J. Korean Math. Soc. **54** (2017), No. 3, pp. 877–896
https://doi.org/10.4134/JKMS.j160293
pISSN: 0304-9914 / eISSN: 2234-3008

ON COMPLETE CONVERGENCE AND COMPLETE MOMENT CONVERGENCE FOR A CLASS OF RANDOM VARIABLES

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Abstract. In this paper, the complete convergence and complete moment convergence for a class of random variables satisfying the Rosenthal type inequality are investigated. The sufficient and necessary conditions for the complete convergence and complete moment convergence are provided. As applications, the Baum-Katz type result and the Marcinkiewicz-Zygmund type strong law of large numbers for a class of random variables satisfying the Rosenthal type inequality are established. The results obtained in the paper extend the corresponding ones for some dependent random variables.

1. Introduction

Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants. The limit behavior for the maximum partial sum $\max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni}X_{ni}$ is very important in probability limit theory and mathematical statistics. There exist several versions available in the literature for independent random variables with assumption of control on their moments. If the independent case is classical in the literature, the treatment of dependent variables is more recent.

One of the dependence structure that has attracted the interest of probabilists and statisticians is negative association. The concept of negatively associated random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [10] as follows.

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA, in short) if for every pair of disjoint subsets $A, B \subset$

Received April 25, 2016; Revised July 29, 2016.
2010 *Mathematics Subject Classification.* 60F15.
Key words and phrases. complete convergence, complete moment convergence, stochastic domination, Rosenthal type inequality.
Supported by the National Natural Science Foundation of China (11501004, 11501005, 11671012), the Natural Science Foundation of Anhui Province (1508085J06), the Key Projects for Academic Talent of Anhui Province (gxbjZD2016005), the Provincial Natural Science Research Project of Anhui Colleges (KJ2015A018).

$\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

Primarily motivated by the definition of NA, Chandra and Ghosal [4] introduced the following dependence.

Definition 1.1. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be asymptotically almost negatively associated (AANA, in short) if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\begin{aligned} &Cov(f(X_n), g(X_{n+1}, \ldots, X_{n+k})) \\ &\leq q(n)\{Var(f(X_n))Var(g(X_{n+1}, \ldots, X_{n+k}))\}^{1/2} \end{aligned}$$

for all $n, k \geq 1$ and for all coordinatewise nondecreasing functions f and g whenever the variances exist.

Obviously, the family of AANA random variables contains NA (in particular, independent) random variables (with $q(n) = 0, n \geq 1$) and some more kinds of random variables which are not much deviated from being negatively associated. An example of AANA random variables which are not NA was constructed by Chandra and Ghosal [4]. For various results and applications of AANA random variables, one can refer to Wang et al. [18], Yuan and An [21, 22], Wang et al. [16], Yang et al. [20], Hu et al. [9], Shen et al. [13], Shen [11], Shen and Wu [12], Chen et al. [5], and so on.

The definition of stochastic domination below will paly an important role throughout the paper.

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leq CP(|X| > x)$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

Definition 1.3. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

In this case, we write $U_n \to a$ completely. In view of the Borel-Cantelli lemma, this implies that $U_n \to a$ almost surely (a.s.). The converse is true if random variables $\{U_n, n \geq 1\}$ are independent. This notion was given first by Hsu and Robbins [8]. It proved that arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [7] proved the converse. One of the most important generalizations was provided by Baum and Katz [3] for the strong law of large numbers as follows.

Theorem A. *Let $1/2 < \alpha \leq 1$ and $\alpha p > 1$. Let $\{X_n, n \geq 1\}$ be independent and identically distributed random variables with zero means. Then the following statements are equivalent*:

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty$ *for all* $\varepsilon > 0$.

Up to now, there are many versions of the Baum-Katz type results for independent and dependent random variables. Recently, Wang et al. [17] established the following result on complete convergence for weighted sums of AANA random variables, where $\psi(x) = 1$ or $\psi(x) = \log x$.

Theorem B. *Let $\alpha > \frac{1}{2}$ and $\alpha p \geq 1$. Assume that $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise AANA random variables which is stochastically dominated by a random variable X with $E|X|^p\psi(|X|) < \infty$, $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^q = O(n)$ for some $q > \max\{\frac{\alpha p-1}{\alpha-1/2}, 2\}$. Let $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ if $p \geq 1$. Suppose that the following two conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} q^{s/r}(n) < \infty$ *for some* $r \in (3\cdot 2^{k-1}, 4\cdot 2^{k-1}]$ *and* $r > \frac{\alpha p-1}{\alpha-1/2}$, *where integer number* $k \geq 1$ *if* $\alpha > 1/2$, $\alpha p > 1$ *and* $p \geq 2$;

(ii) $\sum_{n=1}^{\infty} q^2(n) < \infty$ *if* $\alpha > 1/2$, $\alpha p > 1$ *and* $1 \leq p < 2$, *or* $\alpha > 1/2$ *and* $\alpha p = 1$.

Then

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n) P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon n^{\alpha}\right) < \infty \text{ for all } \varepsilon > 0. \tag{1.1}$$

Inspired by the result of Theorem B, we will have the following generalizations:

(i) The array of AANA random variables is extended to a class of random variables satisfying the Rosenthal type maximal inequality, which includes AANA as a special case;

(ii) $\psi(x) = 1$ or $\psi(x) = \log x$ is extended to general slowly varying functions.

(iii) The sufficient condition in Theorem B will be extended to the sufficient and necessary condition.

