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Design of ramp-stress accelerated life test plans for a parallel system with two independent components using masked data

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Abstract: In this paper, we have formulated optimum Accelerated Life Test (ALT) plan for a parallel system with two independent components using masked data with ramp-stress loading scheme and Type-I censoring. Consider a system of two independent and non-identical components connected in parallel. Such a system fails whenever all of its components has failed .The exact component that causes the system to fail is often unknown due to cost and time constraint. For each parallel system at test, we observe its system's failure time and a set of component that includes the component actually causing the system to fail. The stress-life relationship is modelled using inverse power law, and cumulative exposure model is assumed to model the effect of changing stress. The optimal plan consists in finding out the optimum stress rate using D-optimality criterion. The method developed has been explained using a numerical example and sensitivity analysis carried out.

Key Words: *Accelerated Life Test, D-optimality criterion, Fisher information matrix, Masked data, Ramp stress*

1. INTRODUCTION AND MOTIVATION

Accelerated Life Tests help in inducing early failures in high reliability items that are likely to last for several years; by subjecting them to higher than normal stress conditions, say higher temperature, voltage, pressure etc.. Stress under accelerated condition can be applied using **constant-stress**, **step-stress**, **progressive-stress, cyclic stress, random-stress,** or combinations of such loadings. The choice of a stress loading depends on how the product or unit is used in service and other practical and theoretical limitations (Nelson (1980), Elsayed (1996)). **Life tests** under accelerated environmental conditions may be **fully** accelerated or **partially** accelerated. In **fully accelerated life testing** all the test units are run at accelerated condition, while in **partially accelerated life testing** they are run at

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both normal and accelerated conditions. The term fully accelerated life test has been coined by Bhattacharyya and Soejoeti (1989) and the term partially accelerated life test is due to DeGroot and Goel (1979). The terms accelerated life tests and fully accelerated life tests are used interchangeably in the literature (Nelson (1990)) has discussed the statistical models, test plans and methods of data analyses for the ALT.

Design of accelerated life tests plans under different stress loading schemes has been studied extensively in the literature. See for example, Srivastava and Gupta (2015) and the references therein. However, all these tests are suitable for one unit or component or subsystem of a system.

Srivastava and Savita (2016) have described the optimal accelerated life test plan for a parallel system with two dependent components under ramp-stress loading using time-censored data, and the optimal plan consists in finding out the optimum stress rate using D-optimality criterion. Often, due to certain environmental conditions, the exact cause of system failure might be unknown. Instead, it can be ascertained that the cause of system failure is due to the component which belongs to some subset of the *J* components of the system. Such type of observation is referred to as being masked, see Usher and Hodgson (1998). Estimation of component reliability in a parallel system using masked life data has been studied by Sarhan and Ei-Bassiouny (2003). The method developed has been explained using a numerical example and sensitivity analysis carried out.

Acronyms

ALT	accelerated life test
Avar	asymptotic variance
ML	maximum likelihood
Cdf	cumulative distribution function
Pdf	probability distribution function
GAV	generalized asymptotic variance
IWD	Inverse Weibull distribution

Notation

n	Total number of test units
S_i	Set of component that may cause the system failure
T_i	The life time of system i
T_{ij}	The life time of component j in system i
$f_j(t)$	probability distribution function of component $j, j=1,2$
$F_j(t)$	cumulative distribution function of component $j, j=1,2$
^	Maximum Likelihood estimate
$\varepsilon(t)$	Cumulative Exposure at time t
γ_0, γ_1	Parameters of the inverse power law, $\gamma_1 > 0,\ -\infty < \gamma_0 < \infty$
n_1	The number of system failures for which the component 1 where

$S_i = \{1\}$

n_2 The number of system failures for which the component 2 where $S_i = \{2\}$

n_{12} The number of masked observation where $S_i = \{1,2\}$

$f_1(t)$ $\frac{\beta_{11}}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{-(\beta_{11}+1)} e^{-(\frac{t}{\alpha_1})^{-\beta_{11}}}$

