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LIOUVILLE TYPE THEOREMS FOR TRANSVERSALLY HARMONIC AND BIHARMONIC MAPS

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Abstract. In this paper, we study the Liouville type theorems for transversally harmonic and biharmonic maps on foliated Riemannian manifolds.

1. Introduction

Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated Riemannian manifolds and let $\phi : M \to M'$ be a smooth foliated map, i.e., ϕ is a smooth leaf-preserving map. Then ϕ is said to be *transversally harmonic* if the transversal tension field $\tau_b(\phi) = \mathrm{tr}_Q\tilde\nabla_{\rm tr} d_T\phi$ vanishes, where $d_T\phi = d\phi|_Q$ and Q is the normal bundle of $\mathcal{F}$ (see [7], [14], [15] for details). When $\mathcal{F}$ is minimal, a transversally harmonic map is a critical point of the transversal energy $E_B(\phi)$ [7], which is given by

$$E_B(\phi) = \frac{1}{2}\int_M |d_T\phi|^2\mu_M,$$

where μ_M denotes the volume form on M. If $\mathcal{F}$ is not minimal, a transversally harmonic is not a critical point of $E_B(\phi)$. In fact, S. Dragomir and A. Tommasoli [5] called such maps as $(\mathcal{F},\mathcal{F}')$-harmonic maps, i.e., a critical point of the transversal energy. Trivially, two definitions are equivalent when $\mathcal{F}$ is minimal. The smooth map ϕ is said to be *transversally biharmonic* if the transversal bitension field $(\tau_2)_b(\phi) = J_\phi^T(\tau_b(\phi)$ vanishes, where J_ϕ^T is the generalized Jacobi operator along ϕ (see [4], [10] for details). If $\mathcal{F}$ is minimal, then a trasversally biharmonic map is a critical point of the transversal bienergy $E_2(\phi)$, where

$$E_2(\phi) = \frac{1}{2}\int_M |\tau_b(\phi)|^2\mu_M.$$

Transversally harmonic and biharmonic maps are generalizations of harmonic and biharmonic maps because transversally harmonic and biharmonic maps are just harmonic and biharmonic maps on the point foliation, respectively. For more information about transversally harmonic and biharmonic maps, see [4],

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[7], [10], [14], [15]. For harmonic maps, the classical Liouville theorem is well-known. Namely, any bounded harmonic function defined on the whole plane must be constant. The classical Liouville theorem has been improved in several cases [8], [18], [20]. In this article, we study the Liouville type theorems for the transversally harmonic and biharmonic map. Now we consider the following conditions on $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$.

(C1) All leaves of $\mathcal{F}$ are compact and the mean curvature form κ of $\mathcal{F}$ is bounded, coclosed.

(C2) The transversal sectional curvature of $\mathcal{F}'$ is nonpositive.
Then we have the following Liouville type theorem on a foliated Riemannian manifold.

Theorem A. *Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold with $\mathrm{Vol}(M) = \infty$ satisfying $(C1)$ and let $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold satisfying $(C2)$. Assume that the transversal Ricci curvature of $\mathcal{F}$ is non-negative. Then any transversally harmonic map $\phi : M \to M'$ of $E_B(\phi) < \infty$ is transversally constant, i.e., the induced map between leaf spaces is constant.*

Note that any transversally harmonic map is transversally biharmonic. But the converse does not hold. In fact, S. D. Jung [10] proved that on a compact foliated manifold, the converse holds under some condition. For transversally biharmonic map on a complete foliated Riemannian manifold, we have the following theorem.

Theorem B. *Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold with $\mathrm{Vol}(M) = \infty$ satisfying $(C1)$ and let $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold satisfying $(C2)$.*

(1) *Every transversally biharmonic map $\phi : M \to M'$ of $E_2(\phi) < \infty$ is transversally harmonic.*

(2) *If the transversal Ricci curvature of $\mathcal{F}$ is nonnegative, then every transversally biharmonic map $\phi : M \to M'$ of $E_B(\phi) + E_2(\phi) < \infty$ is transversally constant.*

When $\mathcal{F}$ is a point foliation, Theorem A and Theorem B have been found in [18] and [2], respectively.