The concept of slowly varying functions is as follows.

Definition 1.4. A real-valued function $l(x)$, positive and measurable on $(0,\infty)$, is said to be slowly varying at infinity if

$$\lim_{x\to\infty}\frac{l(x\lambda)}{l(x)}=1 \tag{1.2}$$

for each $\lambda>0$.

This work is organized as follows: some preliminary lemmas are provided in Section 2. The complete convergence and complete moment convergence for weighted sums of a class of random variables satisfying the Rosenthal type inequality are presented in Section 3 and Section 4, respectively.

Throughout the paper, $C, C_1, C_2, \ldots$ denote positive constants which may be different in various places. $a_n=O(b_n)$ represents $a_n\le Cb_n$ for all $n\ge 1$ and $I(A)$ is the indicator function on the set A. Set $\log x=\ln\max(x,e)$ and $X^+=XI(X\ge 0)$. $\lfloor x\rfloor$ denotes the integer part of x.

2. Preliminary lemmas

In this section, we will provide some preliminary facts needed for the proof of our main results. The first one is the basic property for stochastic domination. For the proof, one can refer to Wu [19], or Shen et al. [14].

Lemma 2.1. *Let $\{X_{ni}, i\ge 1, n\ge 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For any $\alpha>0$ and $b>0$, the following two statements hold*:

$$E|X_{ni}|^\alpha I\left(|X_{ni}|\le b\right)\le C_1\left[E|X|^\alpha I\left(|X|\le b\right)+b^\alpha P\left(|X|>b\right)\right],$$
$$E|X_{ni}|^\alpha I\left(|X_{ni}|>b\right)\le C_2E|X|^\alpha I\left(|X|>b\right),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_{ni}|^\alpha\le CE|X|^\alpha$, where C is a positive constant.

The following one comes from Bai and Su [2] and will be needed in the proof of Lemma 2.3.

Lemma 2.2. *Let $l(x)>0$ be a slowly varying function at infinity. Then the following statements hold*:

(i) $\lim\limits_{x\to\infty}\frac{l(x+u)}{l(x)}=1$ *for each* $u>0$;

(ii) $\lim\limits_{k\to\infty}\sup\limits_{2^k\le x<2^{k+1}}\frac{l(x)}{l(2^k)}=1$;

(iii) $\lim\limits_{x\to\infty}x^\delta l(x)=\infty$, $\lim\limits_{x\to\infty}x^{-\delta}l(x)=0$ *for each* $\delta>0$;

(iv) *For any $r>0$, $\eta>0$, there exist positive constants c_1 and c_2 such that for any positive number k, $c_1 2^{kr}l(2^k\eta)\le\sum_{j=1}^k 2^{jr}l(2^j\eta)\le c_2 2^{kr}l(2^k\eta)$.*

(v) *For any $r<0$, $\eta>0$, there exist positive constants d_1 and d_2 such that for any positive number k, $d_1 2^{kr}l(2^k\eta)\le\sum_{j=k}^\infty 2^{jr}l(2^j\eta)\le d_2 2^{kr}l(2^k\eta)$.*

With Lemma 2.2 accounted for, we can get the following important property for slowly varying functions, which will be applied to prove the main result of the paper. For the details of the proof, one can refer to Zhou [23] for instance.

Lemma 2.3. *Let $l(x) > 0$ be a slowly varying function at infinity. Then the following statements hold*:
(i) $\sum_{n=1}^{m} n^{s-1} l(n) \le C m^s l(m)$ *for* $s > 0$ *and positive integer* m;
(ii) $\sum_{n=m}^{\infty} n^{s-1} l(n) \le C m^s l(m)$ *for* $s < 0$ *and positive integer* m.

Lemma 2.4. *Let $\{X_n, n \ge 1\}$ be a sequence of random variables such that*

$$E\left|\sum_{i=1}^{n} X_i'\right|^2 \le C_1 \sum_{i=1}^{n} E\left|X_i'\right|^2, \tag{2.1}$$

where $X_i' = I(X_i > x) - EI(X_i > x)$ and $X_i' = I(X_i < -x) - EI(X_i < -x)$. Here, $x \ge 0$ is arbitrary and C_1 is a positive constant. Then there exists a positive constant C such that for any $x \ge 0$ and all $n \ge 1$,

$$\left[1 - P\left(\max_{1\le k\le n} |X_k| > x\right)\right]^2 \sum_{k=1}^{n} P\left(|X_k| > x\right) \le CP\left(\max_{1\le k\le n} |X_k| > x\right). \tag{2.2}$$

Proof. Denote $A_k = (|X_k| > x)$ and

$$\alpha_n = 1 - P\left(\bigcup_{k=1}^{n} A_k\right) = 1 - P\left(\max_{1\le k\le n} |X_k| > x\right).$$

Without loss of generality, we assume that $\alpha_n > 0$. We have by C_r inequality and (2.1) that

$$\begin{aligned}
&E\left[\sum_{k=1}^{n}\left(I(A_k) - EI(A_k)\right)\right]^2\\
&= E\left[\sum_{k=1}^{n}\left(I(X_k > x) - EI(X_k > x)\right) + \sum_{k=1}^{n}\left(I(X_k < -x) - EI(X_k < -x)\right)\right]^2\\
&\le 2E\left[\sum_{k=1}^{n}\left(I(X_k > x) - EI(X_k > x)\right)\right]^2 + 2E\left[\sum_{k=1}^{n}\left(I(X_k < -x) - EI(X_k < -x)\right)\right]^2\\
&\le C\sum_{k=1}^{n} P\left(A_k\right). \qquad (2.3)
\end{aligned}$$