$f_2(t)$ $\frac{\beta_{12}}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{-(\beta_{12}+1)} e^{-(\frac{t}{\alpha_1})^{-\beta_{12}}}$

$F_1(t)$ $e^{-(\frac{t}{\alpha_1})^{-\beta_{11}}}$

$F_2(t)$ $e^{-(\frac{t}{\alpha_1})^{-\beta_{12}}}$

g(t) $e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}\log(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))$

$$+2*e^{2\gamma_0}(\frac{s_0}{k})^{2\gamma_1}(1+\gamma_1)(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{-2+\frac{1}{1+\gamma_1}}(-1+\frac{1}{1+\gamma_1})$$

2. Reliability PARALLEL STRUCTURE AND ITS RELIABILITY

2.1 Parallel System Configuration

Consider a parallel system that consists of *J* independent but non-identical components. This system will fail if all of its components failed.

The graphical representation of a parallel system with two components **also referred to as a parallel system of order 2** is described in Figure 1.

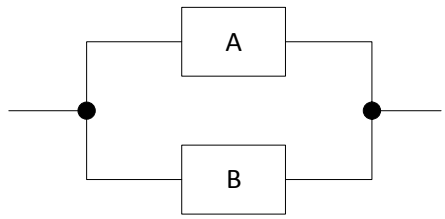

Figure 1. Parallel System

2.2 Reliability Function of a Parallel System of order 2

The life time of ith parallel system, i = 1, 2, ..., n, with two independent and identical components is:

$$T_i = \max(T_{i1}, T_{i2}), \tag{1}$$

where T_{i1} and T_{i2} represent life times of its components.
The reliability of a parallel system of order 2 is:

$$\overline{F}_0(t) = 1 - \overline{F}_1(t)\overline{F}_2(t). \tag{2}$$

3. THE MODEL

Assumptions
The following assumptions are made:
a) The system consists of two independent components connected in parallel.
b) It is assumed that the lifetimes of the components have a Weibull distribution with the failure density:

$$g_X(x;\eta,\beta)=\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-\left(\frac{x}{\eta}\right)^{\beta}}. \tag{3}$$

Here $\beta > 0$ and $\eta > 0$ are the shape and scale parameters. In other words, $X \sim WEI(\beta,\eta)$
c) There are no any two components failing at the same time.
d) Failed parallel systems are not replaced during the test.
e) The occurrence of Masking is independent of the failure cause and time.
f) The cumulative exposure model ([5]) holds for the effect of changing stress.
g) The inverse power law holds for stress-life relationship, i.e.,

$$\eta(s(t))=e^{\gamma_0}\left(\frac{s_0}{s(t)}\right)^{\gamma_1}, \tag{4}$$

where parameters γ_0 and γ_1 are the characteristics of the product, s(t) is a linear function of time in ramp-stress, s(t) = kt, where k is the stress rate .
h) The stress applied to test units is continuously increased with constant rate,k, from zero.

3.1 Inverse Weibull (IW) Distribution Under Ramp-Stress

The random variable $T=\frac{\beta^2}{X}$ has the IW distribution. The life distribution and failure density
of T obtained using (3) are:

$$G_T(t;\eta,\beta) = e^{-(\frac{t}{\eta})^{-\beta}}, \tag{5}$$

and

$$g_T(t;\alpha,\beta) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{-(\beta+1)} e^{-(\frac{t}{\eta})^{-\beta}}, \tag{6}$$

respectively.
Since the components of a parallel system are assumed to be independent, therefore the joint life distribution of ith system is given by:

$$G_{T_{i1},T_{i2}}(t_{i1},t_{i2}) = G_{T_{i1}}(t_{i1})G_{T_{i2}}(t_{i2})$$

$$\Rightarrow G_{T_{i1},T_{i2}}(t_{i1},t_{i2}) = e^{-(\frac{t_{i1}}{\eta_1})^{-\beta_1}} e^{-(\frac{t_{i2}}{\eta_2})^{-\beta_2}} . \tag{7}$$

