J. Korean Math. Soc. **54** (2017), No. 3, pp. 713–732
https://doi.org/10.4134/JKMS.j150771
pISSN: 0304-9914 / eISSN: 2234-3008

COMMUTATORS OF SINGULAR INTEGRAL OPERATOR ON HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENT

Hongbin Wang

Abstract. Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s > 1$ be a homogeneous function of degree zero and b be BMO functions or Lipschitz functions. In this paper, we obtain some boundedness of the Calderón-Zygmund singular integral operator T_Ω and its commutator $[b, T_\Omega]$ on Herz-type Hardy spaces with variable exponent.

1. Introduction

The theory of function spaces with variable exponent has been extensively studied by researchers since the work of Kováčik and Rákosník [7] appeared in 1991, see [2, 4] and the references therein. In [14], the authors defined the Herz-type Hardy spaces with variable exponent and gave their atomic characterizations. Moreover, the authors studied the boundedness of some Calderón-Zygmund integral operators on Herz-type Hardy spaces with variable exponent in [12] and [13], respectively.

Suppose that S^{n-1} denote the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s > 1$ be a homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$. The Calderón-Zygmund singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

Received December 26, 2015.
2010 *Mathematics Subject Classification.* Primary 46E30; Secondary 42B35, 42B20.
Key words and phrases. Herz-type Hardy space, variable exponent, Calderón-Zygmund singular integral, commutator.

Let b be a locally integrable function on $\mathbb{R}^n$. The commutator $[b, T_\Omega]$ generated by the Calderón-Zygmund singular integral operator T_Ω and b is defined by

$$[b, T_\Omega]f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n}[b(x) - b(y)]f(y)dy.$$

Motivated by [9], [12] and [13], we will prove the boundedness of the Calderón-Zygmund singular integral operator T_Ω and its commutator $[b, T_\Omega]$ on Herz-type Hardy spaces with variable exponent, where $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s > 1$. Our results improve and generalize essentially the results in [12] and [13].

Firstly we recall some standard notations and lemmas in variable L^p spaces. Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \longrightarrow [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$.

The space $L^{p(\cdot)}_{\text{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega\}.$$

Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot) : \Omega \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1 ([1])**.** *If* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and satisfies*

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}, \quad |x-y| \le 1/2 \tag{1.2}$$

and

$$|p(x) - p(y)| \le \frac{C}{\log(|x| + e)}, \quad |y| \ge |x|, \tag{1.3}$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$*, that is the Hardy-Littlewood maximal operator* M *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$*.*

Lemma 1.2 ([7]). *Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, then fg is integrable on Ω and*

$$\int_\Omega |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.3 ([5]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1} \quad \text{and}$$
$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$ and χ_S, χ_B are the characteristic functions of S, B, respectively.

Throughout this paper δ_2 is the same as in Lemma 1.2.

Lemma 1.4 ([5]). *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Next we recall the definition of the Herz-type spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

Definition 1.1 ([5]). Let $\alpha \in \mathbb{R}, 0 < p \leq \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

In [14], the authors gave the definition of Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and the atomic decomposition characterizations. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N(f)(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N(f)(x) = \sup_{\phi \in \mathcal{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \le N} |x^\alpha D^\beta \phi(x)| \le 1\}$ and $N > n+1$, ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 1.2 ([14]). Let $\alpha \in \mathbb{R}, 0 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n+1$.

(i) The homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Herz-type Hardy space with variable exponent $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

For $x \in \mathbb{R}$ we denote by $[x]$ the largest integer less than or equal to x.

Definition 1.3 ([14]). Let $n\delta_2 \le \alpha < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and non-negative integer $s \ge [\alpha - n\delta_2]$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom, if it satisfies

(1) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha/n}$.

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0, |\beta| \le s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom of restricted type, if it satisfies the conditions (2), (3) above and

(1)′ $\operatorname{supp} a \subset B(0,r), r \ge 1$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 1.3, then the corresponding central $(\alpha, q(\cdot))$-atom is called a dyadic central $(\alpha, q(\cdot))$-atom.