By (2.3) and Holder's inequality, we have

$$\begin{aligned}
\sum_{k=1}^{n} P\left(A_k\right) &= \sum_{k=1}^{n} P\left(A_k \bigcup_{j=1}^{n} A_j\right) = \sum_{k=1}^{n} E\left[I(A_k) I\left(\bigcup_{j=1}^{n} A_j\right)\right]\\
&= E\left[\left(\sum_{k=1}^{n}\left(I(A_k) - EI(A_k)\right)\right) I\left(\bigcup_{j=1}^{n} A_j\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n} P\left(A_k\right) P\left(\bigcup_{j=1}^{n} A_j\right) \\
\leq\ & \left[E\left(\sum_{k=1}^{n}\left(I(A_k)-EI(A_k)\right)\right)^2 EI\left(\bigcup_{j=1}^{n} A_j\right)\right]^{1/2} \\
&+(1-\alpha_n)\sum_{k=1}^{n} P\left(A_k\right) \\
\leq\ & \left[\frac{C(1-\alpha_n)}{\alpha_n}\alpha_n \sum_{k=1}^{n} P\left(A_k\right)\right]^{1/2}+(1-\alpha_n)\sum_{k=1}^{n} P\left(A_k\right) \\
\leq\ & \frac{1}{2}\left[\frac{C(1-\alpha_n)}{\alpha_n}+\alpha_n \sum_{k=1}^{n} P\left(A_k\right)\right]+(1-\alpha_n)\sum_{k=1}^{n} P\left(A_k\right).
\end{aligned}
$$

By reorganizing the inequality above, we can get the desired result (2.2) immediately. This completes the proof of the lemma. □

The last one comes from Sung [15].

Lemma 2.5. *Let $Y_n, Z_n, n \geq 1$ be random variables. Then for any $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$
\begin{aligned}
&E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_i+Z_i)\right|-\varepsilon a\right)^{+} \\
\leq\ & \left(\frac{1}{\varepsilon^q}+\frac{1}{q-1}\right)\frac{1}{a^{q-1}}E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Y_i\right|^{q}\right)+E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Z_i\right|\right).
\end{aligned}
$$

3. Complete convergence

In this section, we will study the complete convergence for a class of random variables satisfying the Rosenthal type inequality as follows. The first one presents the sufficient condition for complete convergence.

Theorem 3.1. *Let $l(x) > 0$ $(x > 0)$ be a slowly varying function, and $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X with $E|X|^p l(|X|^{1/\alpha}) < \infty$ for some $\alpha > \frac{1}{2}$ and $\alpha p \geq 1$. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying $\sum_{i=1}^{n}|a_{ni}|^q = O(n)$ for some $q > \max\{\frac{\alpha p-1}{\alpha-1/2}, 2\}$. Assume that for any $r \geq 2$, there exists a positive constant C_r depending only on r such that*

$$
E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}(X'_{ni}-EX'_{ni})\right|^{r}\right) \tag{3.1}
$$

$$\leq C_r\left[\sum_{i=1}^{n}|a_{ni}|^r E\left|X'_{ni}\right|^r+\left(\sum_{i=1}^{n}|a_{ni}|^2 E|X'_{ni}|^2\right)^{r/2}\right],$$

where $X'_{ni}=-n^\alpha I(X_{ni}<-n^\alpha)+X_{ni}I(|X_{ni}|\leq n^\alpha)+n^\alpha I(X_{ni}>n^\alpha)$ *or* $X'_{ni}=X_{ni}I(|X_{ni}|\leq n^\alpha)$. *Assume further that* $EX_{ni}=0$ *for all* $i\geq 1$ *and* $n\geq 1$ *if* $p\geq 1$. *Then*

$$\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|>\varepsilon n^\alpha\right)<\infty \quad \textit{for all} \quad \varepsilon>0. \tag{3.2}$$

Proof. The proof is inspired by Wang et al. [17]. Note that the proof for $X'_{ni}=X_{ni}I(|X_{ni}|\leq n^\alpha)$ is similar to that of $X'_{ni}=-n^\alpha I(X_{ni}<-n^\alpha)+X_{ni}I(|X_{ni}|\leq n^\alpha)+n^\alpha I(X_{ni}>n^\alpha)$. So we only need to prove (3.2) for $X'_{ni}=-n^\alpha I(X_{ni}<-n^\alpha)+X_{ni}I(|X_{ni}|\leq n^\alpha)+n^\alpha I(X_{ni}>n^\alpha)$. Without loss of generality, we can assume that $a_{ni}>0$ for all $i\geq 1$ and $n\geq 1$. For fixed $n\geq 1$, denote $X''_{ni}=X_{ni}-X'_{ni}$, $i\geq 1$. We will consider the following three cases.

Case 1: $p>1$.