2. Preliminaries

Let $(M, g, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a complete bundle-like metric g with respect to $\mathcal{F}$. Let TM be the tangent bundle of M, $T\mathcal{F}$ its integrable subbundle given by $\mathcal{F}$, and $Q = TM/T\mathcal{F}$ the corresponding normal bundle of $\mathcal{F}$. Then we have an exact sequence of vector bundles

$$0 \longrightarrow T\mathcal{F} \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0, \tag{2.1}$$

where $\pi : TM \to Q$ is a projection and $\sigma : Q \to T\mathcal{F}^{\perp}$ is a bundle map satisfying $\pi \circ \sigma = id$. Let g_Q be the holonomy invariant metric on Q induced by g, i.e., $L_X g_Q = 0$ for any vector field $X \in T\mathcal{F}$, where L_X is the transverse Lie derivative [12]. Let R^Q, K^Q and Ric^Q be the transversal curvature tensor, transversal sectional curvature and transversal Ricci operator of $\mathcal{F}$ with respect to the transversal Levi-Civita connection $\nabla^Q \equiv \nabla$ in Q [19], respectively. A differential form $\omega \in \Omega^r(M)$ is *basic* if $i(X)\omega = 0$ and $i(X)d\omega = 0$ for all $X \in T\mathcal{F}$. Let $\Omega_B^r(\mathcal{F})$ be the set of all basic r-forms on M. Then $\Omega^r(M) = \Omega_B^r(\mathcal{F}) \oplus \Omega_B^r(\mathcal{F})^\perp$ [1]. Now, we recall the star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ given by [13], [17]

$$\bar{*}\omega = (-1)^{p(q-r)} * (\omega \wedge \chi_{\mathcal{F}}), \quad \forall \omega \in \Omega_B^r(\mathcal{F}), \tag{2.2}$$

where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator associated to g. For any basic forms $\omega, \theta \in \Omega_B^r(\mathcal{F})$, it is well-known [17] that $\omega \wedge \bar{*}\theta = \theta \wedge \bar{*}\omega$ and $\bar{*}^2\omega = (-1)^{r(q-r)}\omega$. Let ν be the transversal volume form, i.e., $*\nu = \chi_{\mathcal{F}}$ and $\langle \cdot, \cdot \rangle$ be the pointwise inner product on $\Omega_B^r(\mathcal{F})$, which is given by

$$\langle \omega, \theta \rangle \nu = \omega \wedge \bar{*}\theta \tag{2.3}$$

for any basic forms $\omega, \theta \in \Omega_B^r(\mathcal{F})$. Trivially $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume form with respect to g. Now, let the operator d_B be the restriction of d to the basic forms, i.e., $d_B = d|_{\Omega_B^*(\mathcal{F})}$. It is well-known that on complete foliated Riemannian manifolds, $d_B \kappa_B = 0$ [16]. Let $d_t = d_B - \kappa_B \wedge$, $\delta_t = (-1)^{q(r+1)+1} \bar{*} d_B \bar{*}$ and

$$\delta_B \omega = (-1)^{q(r+1)+1} \bar{*} d_t \bar{*} \omega = \delta_t \omega + i(\kappa_B^\sharp)\omega, \tag{2.4}$$

where $(\cdot)^\sharp$ is the g_Q-dual vector field of $(\cdot)$ and κ_B is the basic part of the mean curvature form of $\mathcal{F}$. Then $\int_M \langle d_B\omega, \theta \rangle \mu_M = \int_M \langle \omega, \delta_B \theta \rangle \mu_M$ for any $\omega \in \Omega_{B,o}^r(\mathcal{F})$ or $\theta \in \Omega_{B,o}^{r+1}(\mathcal{F})$, where $\Omega_{B,o}^*(\mathcal{F})$ is the subspace of $\Omega_B^*(\mathcal{F})$ composed of forms with compact support. Generally, δ_B is not a restriction of δ on $\Omega_B^r(\mathcal{F})$, i.e., $\delta_B \neq \delta|_{\Omega_B^r(\mathcal{F})}$, where δ is the formal adjoint of d. But $\delta_B \omega = \delta\omega$ for any basic 1-form ω. Hence $\Delta^M|_{\Omega_B^0(\mathcal{F})} = \Delta_B$ [13], where Δ^M is the positive Laplacian on M and Δ_B is the basic Laplacian acting on $\Omega_B^r(\mathcal{F})$ which is given by

