. *Let $\mu = 0$. Suppose that $u_0 \in V$, $v_0 \in H$, $T > 0$, and $g \in L^2(0,T;H)$. Let y be the weak solution of the problem* (1.2)–(1.5)*, and y_m be its approximate solution. Then*

(4.27)
$$\begin{aligned}
&|\dot{y}_m(t) - \dot{y}(t)|^2 + \|y_m(t) - y(t)\|^2 \\
&\le c\left(|P_m v_0 - v_0|^2 + \|P_m u_0 - u_0\|^2 + \|P_m u_0 - u_0\|^4 + \int_0^t |P_m g(s) - g(s)|^2\,ds\right)
\end{aligned}$$

for any $t \in [0,T]$,

Proof. Use (4.12) with $f = g$, $\bar{f} = P_m g$, as well as $\bar{u}_0 = P_m u_0$, $\bar{v}_0 = P_m v_0$. □

5. Vanishing strong damping $\mu \to 0$

Fix the initial conditions u_0, v_0 and the function f in equations (3.1), and (4.1) describing the strong and the weak damping cases. Let the corresponding solutions be $y^{(\mu)}$ and y. The goal of this section is to show that $y^{(\mu)} \to y$ as $\mu \to 0$. Thus the solution in the weak damping case is the limit of the strong damping solutions, when the strong damping is vanishing.

A comparison of the equations (3.1) and (4.1) shows that their difference is the $\mu A\dot{y}^{(\mu)}$ term. To make it approach zero, as $\mu \to 0$, one has to assure that the set $\|\dot{y}^{(\mu)}\|_V$ is either bounded, or its bound does not grow faster than $1/\mu$. However, estimate (3.7) shows that this is not the case. This forces us to consider solutions for some special choices of the initial conditions and f.

The operator $A : V \to V'$ was defined by $\langle Au, v\rangle = \int_\Omega \Delta u \Delta v\, dx,\ u, v \in V$. Define the domain of A by $D(A) = \{v \in V : |Av| < \infty\}$. Let the norm of $v \in D(A)$ be given by $|Av|$. Then $D(A)$ is a Hilbert space, and $A : D(A) \to H$ is an isometry, see [11].

Since $D(A)$ is densely and compactly embedded in V, we can consider the Gelfand triple $D(A) \subset V \subset H$, where V is identified with its dual V', and $[D(A)]'$ with H, see Section 2.4.2 of [11] for details. Within the framework of this new triple, the results for the solutions $y^{(\mu)}$ and y obtained in Sections 3 and 4 can be restated as follows.

Lemma 5.1. *Suppose that $f \in L^2(0,T;V)$, $u_0 \in D(A)$, and $v_0 \in V$. Then both $y^{(\mu)}$ and y satisfy $y^{(\mu)} \in C([0,T];D(A))$, $\dot{y}^{(\mu)} \in C([0,T];V)$, and*

$$|Ay^{(\mu)}(t)|^2 + \|\dot{y}^{(\mu)}(t)\|^2 \le c\Big(\|v_0\|^2 + |Au_0|^2 + |Au_0|^4 + \int_0^t \|f(s)\|^2\, ds\Big) \tag{5.1}$$

for any $0 \le t \le T$. That is, inequality (5.1) is satisfied for any $\mu \ge 0$ for the same $c > 0$.

Lemma 5.2. *Suppose that $f \in L^2(0,T;V)$, $u_0 \in D(A)$, and $v_0 \in V$. Then $y^{(\mu)} \to y$ in $C([0,T];V)$, and $\dot{y}^{(\mu)} \to \dot{y}$ in $C([0,T];H)$ as $\mu \to 0$.*

Proof. Let $z = y^{(\mu)} - y$. Subtract (4.1) from (3.1) to obtain

$$\ddot{z} + \kappa\dot{z} + \alpha Az - \beta Bz - \gamma(Gy^{(\mu)} - Gy) = -\mu A\dot{y}^{(\mu)}, \tag{5.2}$$

with $z(0) = 0$, $\dot{z}(0) = 0$.