Lemma 1.5 ([14]). *Let* $n\delta_2 \le \alpha < \infty$, $0 < p < \infty$ *and* $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then* $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (*or* $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$) *if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \left(or \sum_{k=0}^{\infty} \lambda_k a_k \right), \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k *is a central* $(\alpha, q(\cdot))$*-atom* (*or central* $(\alpha, q(\cdot))$*-atom of restricted type*) *with support contained in* B_k *and* $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ (*or* $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$). *Moreover,*

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \left(\text{or} \quad \|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p} \right),$$

where the infimum is taken over all above decompositions of f.

2. Estimate of the Calderón-Zygmund singular integral operator

A nonnegative locally integrable function $\omega(x)$ on $\mathbb{R}^n$ is said to belong to $A_p (1 < p < \infty)$, if there is a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{1-p'} dx \right)^{p-1} \le C < \infty,$$

where $p' = p/(p-1)$.

The weighted (L^p, L^q) boundedness of T_Ω have been proved by Lu, Ding and Yan [8].

Lemma 2.1 ([8]). *Suppose that* $\Omega \in L^s(\mathrm{S}^{n-1}) (s > 1)$ *satisfies* (1.1). *If* $\omega \in A_{p/s'}$, $s' \le p < \infty$, *then there is a constant* C, *independent of* f, *such that*

$$\int_{\mathbb{R}^n} |T_\Omega(f)(x)|^p \omega(x) dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.$$

Lemma 2.2 ([3]). *Given a family* $\mathcal{F}$ *and an open set* $E \subset \mathbb{R}^n$, *assume that for some* p_0, $0 < p_0 < \infty$ *and for every* $\omega \in A_\infty$,

$$\int_E f(x)^{p_0} \omega(x) dx \le C_0 \int_E g(x)^{p_0} \omega(x) dx, \quad (f, g) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}^0(E)$ *such that* $p(\cdot)$ *satisfies* (1.2) *and* (1.3) *in Lemma 1.1. Then for all* $(f, g) \in \mathcal{F}$ *such that* $f \in L^{p(\cdot)}(E)$,

$$\|f\|_{L^{p(\cdot)}(E)} \le C \|g\|_{L^{p(\cdot)}(E)}.$$

Since $A_{p/s'} \subset A_\infty$, *by Lemma 2.1 and Lemma 2.2 it is easy to get the* $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$*-boundedness of the commutator* T_Ω.

Before stating our result, let us recall the definition of the L^s-Dini condition. We say that satisfies the L^s-Dini condition if $\Omega \in L^s(\mathrm{S}^{n-1})$ with $s \geq 1$ is homogeneous of degree zero on $\mathbb{R}^n$, and

$$\int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta < \infty,$$

where $\omega_s(\delta)$ denotes the integral modulus of continuity of order s of Ω defined by

$$\omega_s(\delta) = \sup_{|\rho|<\delta} \left(\int_{\mathrm{S}^{n-1}} |\Omega(\rho x') - \Omega(\rho x')|^s \, dx' \right)^{1/s}$$

and ρ is a rotation in $\mathbb{R}^n$ and $|\rho| = \|\rho - I\|$.

Next, we will give the corresponding result about the commutator T_Ω on Herz-type Hardy spaces with variable exponent.

Theorem 2.1. *Suppose that* $0 < \beta \leq 1$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.2) *and* (1.3) *in Lemma 1.1 with* $\Omega \in L^s(\mathrm{S}^{n-1})(s > q^+)$ *and satisfies*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\beta}} d\delta < \infty.$$

Let $0 < p_1 \leq p_2 < \infty$ *and* $n\delta_2 \leq \alpha < n\delta_2 + \beta$ (*or* $0 < \max(n\delta_2, \alpha_2) \leq \alpha_1 < n\delta_2 + \beta$). *Then* T_Ω *is bounded from* $H\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)$ (*or* $HK_{q(\cdot)}^{\alpha_1,p_1}(\mathbb{R}^n)$) *to* $\dot{K}_{q(\cdot)}^{\alpha,p_2}(\mathbb{R}^n)$ (*or* $K_{q(\cdot)}^{\alpha_2,p_2}(\mathbb{R}^n)$).

In the proof of Theorem 2.1, we also need the following lemmas.