$$\Delta_B = d_B \delta_B + \delta_B d_B. \tag{2.5}$$

Let $V(\mathcal{F})$ be the space of all transversal infinitesimal automorphisms Y of $\mathcal{F}$, i.e., $[Y, Z] \in T\mathcal{F}$ for all $Z \in T\mathcal{F}$. For any $Y \in V(\mathcal{F})$, we define the bundle map $A_Y : \Gamma(\Lambda^r Q^*) \to \Gamma(\Lambda^r Q^*)$ [12] by

$$A_Y \omega = L_Y \omega - \nabla_Y \omega. \tag{2.6}$$

Then A_Y preserves the basic forms and depends only on $\pi(Y)$. Moreover, for any vector field $Y \in V(\mathcal{F})$, if we define $A_Y : \Gamma Q \to \Gamma Q$ by $\omega(A_Y s) = -(A_Y \omega)(s)$ for any $s \in \Gamma Q$ and $\omega \in \Gamma Q^*$, then $A_Y s = L_Y s - \nabla_Y s$. Since $L_Y s = \pi[Y, Y_s]$ for $Y_s = \sigma(s) \in T\mathcal{F}^\perp$ [12],

$$A_Y s = -\nabla_{Y_s} \pi(Y). \tag{2.7}$$

Let $\{E_a\}(a=1,\dots,q)$ be a local orthonormal basic frame of Q and θ^a a g_Q-dual 1-form to E_a. We define $\nabla_{\rm tr}^*\nabla_{\rm tr}:\Omega_B^r(\mathcal{F})\to\Omega_B^r(\mathcal{F})$ by

$$\nabla_{\rm tr}^*\nabla_{\rm tr}=-\sum_a\nabla^2_{E_a,E_a}+\nabla_{\kappa_B^\sharp}, \tag{2.8}$$

where $\nabla^2_{X,Y}=\nabla_X\nabla_Y-\nabla_{\nabla^M_XY}$ for any $X,Y\in TM$ and ∇^M is the Levi-Civita connection with respect to g. The operator $\nabla_{\rm tr}^*\nabla_{\rm tr}$ is positive definite and formally self adjoint on $\Omega^r_{B,o}(\mathcal{F})$ [9]. Then the generalized Weitzenböck type formula on $\Omega_B^r(\mathcal{F})$ is given by [9]

$$\Delta_B\omega=\nabla_{\rm tr}^*\nabla_{\rm tr}\omega+F(\omega)+A_{\kappa_B^\sharp}\omega \tag{2.9}$$

for any $\omega\in\Omega_B^r(\mathcal{F})$, where $F=\sum_{a,b=1}^q\theta^a\wedge i(E_b)R^Q(E_b,E_a)$. For any basic-harmonic form $\omega\in\Omega_B^*(\mathcal{F})$, i.e., $\Delta_B\omega=0$, we have [9] that

$$-\frac{1}{2}\Delta_B|\omega|^2=|\nabla_{\rm tr}\omega|^2+\langle A_{\kappa_B^\sharp}\omega,\omega\rangle+\langle F(\omega),\omega\rangle. \tag{2.10}$$

3. Generalized maximum principle

Let $(M,g,\mathcal{F})$ be a complete foliated Riemannian manifold, i.e., manifold with a Riemannian foliation $\mathcal{F}$ and a complete bundle-like metric g with respect to $\mathcal{F}$. Now, we consider a smooth function μ on $\mathbb{R}$ satisfying

(i) $0\le\mu(t)\le1$ on $\mathbb{R}$, (ii) $\mu(t)=1$ for $t\le1$, (iii) $\mu(t)=0$ for $t\ge2$.

