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ON THE LARGE DEVIATION FOR THE GCF$_\epsilon$ EXPANSION WHEN THE PARAMETER $\epsilon \in [-1, 1]$

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Abstract. The GCF$_\epsilon$ expansion is a new class of continued fractions induced by the transformation $T_\epsilon : (0,1] \to (0,1]$:

$$T_\epsilon(x) = \frac{-1+(k+1)x}{1+k-k\epsilon x} \text{ for } x \in \big(1/(k+1), 1/k\big].$$

Under the algorithm T_ϵ, every $x \in (0,1]$ corresponds to an increasing digits sequences $\{k_n, n \geq 1\}$. Their basic properties, including the ergodic properties, law of large number and central limit theorem have been discussed in [4], [5] and [7]. In this paper, we study the large deviation for the GCF$_\epsilon$ expansion and show that: $\big\{\frac{1}{n}\log k_n, n \geq 1\big\}$ satisfies the different large deviation principles when the parameter ϵ changes in $[-1,1]$, which generalizes a result of L. J. Zhu [9] who considered a case when $\epsilon(k) \equiv 0$ (i.e., Engel series).

1. Introduction

Let $\epsilon : \mathbb{N} \to \mathbb{R}$ be a parameter function satisfying the condition $\epsilon(k)+k+1 > 0$ and let $T_\epsilon : (0,1] \to (0,1]$ be a transformation defined by

$$T_\epsilon(x) := \frac{-1+(k+1)x}{1+\epsilon(k)-k\epsilon(k)x} \text{ for } x \in B(k) := (1/(k+1), 1/k]. \tag{1.1}$$

Under the algorithm T_ϵ, every $x \in (0,1]$ is attached to an expansion, called generalized continued fraction (GCF$_\epsilon$) expansion (see [4]).

For any $x \in (0,1]$, the digits sequences $\{k_n\}_{n\geq 1}$ of the GCF$_\epsilon$ expansion is defined by

$$k_1 = k_1(x) := \left\lfloor \frac{1}{x} \right\rfloor, \quad \text{and} \quad k_n = k_n(x) := k_1\big(T_\epsilon^{n-1}(x)\big). \tag{1.2}$$

Then $k_n(x)$ satisfies

$$k_{n+1}(x) \geq k_n(x) \quad \text{for all } n \geq 1. \tag{1.3}$$

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It follows from the algorithm (1.1) that

$$x = \frac{A_n + B_n T_\epsilon^n(x)}{C_n + D_n T_\epsilon^n(x)} \text{ for all } n \geq 1,$$

where the numbers A_n, B_n, C_n, D_n are given by the following recursive relations (see [4] for details):

$$\begin{pmatrix} C_n & D_n \\ A_n & B_n \end{pmatrix} = \begin{pmatrix} C_{n-1} & D_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} k_n + 1 & k_n \epsilon(k_n) \\ 1 & 1 + \epsilon(k_n) \end{pmatrix}, \; n \geq 1. \tag{1.4}$$

$$\text{with} \quad \begin{pmatrix} C_0 & D_0 \\ A_0 & B_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

For any increasing integer vector $(k_1, \ldots, k_n)$, define the nth order cylinder as follows

$$B(k_1, \ldots, k_n) = \{x \in (0,1] : k_j(x) = k_j, \forall 1 \leq j \leq n\}.$$

Since there is a one-to-one correspondence between $x \in (0,1]$ and the non-decreasing integer sequence $(k_1, k_2, \ldots,)$, we have [4]

$$P(B(k_1, \ldots, k_n)) = \frac{B_n C_n - A_n D_n}{C_n(C_n k_n + D_n)} \tag{1.5}$$

and

$$P(B(k_1, \ldots, k_n, k_{n+1})) = \frac{B_n C_n - A_n D_n}{(C_n k_{n+1} + D_n)(C_n(k_{n+1} + 1) + D_n)}, \tag{1.6}$$

where $P(\cdot)$ denotes the usual Lebesgue measure. Moreover, for any $0 \leq b \leq \frac{1}{k_n}$,

$$\{x \in [0,1] : k_i(x) = k_i, \; 1 \leq i \leq n, T_\epsilon^n(x) \leq b\} = \Big[\frac{A_n}{C_n}, \frac{A_n + B_n b}{C_n + D_n b}\Big].$$