The joint pdf of ith system is given as:

$$g_{T_{i1},T_{i2}}(t_{i1},t_{i2}) = \frac{\beta_1}{\eta_1}\left(\frac{t_{i1}}{\eta_1}\right)^{-(\beta_1+1)} e^{-(\frac{t_{i1}}{\eta_1})^{-\beta_1}} \frac{\beta_2}{\eta_2}\left(\frac{t_{i2}}{\eta_2}\right)^{-(\beta_2+1)} e^{-(\frac{t_{i2}}{\eta_2})^{-\beta_2}} . \tag{8}$$

The stress at time t is from Assumption j is:

$$s(t) = kt. \tag{9}$$

From the linear cumulative exposure model and the inverse power law, the joint cdf of the lifetime of ith parallel system with independent components under ramp-stress loading is:

$$F_{T_{i1},T_{i2}}(t_{i1},t_{i2}) = G_{T_{i1},T_{i2}}(\varepsilon(t_{i1}),\varepsilon(t_{i2})) , \tag{10}$$

where $G(\cdot,\cdot)$ is the assumed joint inverse cumulative Weibull life distribution function with scale parameter set equal to one, and

$$\varepsilon(t)=\int_0^t \frac{1}{\eta(s(u))}du \tag{11}$$

represents the cumulative exposure (damage) model at t.
Since the components are independent, therefore

$$F_{T_{i1},T_{i2}}(t_{i1},t_{i2}) = G_{T_{i1}}(\varepsilon(t_{i1}))G_{T_{i2}}(\varepsilon(t_{i2})) \tag{12}$$

Hence, the cdf and pdf of the lifetime of the ith systemunder ramp-stress loading are:

$$F_{T_{i1},T_{i2}}(t_{i1},t_{i2}) = \exp[-(\frac{\int_0^{t_{i1}} \frac{1}{\eta(s(u))}du}{\eta_1})^{-\beta_1}] \exp[-(\frac{\int_0^{t_{i2}} \frac{1}{\eta(s(u))}du}{\eta_2})^{-\beta_2}]$$

$$= \exp[-((\frac{t_{i1}}{\alpha_1})^{-\beta_{11}} + (\frac{t_{i2}}{\alpha_1})^{-\beta_{12}})], \tag{13}$$

$$f_{T_{i1},T_{i2}}(t_{i1},t_{i2}) = \frac{\beta_{11}}{\alpha_1}\left(\frac{t_{i1}}{\alpha_1}\right)^{-(\beta_{11}+1)} \exp(-(\frac{t_{i1}}{\alpha_1})^{-\beta_{11}})$$

$$.\frac{\beta_{12}}{\alpha_1}\left(\frac{t_{i2}}{\alpha_1}\right)^{-(\beta_{12}+1)} \exp(-(\frac{t_{i2}}{\alpha_1})^{-\beta_{12}}), \tag{14}$$

respectively, where $\alpha_1=(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{\frac{1}{(1+\gamma_1)}}$ is the scale parameter, $\beta_{11} = \beta_1(1+\gamma_1)$ and $_{12} = {}_2(1+ {}_1)$ are the shape parameters.