Let x_0 be a point in M. For each point $y\in M$, we denote by $\rho(y)$ the distance between leaves through x_0 and y. For any real number $l>0$, we define a Lipschitz continuous function ω_l on M by

$$\omega_l(y)=\mu(\rho(y)/l).$$

Trivially, ω_l is a basic function. Let $B(l)=\{y\in M\,|\,\rho(y)\le l\}$. Then ω_l satisfies the following properties:

$$\begin{array}{ll} 0\le\omega_l(y)\le1 & \text{for any } y\in M\\ \operatorname{supp}\omega_l\subset B(2l) & \\ \omega_l(y)=1 & \text{for any } y\in B(l)\\ \lim_{l\to\infty}\omega_l=1 & \\ |d\omega_l|\le\frac{C}{l} & \text{almost everywhere on } M, \end{array}$$

where C is a positive constant independent of l [22]. Hence $\omega_l\psi$ has compact support for any basic form $\psi\in\Omega_B^*(\mathcal{F})$ and $\omega_l\psi\to\psi$ (strongly) when $l\to\infty$.

Note that for any basic function f, $\Delta^Mf=\Delta_Bf$ [13]. Hence we have the following theorems.

Theorem 3.1. *Let $(M,g,\mathcal{F})$ be a complete foliated Riemannian manifold. If a basic function f is basic-subharmonic, i.e., $\Delta_Bf\le0$, with $\int_M|df|<\infty$, then f is basic-harmonic.*

Proof. This follows from the result in [21, p. 660]. □

Theorem 3.2. *Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold. If a nonnegative basic function f is basic-subharmonic, i.e., $\Delta_B f \le 0$, with $\int_M f^p < \infty$ $(p > 1)$, then f is constant.*

Proof. This follows from Theorem 3 in [21, p. 663]. □

Now we prove the generalized maximum principle.

Theorem 3.3. *Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold whose all leaves are compact. Assume that κ_B is bounded and coclosed. Then a nonnegative basic function f such that $(\Delta_B - \kappa_B^\sharp) f \le 0$ with $\int_M f^p < \infty$ $(p > 1)$ is constant.*

Proof. Let $u = f^{p/2}$. By a direct calculation, we have

$$u(\Delta_B - \kappa_B^\sharp)u = \frac{p}{2} f^{p-1}(\Delta_B - \kappa_B^\sharp) f - \frac{p(p-2)}{4} f^{p-2} |d_B f|^2. \tag{3.1}$$

By the assumption, we have

$$u(\Delta_B - \kappa_B^\sharp)u \le -\frac{p-2}{p} |d_B u|^2. \tag{3.2}$$

On the other hand, we have

$$\int_{B(2l)} \langle \omega_l^2 u, \Delta_B u \rangle = 2 \int_{B(2l)} \langle \omega_l d_B u, u d_B \omega_l \rangle + \int_{B(2l)} |\omega_l d_B u|^2. \tag{3.3}$$

So, from (3.2) and (3.3), we have that

$$\frac{2(p-1)}{p} \int_{B(2l)} |\omega_l d_B u|^2 \le -2 \int_{B(2l)} \langle \omega_l d_B u, u d_B \omega_l \rangle + \int_{B(2l)} \langle \omega_l^2 u, \kappa_B^\sharp(u) \rangle. \tag{3.4}$$