The GCF_ϵ transformation provides a big class of continued fractions algorithms which extends our knowledge on one-dimensional dynamical systems. With proper choice of the parameter ϵ, the GCF_ϵ expansions presented different stochastic properties and ergodic properties [4]. Specially, in the case of $-1 < \epsilon \leq 1$ and $\epsilon(k) = ck + c$, the metric properties of GCF_ϵ were derived in [7] and [8], respectively. the "0-1" law and central limit theorem were studied by L. Shen and Y. Zhou [5]. In the present paper, we consider the large deviation for the GCF_ϵ expansion and show that: $\left\{\frac{1}{n} \log k_n, n \geq 1\right\}$ satisfies the different large deviation principles when the parameter ϵ changes in $\epsilon \in [-1, 1]$, which generalizes a result of L. J. Zhu, [9] who considered a case when $\epsilon(k) \equiv 0$ (i.e., Engel series).

Now we introduce the large deviation principles. Let $\{X_n, n \geq 1\}$ be a sequence of the real valued random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. A function $I : \mathbb{R} \to [0, \infty]$ is called a good rate function if it is lower semi continuous and has compact level sets. We say that the sequence

$\{X_n, n \geq 1\}$ satisfies a large deviation principle with speed n and good rate function I under P, if for any Borel set Γ, we have

$$-\inf_{x\in\Gamma^o} I(x) \leq \liminf_{n\to\infty} \frac{1}{n}\log P(x_n \in \Gamma) \leq \limsup_{n\to\infty} \frac{1}{n}\log P(x_n \in \Gamma) \leq -\sup_{x\in\bar{\Gamma}} I(x),$$

where Γ^o and $\bar{\Gamma}$ denotes the interior and the closure of Γ respectively. For general theory of the large deviations, we can refer to Dembo and Zeitouni [1] and Varadhan [6].

In this paper, we denote by $(\Omega, \mathcal{F}, P)$ a probability space, where $\Omega = (0, 1]$, $\mathcal{F}$ is the Borel σ-algebra on Ω and P denotes the Lebesgue measure on $(\Omega, \mathcal{F})$. And $k_n(x)$ always denotes the n-th digit of GCF$_\epsilon$ defined by (1.2); A_n, B_n, C_n, D_n the numbers recursively defined by (1.4); and the parameters ϵ always satisfies $-1 \leq \epsilon(k) \leq 1$.

2. Preliminary

In this section, we present some fundamental properties about GCF$_\epsilon$ expansion for later use. The first lemma concerns the relationships between A_n, B_n, C_n, D_n which are recursively defined by (1.4).

Lemma 2.1 ([4, 7]). *For all $n \geq 1$ we have*

(i) $C_n = (k_n + 1)C_{n-1} + D_{n-1} > 0,\ C_0 = 1.$
(ii) $D_n = k_n\epsilon(k_n)C_{n-1} + (1 + \epsilon(k_n))D_{n-1},\ D_0 = 0.$
(iii) $B_nC_n - A_nD_n = \prod_{i=1}^{n}(k_i + 1 + \epsilon(k_i)) > 0.$
(iv) $k_nC_n + D_n = (k_nC_{n-1} + D_{n-1})(k_n + 1 + \epsilon(k_n)).$
(v) $\epsilon(k_n)C_n - D_n = \epsilon(k_n)C_{n-1} - D_{n-1}.$

Using this lemma, we can derive the following two lemmas

Lemma 2.2. *We have*

$$P(B(\underbrace{1, 1, \ldots, 1}_{n})) = \begin{cases} \frac{1+\epsilon}{(2+\epsilon)^n+\epsilon}, & \text{as } -1 < \epsilon \leq 1; \\ \frac{1}{n+1}, & \text{as } \epsilon = -1. \end{cases}$$