3.2 Likelihood Function Formulation

The test procedure consists in putting 'n' independent and identical 2-component parallel systems (parallel systems of order 2) to ramp-stress accelerated test using time-censored (Type-I censored) data. The two components are independent.
The log-likelihood function of n parallel systems of order 2 is obtained as follows:
The system will fail if all of its components fail. It means that the system fails at time t_i if one of its components fails at t_i and the rest of the components failed before the time t_i Hence the pdf of lifetime T_i of ith system when a component j, j = 1,2, in the system fails at time t_i and the rest of the components failed before t_i is obtained as:

$$\lim_{\Delta t_i \to 0} \frac{P[T_{ik} \le t_i, t_i \le T_{ij} \le t_i + \Delta t_i]}{\Delta t_i}$$

$$= \lim_{\Delta t_i \to 0} \frac{P[T_{ik} \le t_i] P[t_i \le T_{ij} \le t_i + \Delta t_i \mid T_{ik} \le t_i]}{\Delta t_i},$$

where $j \neq k$. Since the components are independent, therefore

$$\lim_{\Delta t_i \to 0} \frac{P[T_{ik} \le t_i, t_i \le T_{ij} \le t_i + \Delta t_i]}{\Delta t_i} = \lim_{\Delta t_i \to 0} \frac{P[T_{ik} \le t_i] P[t_i \le T_{ij} \le t_i + \Delta t_i]}{\Delta t_i}$$

$$= F_{ik}(t_i) f_{ij}(t_i).$$

Thus,

$$\lim_{\Delta t_i \to 0} \frac{P[T_{ik} \le t_i, t_i \le T_{ij} \le t_i + \Delta t_i]}{\Delta t_i}$$

$$= \begin{cases} F_{i2}(t_i) f_{i1}(t_i), \text{ if component 2 fails before } t_i \text{ and component 1 fails at } t_i, \text{ and} \\ F_{i1}(t_i) f_{i2}(t_i), \text{ if component 1 fails before } t_i \text{ and component 2 fails at } t_i. \end{cases} \tag{15}$$

Using (15), the likelihood function of n parallel systems is:

$$L \equiv \prod_{i|S_i=\{1\}}^{n_1} f_{i1}(t_i) F_{i2}(t_i) \prod_{i|S_i=\{2\}}^{n_2} f_{i2}(t_i) F_{i1}(t_i) \prod_{i|S_i=\{1,2\}}^{n_{12}} \{f_{i1}(t_i) F_{i2}(t_i) + f_{i2}(t_i) F_{i1}(t_i)\}$$

$$L = \prod_{i|S_i=\{1\}}^{n_1} \left(\frac{\beta_{11}}{\alpha_1} \left(\frac{t_i}{\alpha_1} \right)^{-(\beta_{11}+1)} e^{-\sum_{j=1}^{2} (\frac{t_i}{\alpha_1})^{-\beta_{1j}}} \right) \prod_{i|S_i=\{2\}}^{n_2} \left(\frac{\beta_{12}}{\alpha_1} \left(\frac{t_i}{\alpha_1} \right)^{-(\beta_{12}+1)} e^{-\sum_{j=1}^{2} (\frac{t_i}{\alpha_1})^{-\beta_{1j}}} \right)$$

$$\times \prod_{i|S_i=\{1,2\}}^{n_{12}} \left\{ \sum_{j=1}^{2} \frac{\beta_{1j}}{\alpha_1} \left(\frac{t_i}{\alpha_1} \right)^{-(\beta_{1j}+1)} e^{-\sum_{j=1}^{2} (\frac{t_i}{\alpha_1})^{-\beta_{1j}}} \right\} \tag{16}$$

$\Rightarrow$ log-likelihood function is:

$$\log L = \sum_{i|S_i=\{1\}}^{n_1} \log\left(\frac{\beta_{11}}{\alpha_1}\left(\frac{t_i}{\alpha_1}\right)^{-(\beta_{11}+1)} e^{-\sum_{j=1}^{2}\left(\frac{t_i}{\alpha_1}\right)^{-\beta_{1j}}}\right) + \sum_{i|S_i=\{2\}}^{n_2} \log\left(\frac{\beta_{12}}{\alpha_1}\left(\frac{t_i}{\alpha_1}\right)^{-(\beta_{12}+1)} e^{-\sum_{j=1}^{2}\left(\frac{t_i}{\alpha_1}\right)^{-\beta_{1j}}}\right)$$

$$+ \sum_{i|S_i=\{1,2\}}^{n_{12}} \left\{\sum_{j=1}^{2} \frac{\beta_{1j}}{\alpha_1}\left(\frac{t_i}{\alpha_1}\right)^{-(\beta_{1j}+1)} e^{-\sum_{j=1}^{2}\left(\frac{t_i}{\alpha_1}\right)^{-\beta_{1j}}}\right\}, \tag{17}$$

where $n = n_1 + n_2 + n_{12}$ and n is fixed.