By Lemma 5.1, $\dot{z} \in L^2(0,T;V)$. Take the inner product of (5.2) and $\dot{z}$ in H, and then use Lemma 3.3 to obtain

$$\frac{1}{2}\frac{d}{dt}\{|\dot{z}|^2 + \alpha\|z\|^2\} = -\mu\langle A\dot{y}^{(\mu)}, \dot{z}\rangle - \kappa|\dot{z}|^2 + \beta(\Delta z, \dot{z}) + \gamma(Gy^{(\mu)} - Gy, \dot{z}).$$

Integrate this equality from 0 to t, and use the Lipschitz continuity of G on bounded subsets of V, as shown in (2.4), to get

$$|\dot{z}|^2 + \alpha\|z\|^2 \le c\bigg(\mu\|A\dot{y}^{(\mu)}\|_{V'} + \int_0^t \big(|\dot{z}|^2 + \|z\|^2\big)\, ds\bigg). \tag{5.3}$$

Let $w \in V$ with $\|w\| \leq 1$. From the definition of A

$$|\langle A\dot{y}^{(\mu)}, w\rangle| \leq \|\dot{y}^{(\mu)}\| \, \|w\| \leq \|\dot{y}^{(\mu)}\| \text{ implies } \|A\dot{y}^{(\mu)}\|_{V'} \leq \|\dot{y}^{(\mu)}\|.$$

From (5.1) we get $\|\dot{y}^{(\mu)}(t)\| \leq c, 0 \leq t \leq T$. Finally, Gronwall's inequality applied to (5.3) gives

$$|\dot{z}|^2 + \alpha\|z\|^2 \leq c\mu\|A\dot{y}^{(\mu)}\|_{V'} \leq c\mu.$$

The continuity follows by letting $\mu \to 0$. □

The main result of this section is:

Theorem 5.3. *Let $y^{(\mu)}$, $\mu > 0$, and $y^{(0)}$ be the solutions of the problem* (1.2)–(1.5) *in the strong and the weak damping cases, correspondingly. Then the mappings $\mu \to y^{(\mu)}$ from $[0,\infty)$ into $C([0,T];V)$, and $\mu \to \dot{y}^{(\mu)}$ from $[0,\infty)$ into $C([0,T];H)$, are continuous.*

Proof. The difficult case is $\mu \to 0$. For simplicity, we will break it into two steps.

Assume $f \in L^2(0,T;V)$, $u_0 \in V$, $v_0 \in H$. Choose sequences $u_n \in D(A)$, such that in $u_n \to u_0$ in V, and $v_n \in V$, such that $v_n \to v_0$ in H as $n \to \infty$. For example, we can let $u_n = P_n u_0$, and $v_n = P_n v_0$, since the eigenfunctions $w_k \in D(A)$ for any $k \in \mathbb{N}$. Then

(5.4)
$$\begin{aligned}
&\|y^{(\mu)}(u_0,v_0,f) - y^{(0)}(u_0,v_0,f)\|_V \\
&\leq \|y^{(\mu)}(u_0,v_0,f) - y^{(\mu)}(u_n,v_n,f)\|_V + \|y^{(\mu)}(u_n,v_n,f) - y^{(0)}(u_n,v_n,f)\|_V \\
&\quad + \|y^{(0)}(u_n,v_n,f) - y^{(0)}(u_0,v_0,f)\|_V.
\end{aligned}$$

Let $\epsilon > 0$. The first and the third terms in the right side of (5.4) are less than ϵ, for sufficiently large n, by (3.12), and (4.12). The second term is less than ϵ, for sufficiently small μ, by Lemma 5.2. Thus we have the required convergence for $f \in L^2(0,T;V)$, and any $u_0 \in V$, $v_0 \in H$.