From (3.4) and the Schwarz's inequality, we have that for any real number $\epsilon > 0$,

$$\begin{aligned}
&\frac{p-1}{p} \int_{B(2l)} |\omega_l d_B u|^2 \\
&\le \int_{B(2l)} |\langle \omega_l d_B u, u d_B \omega_l \rangle| + \frac{1}{2} \int_{B(2l)} \langle \omega_l^2 u, \kappa_B^\sharp(u) \rangle \\
&\le \frac{\epsilon}{2} \int_{B(2l)} |\omega_l d_B u|^2 + \frac{1}{2\epsilon} \int_{B(2l)} |u d_B \omega_l|^2 + \frac{1}{2} \int_{B(2l)} \langle \omega_l^2 u, \kappa_B^\sharp(u) \rangle \\
&\le \frac{\epsilon}{2} \int_{B(2l)} |\omega_l d_B u|^2 + \frac{C^2}{2\epsilon l^2} \int_{B(2l)} u^2 + \frac{1}{2} \int_{B(2l)} \langle \omega_l^2 u, \kappa_B^\sharp(u) \rangle.
\end{aligned}$$

Hence we have

$$\left(\frac{p-1}{p} - \frac{\epsilon}{2}\right) \int_{B(2l)} |\omega_l d_B u|^2 \le \frac{C^2}{2\epsilon l^2} \int_{B(2l)} u^2 + \frac{1}{2} \int_{B(2l)} \langle \omega_l^2 u, \kappa_B^\sharp(u) \rangle. \tag{3.5}$$

On the other hand, since $\langle\omega_l^2 u, \kappa_B^\sharp(u)\rangle = \frac{1}{2}\{\kappa_B^\sharp(\omega_l^2 u^2) - 2\langle d\omega_l, (\omega_l u^2)\kappa_B\rangle\}$, we have that from the assumptions of κ_B, i.e., $|\kappa_B| < \infty$ and $\delta_B\kappa_B = 0$,

$$\begin{aligned}\int_{B(2l)} |\langle\omega_l^2 u, \kappa_B^\sharp(u)\rangle| &\le \int_{B(2l)} |d\omega_l||(\omega_l u^2)\kappa_B| \\ &\le \frac{C}{l}\max(|\kappa_B|)\int_{B(2l)} \omega_l u^2.\end{aligned}$$

Since $\int_M u^2 = \int_M f^p < \infty$, if we let $l \to \infty$, then

$$\int_{B(2l)} \langle\omega_l^2 u, \kappa_B^\sharp(u)\rangle \to 0. \tag{3.6}$$

From (3.5) and (3.6), if we let $l \to \infty$, then

$$(\frac{p-1}{p} - \frac{\epsilon}{2})\int_M |d_B u|^2 \le 0. \tag{3.7}$$

If we choose $0 < \epsilon < \frac{2(p-1)}{p}$, then from (3.7),

$$\int_M |d_B u|^2 = 0.$$

Hence $d_B u = 0$ and so $d_B f = 0$, i.e., f is constant. □

Remark. On a compact foliated Riemannian manifold, Theorem 3.3 was proved in [11].

4. The proof of Theorem A

Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two Riemannian manifolds with foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. Let ∇ and ∇' be the transverse Levi-Civita connections on Q and Q', respectively. Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map, i.e., ϕ is a smooth leaf-preserving map. Equivalently, $d\phi(T\mathcal{F}) \subset T\mathcal{F}'$. We define $d_T\phi : Q \to Q'$ by

$$d_T\phi := \pi' \circ d\phi \circ \sigma. \tag{4.1}$$

Then $d_T\phi$ is a section in $Q^* \otimes \phi^{-1}Q'$, where $\phi^{-1}Q'$ is the pull-back bundle on M. Let ∇^ϕ and $\tilde\nabla$ be the connections on $\phi^{-1}Q'$ and $Q^* \otimes \phi^{-1}Q'$, respectively. The *transversal tension field* of ϕ is defined by

$$\tau_b(\phi) = \mathrm{tr}_Q \tilde\nabla d_T\phi = \sum_{a=1}^q (\tilde\nabla_{E_a} d_T\phi)(E_a), \tag{4.2}$$

where $\{E_a\}(a = 1, \ldots, q)$ is a local orthonormal basic frame on Q. Trivially, the transversal tension field $\tau_b(\phi)$ is a section of $\phi^{-1}Q'$. A foliated map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is said to be *transversally harmonic* if the transversal tension field vanishes, i.e., $\tau_b(\phi)=0$ [7]. And the *transversal energy* of ϕ on a compact domain Ω is defined by