3.3 Parameter Estimation

The MLEs of $\gamma_0, \gamma_1, \beta_1$ and β_2 are obtained by using *NMaximize* option of *Mathematica* 10.0.

3.4 Fisher Information Matrix

It is the 4×4 symmetric matrix of expectation of negative second order partial derivatives of the log likelihood function with respect to γ_0, γ_1, β_1 and β_2 .

Thus, the Fisher information matrix F is:

$$F \equiv F(\gamma_0,\gamma_1,\beta_{11},\beta_{12}) = \begin{bmatrix} E\left[-\frac{\partial^2 L}{\partial \gamma_0^2}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_0 \partial \gamma_1}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_0 \partial \beta_{11}}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_0 \partial \beta_{12}}\right] \\ E\left[-\frac{\partial^2 L}{\partial \gamma_1 \partial \gamma_0}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_1^2}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_1 \partial \beta_{11}}\right] & E\left[-\frac{\partial^2 L}{\partial \gamma_1 \partial \beta_{22}}\right] \\ E\left[-\frac{\partial^2 L}{\partial \beta_{11} \partial \gamma_0}\right] & E\left[-\frac{\partial^2 L}{\partial \beta_{11} \partial \gamma_1}\right] & E\left[-\frac{\partial^2 L}{\partial^2 L / \partial \beta_{11}^2}\right] & E\left[-\frac{\partial^2 L}{\partial^2 L / \partial \beta_{11} \partial \beta_{22}}\right] \\ E\left[-\frac{\partial^2 L}{\partial \beta_{22} \partial \gamma_0}\right] & E\left[-\frac{\partial^2 L}{\partial \beta_{22} \partial \gamma_1}\right] & E\left[-\frac{\partial^2 L}{\partial \beta_{22} \partial \beta_{11}}\right] & E\left[-\frac{\partial^2 L}{\partial \beta_{22}^2}\right] \end{bmatrix} \tag{18}$$

The calculation of the element of the F in (18) is shown in the Appendix. These elements have been obtained using the following results:

(i) $n_1 \sim \text{Bin}(n, p_1)$,

where using (15)

$$p_1 = \int_0^\infty f_{i1}(t_i) F_{i2}(t_i) dt_i$$

$$\Rightarrow \quad E[n_1] = n \int_0^\infty f_1(t_i) F_2(t_i) dt_i \tag{19}$$

(ii) $n_2 \sim \text{Bin}(n, p_2)$,

where using (15)

$$p_2 = \int_0^\infty f_{i2}(t_i)F_{i1}(t_i)dt_i \ .$$

$$\Rightarrow \quad E[n_2] = n\int_0^\infty f_2(t_i)F_1(t_i)dt_i \tag{20}$$

(iii) $n_{12} \sim \text{Bin}(n, p_{12})$,

where using (15)

$$p_{12} = \int_0^\infty \left(f_{i1}(t_i)F_{i2}(t_i) + f_{i2}(t_i)F_{i1}(t_i)\right)dt_i \ .$$

$$\Rightarrow \quad E[n_{12}] = n\int_0^\infty \left(f_1(t_i)F_2(t_i) + f_2(t_i)F_1(t_i)\right)dt_i \ . \tag{21}$$

Since for some set of parameters $\{\gamma_0, \gamma_1, s_0, \beta_1, \beta_2, k\}$; $|F|$ or variance function may be negative, therefore, we choose only that parametric set for which $|F| > 0$ and variance function, is positive.