Noting that $EX_{ni}=0$ and $X_{ni}=X'_{ni}+X''_{ni}$, we can easily get that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|>\varepsilon n^\alpha\right)\\
&\leq \sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}(X'_{ni}-EX'_{ni})\right|>\varepsilon n^\alpha/2\right)\\
&\quad+\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}(X''_{ni}-EX''_{ni})\right|>\varepsilon n^\alpha/2\right)\\
&\doteq I+J.
\end{aligned}$$

It is easily seen that for all $0<\gamma\leq q$,

$$\frac{1}{n}\sum_{i=1}^{n}a_{ni}^\gamma\leq\left(\frac{1}{n}\sum_{i=1}^{n}a_{ni}^q\right)^{\gamma/q},$$

which together with $\sum_{i=1}^{n}a_{ni}^q=O(n)$ implies that

$$\sum_{i=1}^{n}a_{ni}^\gamma=O(n) \quad \text{for all} \quad 0<\gamma\leq q. \tag{3.3}$$

Noting that $|X''_{ni}|\leq|X_{ni}|I(|X_{ni}|>n^\alpha)$, we have by Markov's inequality, (3.3) (taking $\gamma=1$), Lemmas 2.1 and 2.3(i) that

$$J\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}l(n)\sum_{i=1}^{n}a_{ni}E|X''_{ni}|$$

$$\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) E|X| I(|X| > n^{\alpha})$$
$$= C\sum_{j=1}^{\infty} E|X| I(j < |X|^{1/\alpha} \le j+1) \sum_{n=1}^{j} n^{\alpha p-1-\alpha} l(n)$$
$$\le C\sum_{j=1}^{\infty} j^{\alpha p-\alpha} l(j) E|X| I(j < |X|^{1/\alpha} \le j+1)$$
$$\le CE|X|^p l(|X|^{1/\alpha}) < \infty. \tag{3.4}$$

For I, it follows by Markov's inequality, (3.1) and Jensen's inequality that for any $r \ge 2$,

$$I \le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni})\right|^r\right)$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \sum_{i=1}^{n} a_{ni}^r E|X'_{ni}|^r$$
$$+ C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \left(\sum_{i=1}^{n} a_{ni}^2 E|X'_{ni}|^2\right)^{r/2}$$
$$\doteq I_1 + I_2. \tag{3.5}$$

We consider the following three cases.

(i) If $\alpha > 1/2$, $\alpha p > 1$ and $p \ge 2$, then we take $r = q$. Noting that $q > \max\{\frac{\alpha p-1}{\alpha-1/2}, 2\}$, we can see that $q > p$ and $\alpha p - 2 - \alpha q + q/2 < -1$. It follows by C_r inequality, Lemma 2.1, Lemma 2.3(ii) and (3.4) that

$$I_1 \le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \sum_{i=1}^{n} a_{ni}^q \left[E|X_{ni}|^q I(|X_{ni}| \le n^{\alpha}) + n^{\alpha q} P(|X_{ni}| > n^{\alpha})\right]$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} l(n) E|X|^q I(|X| \le n^{\alpha})$$
$$+ C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) E|X| I(|X| > n^{\alpha})$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha(p-q)-1} l(n) \sum_{j=1}^{n} j^{\alpha q} P(j-1 < |X|^{1/\alpha} \le j) + CE|X|^p l(|X|^{1/\alpha})$$
$$= C\sum_{j=1}^{\infty} j^{\alpha q} P(j-1 < |X|^{1/\alpha} \le j) \sum_{n=j}^{\infty} n^{\alpha(p-q)-1} l(n) + CE|X|^p l(|X|^{1/\alpha})$$
$$\le C\sum_{j=1}^{\infty} j^{\alpha p} l(j) P(j-1 < |X|^{1/\alpha} \le j) + CE|X|^p l(|X|^{1/\alpha})$$

$$\leq CE|X|^p l(|X|^{1/\alpha}) < \infty. \tag{3.6}$$

For I_2, noting that $EX^2 < \infty$ and $|X'_{ni}| \leq |X_{ni}|$, we have by (3.3) and Lemma 2.1 that

$$\begin{aligned} I_2 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \left(\sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2 \right)^{q/2} \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \left(\sum_{i=1}^{n} a_{ni}^2 EX^2 \right)^{q/2} \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2} l(n) < \infty. \end{aligned}$$

(ii) If $\alpha > 1/2$, $\alpha p > 1$ and $1 < p < 2$, then we take $r = 2$. Similar to the proofs of (3.5), (3.6) and (3.4), we have that

$$\begin{aligned} I &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n) \sum_{i=1}^{n} a_{ni}^2 \left[EX_{ni}^2 I(|X_{ni}| \leq n^\alpha) + n^{2\alpha} P(|X_{ni}| > n^\alpha) \right] \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} l(n) EX^2 I(|X| \leq n^\alpha) \\ &\quad + C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) E|X| I(|X| > n^\alpha) \end{aligned}$$

$$\leq CE|X|^p l(|X|^{1/\alpha}) < \infty. \tag{3.7}$$

(iii) If $\alpha > 1/2$, $\alpha p = 1$ and $p > 1$, then we take $r = 2$. Noting that $1 < p = 1/\alpha < 2$, we have by the proof of (3.7) that $I < \infty$.