$$E_B(\phi;\Omega)=\frac{1}{2}\int_\Omega |d_T\phi|^2\mu_M, \tag{4.3}$$

where $|d_T\phi|^2=\sum_a g_{Q'}(d_T\phi(E_a),d_T\phi(E_a))\in\Omega_B^0(\mathcal{F})$ [6]. Then we have the first variational formula [7]

$$\frac{d}{dt}E_B(\phi_t;\Omega)|_{t=0}=-\int_\Omega\langle V,\tau_b(\phi)-d_T\phi(\kappa_B^\sharp)\rangle\mu_M, \tag{4.4}$$

where $V=\frac{d\phi_t}{dt}|_{t=0}$ is the normal variation vector field with a foliated variation $\{\phi_t\}$ of ϕ. Hence if $\mathcal{F}$ is minimal, then the transversal harmonic map is a critical point of the transversal energy $E_B(\phi;\Omega)$ of ϕ supported in a compact domain Ω. Let $\Omega_B^*(E)\equiv\Omega_B^*(\mathcal{F})\otimes E$, where $E\equiv\phi^{-1}Q'$. Then we define $d_\nabla:\Omega_B^r(E)\to\Omega_B^{r+1}(E)$ by

$$d_\nabla(\omega\otimes s)=d_B\omega\otimes s+(-1)^r\omega\wedge\nabla_{\rm tr}^\phi s \tag{4.5}$$

for any $\omega\in\Omega_B^r(\mathcal{F})$ and $s\in\Gamma E$. Let δ_∇ be the formal adjoint of d_∇ on $\Omega_{B,o}^*(E)$, the space of the compact supports. We define the Laplacian Δ on $\Omega_B^r(E)$ by

$$\Delta=d_\nabla\delta_\nabla+\delta_\nabla d_\nabla. \tag{4.6}$$

The operator A_Y is extended to $\Omega_B^r(E)$ [7]. That is, for any $\omega\otimes s\in\Omega_B^r(E)$, $A_Y(\omega\otimes s)=A_Y\omega\otimes s$. Then we have the generalized Weitzenböck type formula.

Theorem 4.1 ([7])**.** *Let $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ be a smooth foliated map. Then*

$$\frac{1}{2}\Delta_B|d_T\phi|^2=\langle\Delta d_T\phi,d_T\phi\rangle-|\tilde\nabla_{\rm tr}d_T\phi|^2-\langle A_{\kappa_B^\sharp}d_T\phi,d_T\phi\rangle-\langle F(d_T\phi),d_T\phi\rangle,$$

where

$$\begin{aligned}\langle F(d_T\phi),d_T\phi\rangle=&\sum_a g_{Q'}(d_T\phi({\rm Ric}^Q(E_a)),d_T\phi(E_a))\\&-\sum_{a,b}g_{Q'}(R^{Q'}(d_T\phi(E_a),d_T\phi(E_b))d_T\phi(E_b),d_T\phi(E_a)).\end{aligned} \tag{4.7}$$

Note that for a smooth foliated map $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ [7],

$$d_\nabla d_T\phi=0,\quad \delta_\nabla d_T\phi=-\tau_b(\phi)+i(\kappa_B^\sharp)d_T\phi. \tag{4.8}$$

Hence we have the following corollary.