Proof. When $k_i \equiv k$ and $\epsilon(k) \equiv \epsilon$, Lemma 2.1(iii), (iv) and (v) give that

$$\begin{aligned} B_nC_n - A_nD_n &= (k + 1 + \epsilon)^n, \\ kC_n + D_n &= (kC_0 + D_0)(k + 1 + \epsilon)^n = k(k + 1 + \epsilon)^n, \\ \epsilon C_n - D_n &= \epsilon C_0 - D_0 = \epsilon. \end{aligned}$$

So we have when $k_i \equiv 1$ and $\epsilon \in [-1, 1]$,

$$\frac{B_nC_n - A_nD_n}{k_nC_n + D_n} = 1; \qquad C_n(1 + \epsilon) = (2 + \epsilon)^n + \epsilon.$$

Then by (1.5), we get when $\epsilon \in (-1, 1]$,

$$P(B(\underbrace{1, 1, \ldots, 1}_{n})) = \frac{1}{C_n} = \frac{1 + \epsilon}{(2 + \epsilon)^n + \epsilon}. \tag{2.1}$$

But when $\epsilon = -1$, the equality $C_n(1+\epsilon) = (2+\epsilon)^n + \epsilon$ cannot be used, and $\epsilon C_n - D_n = \epsilon$ becomes $C_n + D_n = 1$. Using $C_n + D_n = 1$ and Lemma 2.1(i), we get

$$C_n = 2C_{n-1} + D_n = C_{n-1} + 1 = C_0 + n = n + 1.$$

So when $\epsilon = -1$, we have

$$P(B(\underbrace{1, 1, \dots, 1}_{n})) = \frac{1}{C_n} = \frac{1}{n+1}. \tag{2.2}$$

Together (2.1) and (2.2) give the desired result. □

Since the sequence $\{k_n\}_{n \geq 1}$ is not a Markov chain, so it's difficult to get the exact probability of $(k_n \leq N)$ by using the nice method in [2]. However, the next lemma can give an estimate of $P(k_n \leq N)$.

Lemma 2.3. *For any positive number $N > 1$, when $-1 < \epsilon \leq 1$ we have*

$$\frac{1+\epsilon}{(2+\epsilon)^n + \epsilon} \leq P\Big(k_n \leq N\Big) \leq (1+n)^{N-1} \cdot \frac{1+\epsilon}{(2+\epsilon)^n + \epsilon};$$

and when $\epsilon = -1$ we have

$$\frac{1}{1+n} \leq P\Big(k_n \leq N\Big) \leq (1+n)^{N-2}.$$

Proof. First we check the number of all the nth order cylinders of $(k_n = j)$, which is denoted by $\sharp(k_n = j)$. We first show that

$$(1^\circ) \qquad \sharp(k_n = 1) = 1.$$
$$(2^\circ) \qquad \sharp(k_n = j) \leq n \cdot (1+n)^{j-2} \quad \text{for all } j \geq 2.$$
$$(3^\circ) \qquad \sharp(k_n \leq j) \leq (1+n)^{j-1} \quad \text{for all } j \geq 1.$$

In fact, by the increase of $k_n \geq 1$, we have $k_n = 1 = B(1, 1, \dots, 1)$ contains only one cylinder, thus (1°) is true.

Second, we prove (2°) by induction. Notice that, each cylinder $B(k_1, \dots, k_{n-1}, k_n)$ of $(k_n \leq j-1)$ corresponds to n cylinders of $(k_n = j)$ as

$$B(k_1, \dots, k_{n-1}, j),\ B(k_1, \dots, j, j),\ \dots, B(j, \dots, j, j).$$

Thus

$$(k_n = j) \leq n \cdot \sharp(k_n \leq j-1),$$

here "$\leq$" is actually "$<$", because the right side of it contains some double-counted cylinders.

Then with $\sharp(k_n = 1) = 1$, it is obvious that $\sharp(k_n = 2) = n \leq n \cdot (1+n)^{2-2}$. So (2°) is true for $j = 2$.