Lemma 2.3 ([2])**.** *Given* E *and* $p(\cdot) \in \mathcal{P}(E)$, *let* $f : E \times E \to \mathbb{R}$ *be a measurable function* (*with respect to product measure*) *such that for almost every* $y \in E$, $f(\cdot, y) \in L^{p(\cdot)}(E)$. *Then*

$$\left\| \int_E f(\cdot, y) dy \right\|_{L^{p(\cdot)}(E)} \leq C \int_E \|f(\cdot, y)\|_{L^{p(\cdot)}(E)} \, dy.$$

Lemma 2.4 ([10])**.** *Define a variable exponent* $\tilde{q}(\cdot)$ *by* $\frac{1}{p(x)} = \frac{1}{\tilde{q}(x)} + \frac{1}{q}$ $(x \in \mathbb{R}^n)$. *Then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \|fg\|_{L^q(\mathbb{R}^n)}$$

for all measurable functions f *and* g.

Lemma 2.5 ([4])**.** *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy conditions* (1.2) *and* (1.3) *in Lemma 1.1. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}} & \text{if } |Q| \leq 2^n \text{ and } x \in Q, \\ |Q|^{\frac{1}{p(\infty)}} & \text{if } |Q| \geq 1 \end{cases}$$

for every cube (*or ball*) $Q \subset \mathbb{R}^n$, *where* $p(\infty) = \lim_{x\to\infty} p(x)$.

Lemma 2.6 ([8]). *Suppose that Ω satisfies the L^s-Dini condition $(1 \le s < \infty)$. Then for any $R > 0$ and $x \in \mathbb{R}^n$, when $|y| < R/2$, there is a constant $C > 0$ such that*

$$\left(\int_{R<|x|<2R} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right|^s dx \right)^{1/s} \le CR^{\frac{n}{s}-n} \left\{ \frac{|y|}{R} + \int_{|y|/2R<\delta<|y|/R} \frac{\omega_s(\delta)}{\delta} d\delta \right\}.$$

Proof of Theorem 2.1. We only prove the homogeneous case. In [15], the authors proved $K_{q(\cdot)}^{\alpha_1,p_2}(\mathbb{R}^n) \subset K_{q(\cdot)}^{\alpha_2,p_2}(\mathbb{R}^n)$ for $0 < \alpha_2 \le \alpha_1$. So the non-homogeneous case can be proved in the same way. Let $f \in H\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)$. By Lemma 1.5 we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha, q(\cdot))$-atom with the support B_j. Note that $p_1 \le p_2$, we have

$$\begin{aligned} \|T_\Omega(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p_2}(\mathbb{R}^n)}^{p_1} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \|T_\Omega(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_2} \right\}^{p_1/p_2} \\ &\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \|T_\Omega(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \\ &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|T_\Omega(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|T_\Omega(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &=: I_1 + I_2. \end{aligned} \tag{2.1}$$

We first estimate I_1. For each $k \in \mathbb{Z}, j \le k-2$ and a.e. $x \in A_k$, using Lemma 2.3, the Minkowski inequality and the vanishing moments of a_j we have

$$\|T_\Omega(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy.$$

Noting $s > q^+$, we denote $\tilde{q}(\cdot) > 1$ and $\frac{1}{q(x)} = \frac{1}{\tilde{q}(x)} + \frac{1}{s}$. By Lemma 2.4 we have

$$\begin{aligned} &\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ \le & \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{s}(\mathbb{R}^n)} \|\chi_k(\cdot)\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \end{aligned}$$

$$\leq C \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}.$$

When $|B_k| \leq 2^n$ and $x_k \in B_k$, by Lemma 2.5 we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}(x_k)}} \approx \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}}.$$

When $|B_k| \geq 1$ we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}(\infty)}} \approx \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}}.$$

So we obtain $\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}}$.