Now assume that $f, \dot{f} \in L^2(0,T;V')$. Approximate f by functions $g_n \in L^2(0,T;V)$, such that $g_n \to f$, and $\dot{g}_n \to \dot{f}$ in $L^2(0,T;V')$ as $n \to \infty$. Then

(5.5)
$$\begin{aligned}
&\|y^{(\mu)}(u_0,v_0,f) - y^{(0)}(u_0,v_0,f)\|_V \\
&\leq \|y^{(\mu)}(u_0,v_0,f) - y^{(\mu)}(u_0,v_0,g_n)\|_V + \|y^{(\mu)}(u_0,v_0,g_n) - y^{(0)}(u_0,v_0,g_n)\|_V \\
&\quad + \|y^{(0)}(u_0,v_0,g_n) - y^{(0)}(u_0,v_0,f)\|_V.
\end{aligned}$$

Let $\epsilon > 0$. The first and the third terms in the right side of (5.5) are less than ϵ, for sufficiently large n, by (3.14), and (4.12). The second term is less than ϵ, for sufficiently small μ, as was shown in (5.4).

A similar argument goes for the derivatives $\dot{y}^{(\mu)}$, and $\dot{y}^{(0)}$. This proves the result for $\mu \to 0$.

If $\mu \to \nu > 0$, then estimate (3.7) shows that the derivatives $\dot{y}^{(\mu)}$ are uniformly bounded in $L^2(0,T;V)$ for μ on any interval in $\mathbb{R}^+$, bounded away from zero. Then the required continuity results are obtained in a straightforward manner, as in the proof of Lemma 5.2. □

6. Car motion on a bridge

We model a long bridge as an arch over the interval $\Omega = (0, \pi)$. Its shape $y(x,t)$, $x \in \Omega$, $t \geq 0$ is required to satisfy the one-dimensional arch equation (1.2). For definiteness, let us assume that it satisfies the hinged boundary conditions $y(0,t) = y''(0,t) = 0$, and $y(\pi,t) = y''(\pi,t) = 0$. The proper variational setting for this problem is described in Definition 3.1, with $V = H_0^1(\Omega) \cap H^2(\Omega)$.

We are interested in the dynamics of the bridge when cars move across it. The cars are modeled by concentrated loads represented by a forcing function $f(t) \in V'$ for $t \geq 0$. For simplicity we assume that all cars have the same mass.

Since any $w \in V$ can be considered to be continuous on Ω, let us define a linear functional $\delta_h : V \to \mathbb{R}$, $h \in \mathbb{R}$ by

$$\langle \delta_h, w \rangle = \begin{cases} w(h), & h \in \Omega, \\ 0, & h \notin \Omega. \end{cases}$$

For $h \in \Omega$, we have $|\langle \delta_h, w \rangle| \leq |w(h)| \leq c\|w\|$, so we conclude that $\delta_h \in V'$. For convenience, we will also use the more common notation $\delta_h = \delta(x-h)$.

Let a car move across the bridge with the velocity $v > 0$. This motion is modeled by the forcing function $f(t) = \delta(x - vt)$. Of course, $f(t) = 0$ for $t \leq 0$, and $t \geq \pi/v$. As we have just shown, $f(t) \in V'$ for $t \geq 0$.

Let $t_1, t_2 \in \Omega$. Then for any $w \in V$

$$\begin{aligned} |\langle \delta(x - vt_1) - \delta(x - vt_2), w \rangle| = |w(vt_1) - w(vt_2)| &\leq \int_{vt_1}^{vt_2} |w'(s)|\, ds \\ &\leq cv|t_1 - t_2|\, \|w\|_V. \end{aligned}$$

Thus $f \in C_b(\mathbb{R}; V')$. By Theorems 3.6 and 4.4, the solutions for the bridge motion $y(x,t)$ exist on any interval $[0,T]$ in the weak and the strong damping cases.

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