Case 2: $p = 1$**.**

Since $\alpha p \geq 1$, we have $\alpha \geq 1$. Noting that $E|X| < \infty$, we have by $EX_{ni} = 0$, Lemma 2.1 and (3.3) that

$$\begin{aligned} & n^{-\alpha} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} EX'_{ni} \right| \\ &\leq n^{-\alpha} \sum_{i=1}^{n} a_{ni} E|X_{ni}| I(|X_{ni}| > n^\alpha) + \sum_{i=1}^{n} a_{ni} P(|X_{ni}| > n^\alpha) \\ &\leq Cn^{1-\alpha} E|X| I(|X| > n^\alpha) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies that for all n large enough,

$$n^{-\alpha} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} EX'_{ni} \right| < \frac{\varepsilon}{2}. \tag{3.8}$$

Hence,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha-2} l(n) P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| > \varepsilon n^{\alpha}\right)\\
\le\ & \sum_{n=1}^{\infty} n^{\alpha-2} l(n) \sum_{i=1}^{n} P(|X_{ni}| > n^{\alpha})\\
&+ \sum_{n=1}^{\infty} n^{\alpha-2} l(n) P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni} X'_{ni}\right| > \varepsilon n^{\alpha}\right)\\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha-1} l(n) P(|X| > n^{\alpha})\\
&+ C\sum_{n=1}^{\infty} n^{\alpha-2} l(n) P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni} (X'_{ni} - EX'_{ni})\right| > \frac{\varepsilon n^{\alpha}}{2}\right)\\
\doteq\ & CI'_1 + CI'_2. \qquad (3.9)
\end{aligned}
$$

For I'_1, we have by Lemma 2.3(i) and $E|X|l(|X|^{1/\alpha}) < \infty$ that

$$
\begin{aligned}
I'_1 &= \sum_{i=1}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha}) \sum_{n=1}^{i} n^{\alpha-1} l(n)\\
&\le C\sum_{i=1}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha}) i^{\alpha} l(i)\\
&\le CE|X|l(|X|^{1/\alpha}) < \infty. \qquad (3.10)
\end{aligned}
$$

For I'_2, we have by Markov's inequality, Jensen's inequality, (3.1), Lemma 2.1, (3.3), (3.10) and Lemma 2.3(ii) that

$$
\begin{aligned}
I'_2 &\le C\sum_{n=1}^{\infty} n^{-\alpha-2} l(n) E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni})\right|^2\right)\\
&\le C\sum_{n=1}^{\infty} n^{-\alpha-2} l(n)\left[\sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2 I(|X_{ni}| \le n^{\alpha}) + n^{2\alpha}\sum_{i=1}^{n} a_{ni}^2 P(|X_{ni}| > n^{\alpha})\right]\\
&\le C\sum_{n=1}^{\infty} n^{-\alpha-1} l(n) EX^2 I(|X| \le n^{\alpha}) + C\sum_{n=1}^{\infty} n^{\alpha-1} l(n) P(|X| > n^{\alpha})\\
&\le C\sum_{n=1}^{\infty} n^{-\alpha-1} l(n) \sum_{k=1}^{n} EX^2 I((k-1)^{\alpha} < |X| \le k^{\alpha}) + C\\
&= C\sum_{k=1}^{\infty} EX^2 I((k-1)^{\alpha} < |X| \le k^{\alpha}) \sum_{n=k}^{\infty} n^{-\alpha-1} l(n) + C
\end{aligned}
$$

$$\leq C\sum_{k=1}^{\infty}k^{-\alpha}l(k)EX^2I((k-1)^\alpha<|X|\leq k^\alpha)+C$$

$$\leq CE|X|l(|X|^{1/\alpha})+C<\infty. \tag{3.11}$$

The desired result (3.2) follows by (3.9)-(3.11) immediately.

Case 3: $0<p<1$**.**

For fixed $n\geq 1$, denote for $1\leq j\leq n$ that

$$\sum_{i=1}^{j}a_{ni}X_{ni}=\sum_{i=1}^{j}a_{ni}X_{ni}I(|X_{ni}|\leq n^\alpha)+\sum_{i=1}^{j}a_{ni}X_{ni}I(|X_{ni}|>n^\alpha)$$
$$\doteq S'_{nj}+S''_{nj}. \tag{3.12}$$

Noting that $E|X|^pl(|X|^{1/\alpha})<\infty$, we have by Markov's inequality, Lemma 2.1, Lemma 2.3 and (3.3) that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P\left(\max_{1\leq j\leq n}|S'_{nj}|>\varepsilon n^\alpha\right)\\
&\leq \varepsilon^{-1}\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}l(n)E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}I(|X_{ni}|\leq n^\alpha)\right|\right)\\
&\leq \varepsilon^{-1}\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}l(n)\sum_{i=1}^{n}a_{ni}E|X_{ni}|I(|X_{ni}|\leq n^\alpha)\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha}l(n)E|X|I(|X|\leq n^\alpha)+C\sum_{n=1}^{\infty}n^{\alpha p-1}l(n)P(|X|>n^\alpha)\\
&= C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha}l(n)\sum_{j=1}^{n}E|X|I(j-1<|X|^{1/\alpha}\leq j)\\
&\quad +C\sum_{n=1}^{\infty}n^{\alpha p-1}l(n)\sum_{j=n}^{\infty}P(j<|X|^{1/\alpha}\leq j+1)\\
&\leq C\sum_{j=1}^{\infty}j^{\alpha}P(j-1<|X|^{1/\alpha}\leq j)\sum_{n=j}^{\infty}n^{\alpha p-1-\alpha}l(n)\\
&\quad +C\sum_{j=1}^{\infty}P(j<|X|^{1/\alpha}\leq j+1)\sum_{n=1}^{j}n^{\alpha p-1}l(n)\\
&\leq C\sum_{j=1}^{\infty}j^{\alpha p}l(j)P(j-1<|X|^{1/\alpha}\leq j)\\
&\quad +C\sum_{j=1}^{\infty}j^{\alpha p}l(j)P(j<|X|^{1/\alpha}\leq j+1)
\end{aligned}$$