Now we suppose that (2°) is true for all of $j \leq i$, then for $j = i+1$,

$$\begin{aligned} \sharp(k_n = i+1) &\leq n\big(\sharp(k_n = 1) + \sharp(k_n = 2) + \sharp(k_n = 3) + \cdots + \sharp(k_n = i)\big) \\ &\leq n\big(1 + n + n(1+n) + n(1+n)^2 + \cdots + n(1+n)^{i-2}\big) \end{aligned}$$

$$= n\Big(1 + n + n(1+n) \cdot \frac{1-(1+n)^{i-2}}{1-(1+n)}\Big) = n(1+n)^{i-1},$$

which shows that (2°) is also true for $j = i + 1$. So (2°) is proved by math induction.

Third, (3°) is follows from (2°) that,

$$\sharp(k_n \le j) = \sharp(k_n = 1) + \sharp(k_n = 2) + \cdots + \sharp(k_n = j) \le (1+n)^{j-1}. \tag{2.3}$$

Now we can come to estimate $P(k_n \le N)$. It's easy to see that,

$$P(B(k_1, k_2, \ldots, k_n)) \le P(B(1, 1, \ldots, 1)) \text{ and}$$
$$P(B(1, 1, \ldots, 1)) \le P(k_n \le N) \le \sharp(k_n \le N) \cdot P(B(1, 1, \ldots, 1)).$$

Combining this with (2.3) and Lemma 2.2, we get the desired result. □

In older to overcome the inadequacies of that the sequence $\{k_n, n \ge 1\}$ is not a Markov chain, we also need the following lemma.

Lemma 2.4 ([7]). *Let* $y_n := \frac{D_n}{C_n}$ *for all* $n \ge 1$. *Then*

$$-1 < \epsilon(k) \le 1 \Rightarrow -1 < y_n \le 1.$$

Using this lemma, we can get the following estimate:

Lemma 2.5. *The conditional probability* $P(k_{n+1} = k | k_n = j)$ *satisfies that*

$$\frac{j-1}{(k-1)(k+2)} < P(k_{n+1} = k | k_n = j) \le \frac{j+1}{(k+1)k}. \tag{2.4}$$

Proof. From (2.1) and (2.2), we can see that in every cylinder $B(k_1, \ldots, k_{n-1})$,

$$\begin{aligned} P(k_{n+1} = k | k_n = j) &= \frac{P(B(k_1, \ldots, k_{n-1}, j, k))}{P(B(k_1, \ldots, k_{n-1}, j))} \\ &= \frac{C_n(jC_n + D_n)}{(kC_n + D_n)((k+1)C_n + D_n)} \\ &= \frac{(j + y_n)}{(k + y_n)(k + 1 + y_n)}, \quad \text{where } y_n = \frac{D_n}{C_n}. \end{aligned}$$

So by $-1 < y_n \le 1$ and using the monotone property of $\frac{j+y_n}{k+y_n}$, we get

$$\frac{j-1}{(k-1)(k+2)} < \frac{j + y_n}{(k + y_n)(k + y_n + 1)} \le \frac{j+1}{(k+1)k}.$$

Thus (2.4) is proved. □

Further, we have:

Lemma 2.6. *Let* $N = \max\{\frac{2-\theta}{\delta}, \frac{2}{\delta}\}$ *and* $\theta < 1$. *Then for all* $j \ge N$, *we have*:

$$\frac{1-\delta}{1-\theta} \le \sum_{k \ge j} \Big(\frac{k}{j}\Big)^{\theta} P(k_{n+1} = k | k_n = j) \le \frac{1+\delta}{1-\theta}.$$

Proof. From (2.4) we have,

$$\begin{aligned}
\sum_{k\geq j}\Big(\frac{k}{j}\Big)^{\theta}P(k_{n+1}=k|k_n=j) &\leq \frac{1}{j}+\sum_{k\geq j+1}\Big(\frac{k^{\theta}}{k(k+1)}\Big)\cdot\frac{j+1}{j^{\theta}}\\
&\leq \frac{1}{j}+\frac{j+1}{j^{\theta}}\sum_{k\geq j+1}\frac{1}{k^{2-\theta}}\\
&\leq \frac{1}{j}+\frac{j+1}{j^{\theta}}\int_j^{\infty}\frac{1}{x^{2-\theta}}dx\\
&= \frac{1}{j}+\frac{j+1}{j^{\theta}}\frac{j^{\theta-1}}{1-\theta}=\frac{2+j-\theta}{j}\frac{1}{1-\theta}\\
&\leq \frac{1+\delta}{1-\theta}\qquad \text{for } j\geq\frac{2-\theta}{\delta}.
\end{aligned}$$