Meanwhile, by Lemma 2.6 we have

$$\begin{aligned}
&\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} \\
&\leq 2^{(k-1)(\frac{n}{s}-n)} \left\{ \frac{|y|}{2^k} + \int_{|y|/2^k}^{|y|/2^{k-1}} \frac{\omega_s(\delta)}{\delta} d\delta \right\} \\
&\leq 2^{(k-1)(\frac{n}{s}-n)} \left(2^{j-k+1} + 2^{(j-k+1)\beta} \int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta \right) \\
&\leq C 2^{(k-1)(\frac{n}{s}-n)} 2^{(j-k)\beta}.
\end{aligned}$$

So by Lemma 1.3, Lemma 1.4 and the generalized Hölder inequality we have

$$\begin{aligned}
\|T_\Omega(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &= \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\
&\leq C 2^{(k-1)(\frac{n}{s}-n)} 2^{(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}} \int_{B_j} |a_j(y)| dy \\
&\leq C 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leq C 2^{(j-k)\beta} \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\
&\leq C 2^{-j\alpha+(j-k)(\beta+n\delta_2)}.
\end{aligned}$$

So we have

$$\begin{aligned}
I_1 &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{-j\alpha+(j-k)(\beta+n\delta_2)} \right)^{p_1} \\
&= C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)(\beta+n\delta_2-\alpha)} \right)^{p_1}.
\end{aligned}$$

When $1 < p_1 < \infty$, take $1/p_1 + 1/p_1' = 1$. Since $\beta + n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} I_1 &\le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right) \\ &\qquad \times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1'/2} \right)^{p_1/p_1'} \\ &\le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right) \\ &= C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right) \\ &\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{2.2}$$

When $0 < p_1 \le 1$, we have

$$\begin{aligned} I_1 &\le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \right) \\ &= C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \right) \\ &\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{2.3}$$

Next we estimate I_2, by the $(L^{q(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator T_Ω we have

$$\begin{aligned} I_2 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \right)^{p_1}. \end{aligned}$$

If $0 < p_1 \le 1$, then we have

$$I_2 \le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \right) \le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \tag{2.4}$$

If $1 < p_1 < \infty$, by the Hölder inequality we have

$$
\begin{aligned}
I_2 &\le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha p_1/2} \right) \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1'/2} \right)^{p_1/p_1'} \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{aligned} \tag{2.5}
$$

Thus, by (2.1)-(2.5) we complete the proof of Theorem 2.1. □

3. BMO estimate for the commutator of Calderón-Zygmund singular integral operator

Let us first recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$
\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,
$$

where $f_Q = |Q|^{-1} \int_Q f(y)dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

Lemma 3.1 ([6])**.** *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, k be a positive integer and B be a ball in $\mathbb{R}^n$. Then we have that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ with $j > i$,*

$$
\frac{1}{C}\|b\|_*^k \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^k \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^k,
$$

$$
\|(b - b_{B_i})^k \chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j-i)^k \|b\|_*^k \|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},
$$

where $B_i = \{x \in \mathbb{R}^n : |x| \le 2^i\}$ and $B_j = \{x \in \mathbb{R}^n : |x| \le 2^j\}$.

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. The weighted (L^p, L^q) boundedness of $[b, T_\Omega]$ have been proved by Lu, Ding and Yan [8].

Lemma 3.2 ([8])**.** *Suppose that $\Omega \in L^s(\mathrm{S}^{n-1})(s > 1)$ satisfies (1.1). If $\omega \in A_{p/s'}$, $s' \le p < \infty$, then there is a constant C, independent of f, such that*

$$
\int_{\mathbb{R}^n} |[b, T_\Omega](f)(x)|^p \omega(x) dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.
$$

Since $A_{p/s'} \subset A_\infty$, by Lemma 3.2 and Lemma 2.2 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, T_\Omega]$.

Next, we will give the corresponding result about the commutator $[b, T_\Omega]$ on Herz-type Hardy spaces with variable exponent.

Theorem 3.1. *Suppose that* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \beta \le 1$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.2) *and* (1.3) *in Lemma 1.1 with* $\Omega \in L^s(\mathrm{S}^{n-1})(s > q^+)$ *and satisfies*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\beta}} d\delta < \infty.$$

Let $0 < p_1 \le p_2 < \infty$ *and* $n\delta_2 \le \alpha < n\delta_2 + \beta$ (*or* $0 < \max(n\delta_2, \alpha_2) \le \alpha_1 < n\delta_2 + \beta$). *Then* $[b, T_\Omega]$ *is bounded from* $H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ (*or* $HK^{\alpha_1,p_1}_{q(\cdot)}(\mathbb{R}^n)$) *to* $\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$ (*or* $K^{\alpha_2,p_2}_{q(\cdot)}(\mathbb{R}^n)$).