$$(3.13)\quad \leq E|X|^p l(|X|^{1/\alpha}) < \infty$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\leq j\leq n} |S''_{nj}| > \epsilon n^{\alpha}\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p/2-2} l(n) E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}I(|X_{ni}|>n^{\alpha})\right|^{p/2}\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p/2-2} l(n) \sum_{i=1}^{n} a_{ni}^{p/2} E|X_{ni}|^{p/2} I(|X_{ni}|>n^{\alpha}) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p/2-1} l(n) E|X|^{p/2} I(|X|>n^{\alpha}) \\
&= C\sum_{n=1}^{\infty} n^{\alpha p/2-1} l(n) \sum_{j=n}^{\infty} E|X|^{p/2} I(j<|X|^{1/\alpha}\leq j+1) \\
&\leq C\sum_{j=1}^{\infty} j^{\alpha p/2} P(j<|X|^{1/\alpha}\leq j+1) \sum_{n=1}^{j} n^{\alpha p/2-1} l(n) \\
&\leq C\sum_{j=1}^{\infty} j^{\alpha p} l(j) P(j-1<|X|^{1/\alpha}\leq j)
\end{aligned}$$

$$(3.14)\qquad \leq CE|X|^p l(|X|^{1/\alpha}) < \infty.$$

Hence, the desired result (3.2) follows by (3.12)-(3.14) immediately. This completes the proof of the theorem. □

The next one presents the necessary condition for complete convergence.

Theorem 3.2. *Let $\alpha > \frac{1}{2}$ and $\alpha p \geq 1$. Let $l(x) > 0$ ($x > 0$) be a slowly varying function, and $\{X_n, n \geq 1\}$ be a sequence of random variables such that for all $x \geq 0$ and $n \geq 1$,*

$$(3.15)\qquad P(|X_n| > x) \geq CP(|X| > x),$$

and (2.1) holds. Assume further that $l(x) \geq C$ for some positive constant C when $\alpha p = 1$. If

$$(3.16)\qquad \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty \quad \text{for all} \quad \varepsilon > 0,$$

then $E|X|^p l(|X|^{1/\alpha}) < \infty$.

Proof. Denote $S_j = \sum_{i=1}^{j} X_i$ for $1 \leq j \leq n$ and $S_0 = 0$. It follows by (3.16) that for any $\varepsilon > 0$,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\leq j\leq n} |X_j| > \varepsilon n^{\alpha}\right) \\ \leq &\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\leq j\leq n} |S_j| > \frac{1}{2}\varepsilon n^{\alpha}\right) \\ &+ \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\leq j\leq n} |S_{j-1}| > \frac{1}{2}\varepsilon n^{\alpha}\right) \\ &< \infty. \end{aligned} \tag{3.17}$$

Note that $\alpha p \geq 1$ and $l(x) \geq C > 0$ when $\alpha p = 1$. Hence, we have for all n large enough that

$$\begin{aligned} \sum_{i=n}^{2n} i^{\alpha p-2} l(i) P\left(\max_{1\leq j\leq i} |X_j| > \left(\frac{\varepsilon}{2^{\alpha}}\right) i^{\alpha}\right) &\geq C n^{\alpha p-1} l(n) P\left(\max_{1\leq j\leq n} |X_j| > \varepsilon n^{\alpha}\right) \\ &\geq CP\left(\max_{1\leq j\leq n} |X_j| > \varepsilon n^{\alpha}\right), \end{aligned}$$

which together with (3.17) yields that

$$P\left(\max_{1\leq j\leq n} |X_j| > \varepsilon n^{\alpha}\right) \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Hence, we have that for all n large enough,

$$P\left(\max_{1\leq j\leq n} |X_j| > \varepsilon n^{\alpha}\right) < \frac{1}{2}. \tag{3.19}$$

It follows by Lemma 2.4, (3.19) and (3.15) that for any $\varepsilon > 0$ and all n large enough,

$$P\left(\max_{1\leq j\leq n} |X_j| > \varepsilon n^{\alpha}\right) \geq C \sum_{i=1}^{n} P\left(|X_i| > \varepsilon n^{\alpha}\right) \geq CnP\left(|X| > \varepsilon n^{\alpha}\right). \tag{3.20}$$

Take $\varepsilon = 1$. It follows from (3.17), (3.20) and Lemma 2.2 that

$$\begin{aligned} \infty > \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\leq j\leq n} |X_j| > n^{\alpha}\right) \ &\geq\ C \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P\left(|X| > n^{\alpha}\right) \\ = C \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) \sum_{j=n}^{\infty} P\left(j < |X|^{1/\alpha} \leq j+1\right) \\ = C \sum_{j=1}^{\infty} P\left(j < |X|^{1/\alpha} \leq j+1\right) \sum_{n=1}^{j} n^{\alpha p-1} l(n) \end{aligned}$$

$$\geq C\sum_{j=1}^{\infty}P\left(j<|X|^{1/\alpha}\leq j+1\right)\sum_{i=1}^{\lfloor\log_2 j\rfloor}\sum_{n=2^{i-1}}^{2^i-1}n^{\alpha p-1}l(n)$$
$$\geq C\sum_{j=1}^{\infty}P\left(j<|X|^{1/\alpha}\leq j+1\right)\sum_{i=1}^{\lfloor\log_2 j\rfloor}2^{i\alpha p}l\left(2^i\right)$$
$$\geq C\sum_{j=1}^{\infty}P\left(j<|X|^{1/\alpha}\leq j+1\right)2^{(\lfloor\log_2 j\rfloor)\alpha p}l\left(2^{\lfloor\log_2 j\rfloor}\right)$$
$$\geq C\sum_{j=1}^{\infty}P\left(j<|X|^{1/\alpha}\leq j+1\right)j^{\alpha p}l(j)$$
$$\geq CE|X|^p l\left(|X|^{1/\alpha}\right).$$

This completes the proof of the theorem. □

Combining Theorems 3.1 and 3.2, we can get the sufficient and necessary condition of complete convergence, i.e., Baum-Katz type result for a class of random variables satisfying the Rosenthal type inequality as follows.