And

$$\begin{aligned}
\sum_{k\geq j}\Big(\frac{k}{j}\Big)^{\theta}P(k_{n+1}=k|k_n=j) &\geq \frac{j-1}{j^{\theta}}\sum_{k\geq j}\frac{k^{\theta}}{(k+2)(k-1)}\\
&\geq \frac{j-1}{j^{\theta}}\sum_{k\geq j}k^{\theta-2}\frac{k}{k+1}\\
&\geq \frac{j-1}{j^{\theta}}\frac{j}{j+1}\int_j^{\infty}\frac{1}{x^{2-\theta}}dx\\
&= \frac{j-1}{j+1}\frac{1}{1-\theta}\geq\frac{1-\delta}{1-\theta}\qquad \text{for } j\geq\frac{2}{\delta}.
\end{aligned}$$

□

3. Proof of the main result

Before we go to the statement and proof of the large deviations result for GCF_ϵ expansions, let us first state and prove the following lemma.

Lemma 3.1. *Let $\{k_n, n\geq 1\}$ be the digits sequence of GCF_ϵ expansion. Then in the case of $-1<\epsilon\leq 1$,*

$$\lim_{n\to\infty}\frac{1}{n}\log E\big(k_n^{\theta}\big)=\begin{cases}+\infty, & \textit{when } \theta\geq 1,\\ \max\big\{\log\frac{1}{2+\epsilon},\ \log\frac{1}{1-\theta}\big\} & \textit{when } \theta<1;\end{cases}$$

and in the case of $\epsilon=-1$,

$$\lim_{n\to\infty}\frac{1}{n}\log E\big(k_n^{\theta}\big)=\begin{cases}+\infty, & \textit{when } \theta\geq 1,\\ \log\frac{1}{1-\theta} & \textit{when } \theta<1.\end{cases}$$

Proof. First, for any $\theta\geq 1$, from (1.4) and (1.5) we get $P(k_1=k)=\frac{B_1C_1-A_1D_1}{C_1(k_1C_1+D_1)}$ $=\frac{1}{k(k+1)}$, then by $k_n\geq k_1$ we have

$$E\big(e^{\theta\log k_n}\big)=E\big(k_n^{\theta}\big)\geq E\big(k_1^{\theta}\big)=\sum_{k=1}^{\infty}\frac{1}{k(k+1)}k^{\theta}=+\infty.$$

Next, for any $\theta < 1$, we divide the average into two terms:

$$\sum_{k=1}^{\infty} P(k_n = k)k^\theta = \sum_{k=1}^{N-1} P(k_n = k)k^\theta + \sum_{k=N}^{\infty} P(k_n = k)k^\theta, \tag{3.1}$$

and prove the results when $-1 < \epsilon \le 1$ and $\epsilon = -1$, respectively.

Part 1: In the case of $\theta < 1$ and $-1 < \epsilon \le 1$:

(1) Lower bound

For the first term in the sum of (3.1), it follows from (2.1) that

$$\sum_{k=1}^{N-1} P(k_n = k)k^\theta \ge P(k_n = 1) \cdot 1^\theta = P(B(\underbrace{1, 1, \ldots, 1}_{n})) = \frac{1+\epsilon}{(2+\epsilon)^n + \epsilon}. \tag{3.2}$$

For the second term in the sum of (3.1), it is clear that

$$\sum_{k=N}^{\infty} P(k_n = k)k^\theta \ge \sum_{j=N}^{\infty} P(k_{n-1} = j)j^\theta \sum_{k=j}^{\infty} P(k_n = k | k_{n-1} = j) \cdot \Big(\frac{k}{j}\Big)^\theta.$$

Then by Lemma 2.6, we get a recursive relation:

$$\sum_{k=N}^{\infty} P(k_n = k)k^\theta \ge \Big(\frac{1-\delta}{1-\theta}\Big) \sum_{j=N}^{\infty} P(k_{n-1} = j)j^\theta.$$