Corollary 4.2 ([7])**.** *Let $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ be a transversally harmonic map. Then*

$$\frac{1}{2}(\Delta_B-\kappa_B^\sharp)|d_T\phi|^2=-|\tilde\nabla_{\rm tr}d_T\phi|^2-\langle F(d_T\phi),d_T\phi\rangle. \tag{4.9}$$

The proof of Theorem A. Note that $\frac{1}{2}\Delta_B|d_T\phi|^2 = |d_T\phi|\Delta_B|d_T\phi| - |d_B|d_T\phi||^2$. From Corollary 4.2, we have

$$|d_T\phi|(\Delta_B - \kappa_B^\sharp)|d_T\phi| = |d_B|d_T\phi||^2 - |\tilde{\nabla}_{\rm tr} d_T\phi|^2 - \langle F(d_T\phi), d_T\phi\rangle.$$

Since $|\tilde{\nabla}_{\rm tr} d_T\phi| \geq |d_B|d_T\phi||$ (Kato's inequality [3]), we have

$$|d_T\phi|(\Delta_B - \kappa_B^\sharp)|d_T\phi| \leq -\langle F(d_T\phi), d_T\phi\rangle.$$

By the assumptions of the curvatures, $\langle F(d_T\phi), d_T\phi\rangle \geq 0$, which means $(\Delta_B - \kappa_B^\sharp)|d_T\phi| \leq 0$. Hence, by Theorem 3.3, $|d_T\phi|$ is constant. Since $\mathrm{Vol}(M)$ is infinite and $E_B(\phi) < \infty$, we have $d_T\phi = 0$. Hence ϕ is transversally constant. □

5. The proof of Theorem B

Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. The *transversal bitension field* $(\tau_2)_b(\phi)$ of ϕ is defined by

$$(\tau_2)_b(\phi) = J_\phi^T(\tau_b(\phi)), \tag{5.1}$$

where the generalized Jacobi operator $J_\phi^T : \phi^{-1}Q' \to \phi^{-1}Q'$ along ϕ is defined by

$$J_\phi^T(s) = (\nabla_{\rm tr}^\phi)^*(\nabla_{\rm tr}^\phi)s - \nabla_{\kappa_B^\sharp}^\phi s - \mathrm{tr}_Q R^{Q'}(s, d_T\phi)d_T\phi \tag{5.2}$$

for any $s \in \phi^{-1}(Q')$ [10].

Definition 5.1 ([4])**.** Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then ϕ is said to be transversally biharmonic if the transversal bitension field vanishes, i.e., $(\tau_2)_b(\phi) = 0$.

Trivially, ϕ is a transversally biharmonic map if and only if the transversal tension field $\tau_b(\phi)$ is a generalized Jacobi field along ϕ.

The proof of Theorem B. Note that for any $s \in \phi^{-1}Q'$, we have

$$\frac{1}{2}\Delta_B|s|^2 = \langle(\nabla_{\rm tr}^\phi)^*(\nabla_{\rm tr}^\phi)s, s\rangle - |\nabla_{\rm tr}^\phi s|^2. \tag{5.3}$$

Since $\phi : M \to M'$ is transversally biharmonic, from (5.1) and (5.2),

$$(\nabla_{\rm tr}^\phi)^*(\nabla_{\rm tr}^\phi)\tau_b(\phi) - \nabla_{\kappa_B^\sharp}^\phi \tau_b(\phi) - \mathrm{tr}_Q R^{Q'}(\tau_b(\phi), d_T\phi)d_T\phi = 0. \tag{5.4}$$

From (5.3) and (5.4), we have

$$\frac{1}{2}(\Delta_B - \kappa_B^\sharp)|\tau_b(\phi)|^2 = \langle \mathrm{tr}_Q R^{Q'}(\tau_b(\phi), d_T\phi)d_T\phi, \tau_b(\phi)\rangle - |\nabla_{\rm tr}^\phi \tau_b(\phi)|^2. \tag{5.5}$$

Since $\frac{1}{2}\Delta_B f^2 = f\Delta_B f - |d_B f|^2$ for any basic function f, Eq. (5.5) implies that

$$\begin{aligned}|\tau_b(\phi)|(\Delta_B - \kappa_B^\sharp)|\tau_b(\phi)| = {}& \langle \mathrm{tr}_Q R^{Q'}(\tau_b(\phi), d_T\phi)d_T\phi, \tau_b(\phi)\rangle \\ & + |d_B|\tau_b(\phi)||^2 - |\nabla_{\rm tr}^\phi \tau_b(\phi)|^2.\end{aligned}$$