Theorem 3.3. *Let* $\alpha>\frac{1}{2}$ *and* $\alpha p\geq 1$*. Let* $l(x)>0$ $(x>0)$ *be a slowly varying function, and* $\{X_n,n\geq 1\}$ *be a sequence of random variables such that for all* $x\geq 0$ *and* $n\geq 1$*,*

$$C_1P(|X|>x)\leq P(|X_n|>x)\leq C_2P(|X|>x),$$

where C_1 *and* C_2 *are positive constants, and* $EX_n=0$ *for all* $n\geq 1$ *if* $p\geq 1$*. Assume further that* (2.1) *and* (3.1) *hold, where* $a_{ni}\equiv 1$ *and* $X_{ni}\equiv X_i$ *for all* $i\geq 1$ *and* $n\geq 1$*. Suppose that* $l(x)\geq C$ *for some positive constant* C *when* $\alpha p=1$*. Then the following statements are equivalent*:

(i) $E|X|^p l(|X|^{1/\alpha})<\infty$;

(ii) *for all* $\varepsilon>0$*,*

$$\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_i\right|>\varepsilon n^{\alpha}\right)<\infty.$$

Similar to the proof of Theorem 3.1, we can get the following results on the complete convergence and Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of a class of random variables satisfying the Rosenthal type inequality.

Theorem 3.4. *Let* $\{X_n,n\geq 1\}$ *be a sequence of random variables which is stochastically dominated by a random variable* X *with* $E|X|^p<\infty$ *for some* $\alpha>\frac{1}{2}$ *and* $\alpha p\geq 1$*. Let* $\{a_n,n\geq 1\}$ *be a sequence of real numbers satisfying* $\sum_{i=1}^{n}|a_i|^q=O(n)$ *for some* $q>\max\{\frac{\alpha p-1}{\alpha-1/2},2\}$*. Assume that for any* $r\geq 2$*,* *there exists a positive constant C_r depending only on r such that*

$$
\begin{aligned}
&E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_i(X_i'-EX_i')\right|^r\right)\\
&\le C_r\left[\sum_{i=1}^{n}|a_i|^r E\left|X_i'\right|^r+\left(\sum_{i=1}^{n}|a_i|^2E|X_i'|^2\right)^{r/2}\right], \qquad (3.21)
\end{aligned}
$$

where $X_i' = -n^\alpha I(X_i < -n^\alpha) + X_i I(|X_i| \le n^\alpha) + n^\alpha I(X_i > n^\alpha)$ or $X_i' = X_i I(|X_i| \le n^\alpha)$. Assume further that $EX_n = 0$ for all $n \ge 1$ if $p \ge 1$. Then

$$
\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon n^\alpha\right) < \infty \text{ for all } \varepsilon > 0, \qquad (3.22)
$$

and

$$
n^{-\alpha}\sum_{i=1}^{n} a_i X_i \to 0 \quad a.s. \qquad (3.23)
$$

Proof. Similar to the proof of (3.2), we can get (3.22) immediately by taking $l(x) = 1$. So we only need to prove (3.23). It follows by (3.22) that, for all $\varepsilon > 0$,

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon n^\alpha\right)\\
&= \sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon n^\alpha\right)\\
&\ge \begin{cases} \displaystyle\sum_{k=0}^{\infty}(2^k)^{\alpha p-2}2^k P\left(\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon 2^{(k+1)\alpha}\right) & \text{if } \alpha p \ge 2,\\ \displaystyle\sum_{k=0}^{\infty}(2^{k+1})^{\alpha p-2}2^k P\left(\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon 2^{(k+1)\alpha}\right) & \text{if } 1 \le \alpha p < 2,\end{cases}\\
&\ge \begin{cases} \displaystyle\sum_{k=0}^{\infty} P\left(\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon 2^{(k+1)\alpha}\right) & \text{if } \alpha p \ge 2,\\ \displaystyle\frac{1}{2}\sum_{k=0}^{\infty} P\left(\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon 2^{(k+1)\alpha}\right) & \text{if } 1 \le \alpha p < 2.\end{cases}
\end{aligned}
$$

By the Borel-Cantelli lemma, we obtain that

$$
\frac{\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j} a_i X_i\right|}{2^{(k+1)\alpha}} \to 0 \text{ a.s.} \qquad (3.24)
$$

For all positive integers n, there exists a positive integer k such that $2^{k-1} \leq n \leq 2^k$. We have by (3.24) that

$$n^{-\alpha}\left|\sum_{i=1}^{n} a_i X_i\right| \leq \max_{2^{k-1}\leq n\leq 2^k} n^{-\alpha}\left|\sum_{i=1}^{n} a_i X_i\right| \leq \frac{2^{\alpha}\max_{1\leq j\leq 2^k}|\sum_{i=1}^{j} a_i X_i|}{2^{(k+1)\alpha}} \to 0 \quad \text{a.s.}$$

which implies that

$$n^{-\alpha}\sum_{i=1}^{n} a_i X_i \to 0 \quad \text{a.s.}$$

This completes the proof of the theorem. □

Remark 3.1. We should point out that there are many sequences of random variables satisfying the conditions (2.1) and (3.1), such as independent random variables, NA random variables, negatively supperadditive dependent random variables, φ-mixing random variables, $\tilde{\rho}$-mixing random variables, AANA random variables, and so on. Hence, the main results of the paper also hold for these sequences.