Proof. Similar to Theorem 2.1, we only prove the homogeneous case. Let $f \in H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. By Lemma 1.5 we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha, q(\cdot))$-atom with the support B_j. Note that $p_1 \le p_2$, we have

$$\begin{aligned}
\|[b, T_\Omega](f)\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \|[b, T_\Omega](f)\chi_k\|^{p_2}_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{p_1/p_2} \\
&\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \|[b, T_\Omega](f)\chi_k\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|[b, T_\Omega](a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \qquad (3.1) \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|[b, T_\Omega](a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
&=: J_1 + J_2.
\end{aligned}$$

We first estimate J_1. For each $k \in \mathbb{Z}, j \le k-2$ and a.e. $x \in A_k$, using Lemma 2.3, the Minkowski inequality and the vanishing moments of a_j we have

$$\begin{aligned}
&\|[b, T_\Omega](a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| (b(\cdot) - b(y))\chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\
&\le \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b_{B_j}|\chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\
&\quad + \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |b_{B_j} - b(y)||a_j(y)| dy \\
&=: J_{11} + J_{12}.
\end{aligned}$$

For J_{11}, noting $s > q^+$, we denote $\tilde{q}(\cdot) > 1$ and $\frac{1}{q(x)} = \frac{1}{\tilde{q}(x)} + \frac{1}{s}$. By Lemma 3.1 and Lemma 2.4 we have

$$\begin{aligned}
&\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b_{B_j}| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
\le & \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} \left\| |b(\cdot) - b_{B_j}| \chi_k(\cdot) \right\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \\
\le & C \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} (k-j) \|b\|_* \|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $|B_k| \le 2^n$ and $x_k \in B_k$, by Lemma 2.5 we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}(x_k)}} \approx \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}}.$$

When $|B_k| \ge 1$ we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}(\infty)}} \approx \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}}.$$

So we obtain $\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}}$.

Meanwhile, by Lemma 2.6 we have

$$\begin{aligned}
&\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} \\
\le & 2^{(k-1)(\frac{n}{s} - n)} \left\{ \frac{|y|}{2^k} + \int_{|y|/2^k}^{|y|/2^{k-1}} \frac{\omega_s(\delta)}{\delta} d\delta \right\} \\
\le & 2^{(k-1)(\frac{n}{s} - n)} \left(2^{j-k+1} + 2^{(j-k+1)\beta} \int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta \right) \\
\le & C 2^{(k-1)(\frac{n}{s} - n)} 2^{(j-k)\beta}.
\end{aligned}$$

So by the generalized Hölder inequality we have

$$\begin{aligned}
J_{11} = & \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b_{B_j}| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\
\le & C(k-j) \|b\|_* 2^{(k-1)(\frac{n}{s} - n)} 2^{(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}} \int_{B_j} |a_j(y)| dy \qquad (3.2) \\
\le & C(k-j) \|b\|_* 2^{-kn + (j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For J_{12}, similar to the method of J_{11} we have

$$\begin{aligned}
&\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
\le & \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} \|\chi_k(\cdot)\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \\
\le & C \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq C2^{(k-1)(\frac{n}{s}-n)}2^{(j-k)\beta}\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}$$
$$\leq C2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

So by Lemma 3.1 and the generalized Hölder inequality we have

$$\begin{aligned}
J_{12} &= \int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}|b_{B_j}-b(y)||a_j(y)|dy \\
&\leq C2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\int_{B_j}|b_{B_j}-b(y)||a_j(y)|dy \\
&\leq C2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|(b_{B_j}-b)\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_*2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.3}$$

By (3.2), (3.3), Lemma 1.3 and Lemma 1.4 we have

$$\begin{aligned}
&\|[b,T_\Omega](a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C(k-j)\|b\|_*2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leq C(k-j)\|b\|_*2^{(j-k)\beta}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\
&\leq C(k-j)2^{-j\alpha+(j-k)(\beta+n\delta_2)}\|b\|_*.
\end{aligned}$$