By assumption of the transversal sectional curvature of $\mathcal{F}'$, i.e., $K^{Q'} \leq 0$ and the Kato's inequality [3], we have

$$(\Delta_B - \kappa_B^\sharp)|\tau_b(\phi)| \leq 0. \tag{5.6}$$

Since $\int_M |\tau_b(\phi)|^2 < \infty$, by Theorem 3.3, $|\tau_b(\phi)|$ is constant. Hence $\mathrm{Vol}(M) = \infty$ implies that $\tau_b(\phi) = 0$, i.e., ϕ is transversally harmonic. This complete the proof of (1). For the proof of (2), it is trivial from Theorem A and Theorem B (1). □

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References

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. **10** (1992), no. 2, 179–194.

[2] P. Baird, A. Fardoun, and S. Ouakkas, *Liouville-type theorems for biharmonic maps between Riemannian manifolds*, Adv. Calc. Var. **3** (2010), no. 1, 49–68.

[3] P. Bérard, *A note on Bochner type theorems for complete manifolds*, Manuscripta Math. **69** (1990), no. 3, 261–266.

[4] Y.-J. Chiang and R. A. Wolak, *Transversally biharmonic maps between foliated Riemannian manifolds*, Internat. J. Math. **19** (2008), no. 8, 981–996.

[5] S. Dragomir and A. Tommasoli, *Harmonic maps of foliated Riemannian manifolds*, Geom. Dedicata **162** (2013), 191–229.

[6] A. El Kacimi Alaoui and E. Gallego Gómez, *Applications harmoniques feuilletées*, Illinois J. Math. **40** (1996), no. 1, 115–122.

[7] M. J. Jung and S. D. Jung, *On transversally harmonic maps of foliated Riemannian manifolds*, J. Korean Math. Soc. **49** (2012), no. 5, 977–991.

[8] S. D. Jung, *Harmonic maps of complete Riemannian manifolds*, Nihonkai Math. J. **8** (1997), no. 2, 147–154.

[9] ———, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. **39** (2001), no. 3, 253–264.

[10] ———, *Variation formulas for transversally harmonic and biharmonic maps*, J. Geom. Phys. **70** (2013), 9–20.

[11] S. D. Jung, K. R. Lee, and K. Richardson, *Generalized Obata theorem and its applications on foliations*, J. Math. Anal. Appl. **376** (2011), no. 1, 129–135.

[12] F. W. Kamber and P. Tondeur, *Infinitesimal automorphisms and second variation of the energy for harmonic foliations*, Tôhoku Math. J. **34** (1982), no. 4, 525–538.

[13] ———, *De Rham-Hodge theory for Riemannian foliations*, Math. Ann. **277** (1987), no. 3, 415–431.

[14] J. Konderak and R. A. Wolak, *Transversally harmonic maps between manifolds with Riemannian foliations*, Q. J. Math. **54** (2003), no. 3, 335–354.

[15] ———, *Some remarks on transversally harmonic maps*, Glasg. Math. J. **50** (2008), no. 1, 1–16.

[16] J. S. Pak and S. D. Jung, *A transversal Dirac operator and some vanishing theorems on a complete foliated Riemannian manifold*, Math. J. Toyama Univ. **16** (1993), 97–108.

[17] E. Park and K. Richardson, *The basic Laplacian of a Riemannian foliation*, Amer. J. Math. **118** (1996), no. 6, 1249–1275.

[18] R. Schoen and S. T. Yau, *Harmonic maps and the topology of stable hypersurfaces and manifolds of non-negative Ricci curvature,* Comm. Math. Helv. **51** (1976), no. 3, 333–341.
[19] P. Tondeur, *Geometry of foliations*, Basel: Birkhäuser Verlag, 1997.
[20] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.
[21] ———, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), no. 7, 659–670.
[22] S. Yorozu, *Notes on suqare-integrable cohomology spaces on certain foliated manifolds*, Trans. Amer. Math. Soc. **255** (1979), 329–341.

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