Remark 3.2. Comparing Theorem B with Theorem 3.1, we not only extend the sequences of AANA random variables to the case of a class of random variables satisfying the Rosenthal type inequality, but also extend the function $l(x) = 1$ or $l(x) = \log x$ to the case of general slowly varying functions. In addition, the necessary condition for the complete convergence is investigated in Theorem 3.2.

Remark 3.3. Comparing Theorem A with Theorems 3.3, we have the following generalizations:

(i) Independent and identically distributed random variables are extended to a class of random variables satisfying the Rosenthal type inequality and identical distribution is not needed;

(ii) slowly varying function $l(x)$ is added in Theorems 3.3, while is not in Theorem A; if we take $l(x) = 1$, then Theorem 3.3 yields Theorem A;

(iii) $1/2 < \alpha \leq 1$ and $\alpha p > 1$ are extended to $\alpha > 1/2$ and $\alpha p \geq 1$, respectively.

4. Complete moment convergence

In this section, we will study the complete moment convergence for a class of random variables satisfying the Rosenthal type inequality as follows. The first one presents the sufficient condition for complete moment convergence.

Theorem 4.1. *Suppose that the conditions of Theorem 3.1 hold for $p > 1$. Then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)^{+} < \infty \text{ for all } \varepsilon > 0. \tag{4.1}$$

Proof. It follows by Lemma 2.5 that for any $r \geq 2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)^{+} \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(a_{ni}X'_{ni} - Ea_{ni}X'_{ni})\right|\right)^{r} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(a_{ni}X''_{ni} - Ea_{ni}X''_{ni})\right|\right).
\end{aligned}$$

Similar to the proof of Theorem 3.1 in the case $p > 1$, we can get the desired result (4.1) immediately. The proof is completed. □

The next one presents the necessary condition for complete moment convergence.

Theorem 4.2. *Suppose that the conditions of Theorem 3.2 hold. If*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty \text{ for all } \varepsilon > 0, \tag{4.2}$$

then $E|X|^p l(|X|^{1/\alpha}) < \infty$.

Proof. Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| - \varepsilon n^{\alpha}\right)^{+} \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_0^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| - \varepsilon n^{\alpha} > t\right) dt \\
&\geq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_0^{\varepsilon n^{\alpha}} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| - \varepsilon n^{\alpha} > t\right) dt \\
&\geq \varepsilon \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > 2\varepsilon n^{\alpha}\right). \qquad (4.3)
\end{aligned}$$

Combining (4.2) and (4.3), we can get that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{4.4}$$

The desired result follows by (4.4) and Theorem 3.2 immediately. This completes the proof of the theorem. □

Combining Theorems 4.1 and 4.2, we can get the sufficient and necessary condition of complete moment convergence for a class of random variables satisfying the Rosenthal type inequality as follows.

Theorem 4.3. *Let $\alpha > \frac{1}{2}$, $p > 1$ and $\alpha p \geq 1$. Let $l(x) > 0$ $(x > 0)$ be a slowly varying function, and $\{X_n, n \geq 1\}$ be a sequence of mean zero random variables such that for all $x \geq 0$ and $n \geq 1$,*

$$C_1 P(|X| > x) \leq P(|X_n| > x) \leq C_2 P(|X| > x),$$

where C_1 and C_2 are positive constants. Assume further that (2.1) *and* (3.1) *hold, where $a_{ni} \equiv 1$ and $X_{ni} \equiv X_i$ for all $i \geq 1$ and $n \geq 1$. Suppose that $l(x) \geq C$ for some positive constant C when $\alpha p = 1$. Then the following statements are equivalent*:

(i) $E|X|^p l(|X|^{1/\alpha}) < \infty$;

(ii) *for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} l(n) E\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} X_i \right| - \varepsilon n^{\alpha} \right)^{+} < \infty. \tag{4.5}$$

Remark 4.1. Chow [6] obtained the following complete moment convergence result for independent and identically distributed random variables.

Theorem C. *Suppose that $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $EX_1 = 0$, $\alpha > 1/2$, $p \geq 1$ and $\alpha p > 1$. If $E[|X_1|^p + |X_1| \log(1 + |X_1|)] < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} E\left\{ \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} X_i \right| - \varepsilon n^{\alpha} \right\}^{+} < \infty. \tag{4.6}$$

Comparing Theorem C with Theorem 4.3, we have the following generalizations or improvements:

(i) A sequence of independent and identically distributed random variables is extended to a class of random variables and the identical distribution is not needed;

(ii) Theorem C only considers the sufficient condition for the complete moment convergence, while Theorem 4.3 presents the sufficient and necessary condition for complete moment convergence;

(iii) If we take $l(x) = 1$, then we can see that Theorem 4.3 implies Theorem C; in addition, the condition $E|X|^p < \infty$ is weaker than $E[|X_1|^p + |X_1| \log(1 + |X_1|)] < \infty$.

Acknowledgements. The authors are most grateful to the Editor-in-chief Yun Sung Choi and anonymous referee for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

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