So we have

$$\begin{aligned}
J_1 &\leq C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|(k-j)2^{-j\alpha+(j-k)(\beta+n\delta_2)}\right)^{p_1} \\
&= C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|(k-j)2^{(j-k)(\beta+n\delta_2-\alpha)}\right)^{p_1}.
\end{aligned}$$

When $1<p_1<\infty$, take $1/p_1+1/p_1'=1$. Since $\beta+n\delta_2-\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}
J_1 &\leq C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right) \\
&\qquad\times\left(\sum_{j=-\infty}^{k-2}(k-j)^{p_1'}2^{(j-k)(\beta+n\delta_2-\alpha)p_1'/2}\right)^{p_1/p_1'} \\
&\leq C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right) \\
&= C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right)
\end{aligned}\tag{3.4}$$

$$\leq C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.$$

When $0 < p_1 \leq 1$, we have

$$\begin{aligned} J_1 &\leq C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} (k-j)^{p_1} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \right) \\ &= C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} (k-j)^{p_1} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \right) \qquad (3.5) \\ &\leq C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned}$$

Next we estimate J_2, by the $(L^{q(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, T_\Omega]$ we have

$$\begin{aligned} J_2 &\leq C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\leq C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \right)^{p_1}. \end{aligned}$$

If $0 < p_1 \leq 1$, then we have

$$J_2 \leq C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \leq C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \qquad (3.6)$$

If $1 < p_1 < \infty$, by the Hölder inequality we have

$$\begin{aligned} J_2 &\leq C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha p_1/2} \right) \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1'/2} \right)^{p_1/p_1'} \qquad (3.7) \\ &\leq C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned}$$

Thus, by (3.1), (3.4)-(3.7) we complete the proof of Theorem 3.1. □

4. Lipschitz estimate for the commutator of Calderón-Zygmund singular integral operator

For $0 < \gamma \leq 1$, the Lipschitz space $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ is defined as

$$\mathrm{Lip}_\gamma(\mathbb{R}^n) = \left\{ f : \|f\|_{\mathrm{Lip}_\gamma} = \sup_{x,y\in\mathbb{R}^n; x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\gamma} < \infty \right\}.$$

Let $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$. It is easy to know that $|[b, T_\Omega]| \le C\|b\|_{\mathrm{Lip}_\gamma}|T_{\Omega,\gamma}|$, where

$$T_{\Omega,\gamma}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\gamma}} f(y)dy.$$

In [11], the authors proved that $T_{\Omega,\gamma}$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$ for $1/q_1(x) - 1/q_2(x) = \gamma/n$ and $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying conditions (1.2) and (1.3) in Lemma 1.1 with $q_1^+ < n/\gamma$. So we can get the following theorem.

Theorem 4.1. *Suppose that* $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ *with* $0 < \gamma \le 1$. *If* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.2) *and* (1.3) *in Lemma 1.1 with* $q_1^+ < n/\gamma$, $1/q_1(x) - 1/q_2(x) = \gamma/n$, $\Omega \in L^s(\mathrm{S}^{n-1})(s > q_2^+)$ *with* $1 \le s' < q_1^-$. *Then* $[b, T_\Omega]$ *is bounded from* $L^{q_1(\cdot)}(\mathbb{R}^n)$ *to* $L^{q_2(\cdot)}(\mathbb{R}^n)$.

Next, we will give the Lipschitz estimate about the commutator $[b, T_\Omega]$ on Herz-type Hardy spaces with variable exponent.

Theorem 4.2. *Suppose that* $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ *with* $0 < \gamma \le 1$. *If* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.2) *and* (1.3) *in Lemma 1.1 with* $q_1^+ < n/\gamma$, $1/q_1(x) - 1/q_2(x) = \gamma/n$, $\Omega \in L^s(\mathrm{S}^{n-1})(s > q_2^+)$ *with* $1 \le s' < q_1^-$ *and satisfies*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\gamma}} d\delta < \infty.$$

Let $0 < p_1 \le p_2 < \infty$ *and* $n\delta_2 \le \alpha < n\delta_2 + \gamma$ (*or* $0 < \max(n\delta_2, \alpha_2) \le \alpha_1 < n\delta_2 + \gamma$). *Then* $[b, T_\Omega]$ *maps* $H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ (*or* $HK^{\alpha_1,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$) *continuously into* $\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$ (*or* $K^{\alpha_2,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$).