Iterating this process $n-1$ times until we get that

$$\sum_{k=N}^{\infty} P(k_n = k)k^\theta \ge \Big(\frac{1-\delta}{1-\theta}\Big)^{n-1} \sum_{j=N}^{\infty} P(k_1 = j)j^\theta.$$

And from (1.4) and (1.5) we have, for $\theta < 1$,

$$\sum_{j=N}^{\infty} P(k_1 = j)j^\theta = \sum_{j=N}^{\infty} \frac{j^\theta}{j(j+1)} =: M \quad \text{(convergent)}. \tag{3.3}$$

So we have

$$\sum_{k=N}^{\infty} P(k_n = k)k^\theta \ge \Big(\frac{1-\delta}{1-\theta}\Big)^{n-1} \sum_{j=N}^{\infty} P(k_1 = j)j^\theta = M\Big(\frac{1-\delta}{1-\theta}\Big)^{n-1}. \tag{3.4}$$

Then we get that from (3.2) and (3.4)

$$\begin{aligned}\sum_{k=1}^{\infty} P(k_n = k)k^\theta &\ge \max\Big\{\sum_{k=1}^{N} P(k_n = k)k^\theta,\ \sum_{k=N}^{\infty} P(k_n = k)k^\theta\Big\} \\ &\ge \max\Big\{\frac{1+\epsilon}{(2+\epsilon)^n + \epsilon},\ M\Big(\frac{1-\delta}{1-\theta}\Big)^{n-1}\Big\}.\end{aligned}$$

As a consequence,

$$\liminf_{n\to\infty} \frac{1}{n} \log\Big(\sum_{k=N}^{\infty} P(k_n = k)k^\theta\Big) \ge \max\Big\{\log\frac{1}{2+\epsilon},\ \log\frac{1+\delta}{1-\theta}\Big\}.$$

Since $\delta > 0$ is arbitrary, we get

$$\liminf_{n\to\infty} \frac{1}{n} \log E\big(e^{\theta \log k_n}\big) \geq \max\Big\{ \log \frac{1}{2+\epsilon},\ \log \frac{1}{1-\theta} \Big\}, \tag{3.5}$$

which gives the lower bound of $\liminf_{n\to\infty} \frac{1}{n} \log E\big(e^{\theta \log k_n}\big)$ when $-1 < \epsilon \leq 1$ and $\theta < 1$.

(2) Upper bound

For the first term in the sum of (3.1), it follows from Lemma 2.3 that,

$$\begin{aligned}\sum_{k=1}^{N-1} P(k_n = k) k^\theta &\leq P(k_n \leq N) N^\theta \\ &\leq (1+n)^{N-1} \cdot \frac{1+\epsilon}{(2+\epsilon)^n + \epsilon} N^\theta \leq \frac{2(1+n)^N N^\theta}{(2+\epsilon)^n}. \end{aligned} \tag{3.6}$$

For the second term in the sum of (3.1), it is also can be divided into the sum of the two terms:

$$\begin{aligned}\sum_{k=N}^{\infty} P(k_n = k) k^\theta = &\sum_{j=N}^{\infty} P(k_{n-1} = j) j^\theta \sum_{k=j}^{\infty} P(k_n = k | k_{n-1} = j) \Big(\frac{k}{j}\Big)^\theta \\ &+ \sum_{j=1}^{N} P(k_{n-1} = j) j^\theta \sum_{k=N}^{\infty} P(k_n = k | k_{n-1} = j) \Big(\frac{k}{j}\Big)^\theta. \end{aligned}$$

By Lemma 2.6 and Lemma 2.3, we have for $-1 < \epsilon \leq 1$,

$$\begin{aligned}\sum_{k=N}^{\infty} P(k_n = k) k^\theta &\leq \frac{1+\delta}{1-\theta} \Big(\sum_{j=N}^{\infty} P(k_{n-1} = j) j^\theta + \sum_{j=1}^{N} P(k_{n-1} = j) j^\theta \Big) \\ &\leq \frac{1+\delta}{1-\theta} \Big(\sum_{k=N}^{\infty} P(k_{n-1} = k) k^\theta + \frac{2n^{N-1}}{(2+\epsilon)^{n-1}} \Big). \end{aligned}$$