Proof. Similar to Theorem 2.1, it suffices to prove homogeneous case. Let $f \in H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$. By Lemma 1.5 we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha, q_1(\cdot))$-atom with the support B_j. We have

$$\begin{aligned} \|[b,T_\Omega](f)\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \|[b,T_\Omega](f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right\}^{p_1/p_2} \\ &\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \|[b,T_\Omega](f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \qquad (4.1)\\ &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|[b,T_\Omega](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|[b,T_\Omega](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &=: U_1 + U_2. \end{aligned}$$

We first estimate U_1. For each $k \in \mathbb{Z}$, $j \le k-2$ and a.e. $x \in A_k$, using Lemma 2.3, the Minkowski inequality and the vanishing moments of a_j we have

$$\begin{aligned}
&\|[b,T_\Omega](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
\le\ & \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| (b(\cdot) - b(y))\chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |a_j(y)|dy \\
\le\ & \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b(0)|\chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |a_j(y)|dy \\
& + \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |b(0) - b(y)||a_j(y)|dy \\
=:\ & U_{11} + U_{12}.
\end{aligned}$$

For U_{11}, noting $s > q_2^+$, we denote $\tilde{q}_2(\cdot) > 1$ and $\frac{1}{q_2(x)} = \frac{1}{\tilde{q}_2(x)} + \frac{1}{s}$. By Lemma 2.4 we have

$$\begin{aligned}
&\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b(0)|\chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
\le\ & \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{s}(\mathbb{R}^n)} \||b(\cdot) - b(0)|\chi_k(\cdot)\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\
\le\ & C\|b\|_{\mathrm{Lip}_\gamma} 2^{k\gamma} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{s}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $|B_k| \le 2^n$ and $x_k \in B_k$, by Lemma 2.5 we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}_2(x_k)}} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s} - \frac{\gamma}{n}}.$$

When $|B_k| \ge 1$ we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}_2(\infty)}} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s} - \frac{\gamma}{n}}.$$

So we obtain $\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s} - \frac{\gamma}{n}}$.

Meanwhile, by Lemma 2.6 we have

$$\begin{aligned}
&\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{s}(\mathbb{R}^n)} \\
\le\ & 2^{(k-1)(\frac{n}{s} - n)} \left\{ \frac{|y|}{2^k} + \int_{|y|/2^k}^{|y|/2^{k-1}} \frac{\omega_s(\delta)}{\delta} d\delta \right\} \\
\le\ & 2^{(k-1)(\frac{n}{s} - n)} \left(2^{j-k+1} + 2^{(j-k+1)\gamma} \int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta \right) \\
\le\ & C 2^{(k-1)(\frac{n}{s} - n)} 2^{(j-k)\gamma}.
\end{aligned}$$

So by the generalized Hölder inequality we have

(4.2)

$$\begin{aligned}U_{11} &= \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b(0)| \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\ &\le C\|b\|_{\mathrm{Lip}_\gamma} 2^{k\gamma} 2^{(k-1)(\frac{n}{s}-n)} 2^{(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}-\frac{\gamma}{n}} \int_{B_j} |a_j(y)| dy \\ &\le C\|b\|_{\mathrm{Lip}_\gamma} 2^{-kn+(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

For U_{12}, similar to the method of U_{11} we have

$$\begin{aligned}&\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ \le\ & \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{s}(\mathbb{R}^n)} \|\chi_k(\cdot)\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\ \le\ & \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{s}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\ \le\ & C 2^{(k-1)(\frac{n}{s}-n)} 2^{(j-k)\gamma} \|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\ \le\ & C 2^{-kn+(j-k)\gamma-k\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

So by the generalized Hölder inequality we have

$$\begin{aligned}(4.3)\quad U_{12} &= \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |b(0) - b(y)| |a_j(y)| dy \\ &\le C 2^{-kn+(j-k)\gamma-k\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \int_{B_j} |b(0) - b(y)| |a_j(y)| dy \\ &\le C\|b\|_{\mathrm{Lip}_\gamma} 2^{-kn+2(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\le C\|b\|_{\mathrm{Lip}_\gamma} 2^{-kn+(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

By (4.2), (4.3), Lemma 1.3 and Lemma 1.4 we have

$$\begin{aligned}&\|[b, T_\Omega](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ \le\ & C\|b\|_{\mathrm{Lip}_\gamma} 2^{-kn+(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\ \le\ & C\|b\|_{\mathrm{Lip}_\gamma} 2^{(j-k)\gamma} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \\ \le\ & C 2^{-j\alpha+(j-k)(\gamma+n\delta_2)} \|b\|_{\mathrm{Lip}_\gamma}.\end{aligned}$$

So we have

$$U_1 \le C\|b\|^{p_1}_{\mathrm{Lip}_\gamma} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{-j\alpha+(j-k)(\gamma+n\delta_2)} \right)^{p_1}$$

$$= C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)(\gamma+n\delta_2-\alpha)} \right)^{p_1}.$$

When $1 < p_1 < \infty$, take $1/p_1 + 1/p_1' = 1$. Since $\gamma + n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} U_1 &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1/2} \right) \\ &\qquad \times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1'/2} \right)^{p_1/p_1'} \\ &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1/2} \right) \\ &= C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1/2} \right) \\ &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{4.4}$$

When $0 < p_1 \le 1$, we have

$$\begin{aligned} U_1 &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1} \right) \\ &= C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1} \right) \\ &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{4.5}$$

Next we estimate U_2, by the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, T_\Omega]$ we have

$$\begin{aligned} U_2 &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \right)^{p_1}. \end{aligned}$$

If $0 < p_1 \leq 1$, then we have

$$U_2 \leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \tag{4.6}$$
$$\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.$$

If $1 < p_1 < \infty$, by the Hölder inequality we have

(4.7)

$$U_2 \leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha p_1/2} \right) \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1'/2} \right)^{p_1/p_1'}$$
$$\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.$$

Thus, by (4.1), (4.4)-(4.7) we complete the proof of Theorem 4.1. □

Acknowledgements. The author is very grateful to the referees for their valuable comments.

References

[1] C. Capone, D. Cruz-Uribe, SFO and A. Fiorenza, *The fractional maximal operator and fractional integrals on variable L^p spaces*, Rev. Mat. Iberoamericana **23** (2007), no. 3, 743–770.

[2] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces*: *Foundations and Harmonic Analysis* (*Applied and Numerical Harmonic Analysis*), Springer, Heidelberg, 2013.

[3] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), no. 1, 239–264.

[4] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Math., vol. **2017**, Springer, Heidelberg, 2011.

[5] M. Izuki, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math. **36** (2010), no. 1, 33–50.

[6] ———, *Boundedness of commutators on Herz spaces with variable exponent*, Rend. Circ. Mat. Palermo (2) **59** (2010), no. 2, 199–213.

[7] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), no. 4, 592–618.

[8] S. Lu, Y. Ding, and D. Yan, *Singular Integrals and Related Topics*, World Scientific Press, Beijing, 2011.

[9] S. Lu, Q. Wu, and D. Yang, *Boundedness of commutators on Hardy type spaces*, Sci. China Ser. A **45** (2002), no. 8, 984–997.

[10] E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. **262** (2012), no. 9, 3665–3748.

[11] J. Tan and Z. Liu, *Some boundedness of homogeneous fractional integrals on variable exponent function spaces*, Acta Math. Sinica (Chin. Ser.) **58** (2015), no. 2, 309–320.

[12] H. Wang, *The continuity of commutators on Herz-type Hardy spaces with variable exponent*, Kyoto J. Math. **56** (2016), no. 3, 559–573.

[13] ______, *Boundedness of commutators on Herz-type Hardy spaces with variable exponent*, Jordan J. Math. Stat. **9** (2016), no. 1, 17–30.

[14] H. Wang and Z. Liu, *The Herz-type Hardy spaces with variable exponent and their applications*, Taiwanese J. Math. **16** (2012), no. 4, 1363–1389.

[15] ______, *Some characterizations of Herz-type Hardy spaces with variable exponent*, Ann. Funct. Anal. **6** (2015), no. 2, 224–243.

Hongbin Wang
School of Science
Shandong University of Technology
Zibo 255049, P. R. China
E-mail address: hbwang_2006@163.com