Iterate this process $n-1$ times to get

$$\begin{aligned} &\sum_{k=N}^{\infty} P(k_n = k) k^\theta \\ &\leq \Big(\frac{1+\delta}{1-\theta}\Big)^{n-1} \sum_{k=N}^{\infty} P(k_1 = k) k^\theta + 2n^{N-2} \sum_{i=1}^{n-1} \Big(\frac{1+\delta}{1-\theta}\Big)^i \Big(\frac{1}{2+\epsilon}\Big)^{n-i}, \end{aligned} \tag{3.7}$$

where the geometric series

$$\sum_{i=1}^{n-1} \Big(\frac{1+\delta}{1-\theta}\Big)^i \Big(\frac{1}{2+\epsilon}\Big)^{n-i} = O\bigg(\Big(\frac{1}{2+\epsilon}\Big)^{n-1} - \Big(\frac{1+\delta}{1-\theta}\Big)^{n-1} \bigg). \tag{3.8}$$

Substituting (3.8) and (3.3) into (3.7), and combining with (3.6), we get

$$\sum_{k=1}^{\infty} P(k_n = k) k^\theta \leq 2n^{N-2} M_1 \Big(\frac{1+\delta}{1-\theta}\Big)^{n-1} + (1+n)^N M_2 \Big(\frac{1}{2+\epsilon}\Big)^{n-1},$$

where M_1 and M_2 are two positive constants.

Therefore,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log E\big(e^{\theta\log k_n}\big)\\ \le\ &\limsup_{n\to\infty}\frac{1}{n}\log\Big(2n^{N-2}M_1\Big(\frac{1+\delta}{1-\theta}\Big)^{n-1}+2(1+n)^{N-2}M_2\Big(\frac{1}{2+\epsilon}\Big)^{n-1}\Big)\\ \le\ &\max\Big\{\log\frac{1+\delta}{1-\theta},\ \log\frac{1}{2+\epsilon}\Big\}.\end{aligned}$$

Since $\delta > 0$ is arbitrary, we get

$$\lim_{n\to\infty}\frac{1}{n}\log E\big(e^{\theta\log k_n}\big)\le\max\Big\{\log\frac{1}{2+\epsilon},\ \log\frac{1}{1-\theta}\Big\},$$

which gives the upper bound of $\lim_{n\to\infty}\frac{1}{n}\log E\big(e^{\theta\log k_n}\big)$ when $-1<\epsilon\le 1$ and $\theta<1$.

Combining this upper bound and the lower: (3.5), we obtain when $\theta<1$ and $-1<\epsilon\le 1$,

$$\lim_{n\to\infty}\frac{1}{n}\log E\big(e^{\theta\log k_n}\big)=\max\Big\{\log\frac{1}{2+\epsilon},\ \log\frac{1}{1-\theta}\Big\}.$$

Part 2: In the case of $\theta<1$ and $\epsilon=-1$:

For the first term in the sum of (3.1), Lemma 2.3 gives that

$$\frac{1}{1+n}(N-1)^\theta\le\sum_{k=1}^{N-1}P(k_n=k)k^\theta\le(1+n)^{N-2}(N-1)^\theta.$$

As a consequence,

$$\lim_{n\to\infty}\frac{1}{n}\log\Big(\sum_{k=1}^{N-1}P(k_n=k)k^\theta\Big)=0.$$

So the result of $\lim_{n\to\infty}\frac{1}{n}\log E\big(e^{\theta\log k_n}\big)$ only depends on the second term in the sum of (3.1). So long as we instead using (2.1) by using (2.2) in the proof for the case of $-1<\epsilon\le 1$, by the same proof method, we can get that when $\theta<1$ and $\epsilon=-1$,

$$\lim_{n\to\infty}\frac{1}{n}\log E\big(k_n^\theta\big)=\log\frac{1}{1-\theta}.\qquad\square$$

Now we can prove the following:

Theorem 3.2. *Let $\{k_n\}_{n\ge1}$ be the digits sequence of the GCF_ϵ expansion. Then $\big\{\frac{1}{n}\log k_n, n\ge 1\big\}$ satisfy the large deviation principle with speed n and good rate function $I(x)$ as*

1. *In the case of $-1<\epsilon<1$,*

$$I(x)=\begin{cases}x-1-\log x, & \text{if } x>\frac{1}{2+\epsilon}\\ \log(2+\epsilon)-(1+\epsilon)x, & \text{if } 0\le x<\frac{1}{2+\epsilon}\\ +\infty, & \text{if } x\le 0.\end{cases}$$

2. *In the case of* $\epsilon = -1$,

$$I(x) = \begin{cases} x - 1 - \log x, & \text{if } x > 0 \\ +\infty, & \text{if } x \le 0 \end{cases}$$

under P.

Proof. Lemma 3.2 actually gives that
When $-1 < \epsilon \le 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log E\big(e^{\theta \log k_n}\big) = \begin{cases} +\infty, & \text{when } \theta \ge 1; \\ \log \frac{1}{1-\theta} & \text{when } -1-\epsilon \le \theta < 1; \\ \log \frac{1}{2+\epsilon} & \text{when } \theta < -1-\epsilon. \end{cases}$$

When $\epsilon = -1$,

$$\lim_{n\to\infty} \frac{1}{n} \log E\big(e^{\theta \log k_n}\big) = \begin{cases} +\infty, & \text{when } \theta \ge 1; \\ \log \frac{1}{1-\theta} & \text{when } \theta < 1. \end{cases}$$

By Gartner-Ellis theorem (see e.g. Dembo and Zeitouni [1]), $\big\{\frac{1}{n}\log k_n, n \ge 1\big\}$ satisfies a large deviation principle with rate function

$$I(x) = \sup_{\theta\in\mathbb{R}}\{\theta x - \Gamma(\theta)\},$$

where $\Gamma(\theta) := \frac{1}{n}\log E\big(e^{\theta \log k_n}\big)$ exists. Let $f(\theta) = \theta x - \Gamma(\theta)$, then

1. When $\theta < -1-\epsilon, f(\theta) = \theta x + \log(2+\epsilon)$,

$$\sup_{\theta<-1-\epsilon}\{f(\theta)\} = \begin{cases} f(-1-\epsilon) = -(1+\epsilon)x + \log(2+\epsilon), & \text{if } x > 0 \\ f(-\infty) = \lim_{\theta\to-\infty} \theta x + \log(2+\epsilon) = +\infty, & \text{if } x < 0. \end{cases}$$

2. When $-1-\epsilon \le \theta < 1, f(\theta) = \theta x + \log(1-\theta)$ has maximum points: $\theta = 1 - \frac{1}{x}$. Notice that $-1-\epsilon \le \theta < 1$ and $\theta = 1-\frac{1}{x} \Rightarrow x \ge \frac{1}{2+\epsilon}$, so we have

$$\sup_{-1-\epsilon\le\theta<1}\{f(\theta)\} = f\Big(1-\frac{1}{x}\Big) = x - 1 - \log x \quad \text{for all } x \ge \frac{1}{2+\epsilon}.$$

3. When $\theta \ge 1, f(\theta) = \theta x - \infty$,

$$\sup_{1\le\theta<\infty}\{f(\theta)\} = -\infty \quad \text{for all } -\infty < x < +\infty.$$

Therefore we derive when $-1 < \epsilon \le 1$,

$$I(x) = \begin{cases} x - 1 - \log x, & \text{if } x > \frac{1}{2+\epsilon} \\ \log(2+\epsilon) - (1+\epsilon)x, & \text{if } 0 \le x < \frac{1}{2+\epsilon} \\ +\infty, & \text{if } x \le 0. \end{cases}$$

When $\epsilon = -1$,

$$I(x) = \begin{cases} x - 1 - \log x, & \text{if } x > 0 \\ +\infty, & \text{if } x \le 0. \end{cases}$$

□

We can see that when and only when $\epsilon = 0$, the GCF_ϵ has the same large deviation principle with the Engel expansion; and when $\epsilon = -1$ the GCF_ϵ has the same large deviation principle with Modified ECF expansion and Sylvesters series (see [3] and [9]).

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