4. OPTIMUM PLAN

The optimal plan consists in finding the optimal stress rate using D-optimality criterion wherein the reciprocal of the determinant of F is minimized.a smaller value of the determinant would correspond to a higher (joint) precision of the estimators of parameters (Escobar and Meeker (1995)).

Thus, the optimization problem is:

$$\text{Minimize} \quad \frac{1}{|F|}$$

subject to k> 0.

Optimal k is obtained using the *NMinimize* option of *Mathematica 10.0* .

5. A NUMERICAL ILLUSTRATION AND SENSITIVITY ANALYSIS

5.1 A Numerical Example

In this section, a hypothetical ramp-stress ALT experiment for a parallel system with two independent components is considered to illustrate the method described in this paper with the following data set:

$$n = 30, s_0 = 10, \gamma_0 = -1.86\ \gamma_1 = 0.55,\ \beta_1 = 2.12,\ \beta_2 = 1.12 .$$

The optimal value of k is obtained as $k^* = 0.316$.

5.2 Sensitivity Analysis

To use an optimum test plan, one needs estimates of the design parameters $\gamma_0, \gamma_1, \beta_1$ and β_2. These estimates sometimes may significantly affect the values of the resulting decision variables; therefore, their incorrect choice may give a poor estimate of the quantile at design constant stress. Hence, it is important to conduct sensitivity analysis to evaluate the robustness of the resulting ALT plan.

The percentage deviations of the optimal settings are measured by $PD = (|Z^{**} - Z^{*}|/Z^{*}) \times 100$, where Z^{*} is the setting obtained with the given design parameters, and Z^{**} is the one obtained when the parameter is mis-specified. Table 1 shows the optimal test plans for various deviations from the design parameter estimates. The results show that the optimal setting of Z is robust to the small deviations from baseline parameter estimates.

Table 1. Sensitivity analysis
($n = 30$, $s_0 = 10$, $\gamma_0 = -1.86$, $\gamma_1 = 0.55$, $\beta_1 = 2.12$, $\beta_2 = 1.12$, and $Z^{*} = 0.0000651442$)

Parameter	% change	k	Z^{**}	PD %
β_1	+1%	0.319522	0.0000657636	0.950
β_1	-1%	0.312967	0.0000644979	0.992
β_2	+1%	0.316253	0.0000644245	1.104
β_2	-1%	0.316264	0.000065223	0.120
γ_0	+1%	0.305744	0.0000651442	0
γ_0	-1%	0.328012	0.0000651448	0.001
γ_1	+1%	0.329534	0.000065594	0.690
γ_1	–1%	0.303283	0.0000646995	0.683

6. CONCLUSION

Ramp-stress ALT is characterized by stress that increases linearly with time. This paper deals with design of optimal ramp-stress ALT plan for a parallel system comprising two independent components each with Weibull lifetime. The plan is devised using masked data wherein an observation is said to be masked if the exact component that caused the system to fail is not known due to time and cost constraints. However, it can be ascertained that the cause of system failure is due to the component which belongs to some subset of the system's components. The experiment is terminated when all the test specimens put to test fail or until a pre-specified time whichever is early, i.e., the test is conducted using time-censoring (Type-I censoring) scheme. The optimal plan consists in finding optimal stress rate using D-optimality criterion that is based on minimization of

reciprocal of the determinant of Fisher information matrix. The method develop has been explained using a numerical example and sensitivity analysis carried out. The results of sensitivity analysis show that optimum plan is robust for small deviations in the true values of the model parameters.

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APPENDIX

Calculations of derivatives of the log-likelihood and the elements of Fisher information matrix given in section 3.4 respectively, has been shown following.
The first partial derivatives are,

$$\partial L/\partial\beta_1 = \sum_{i=1}^{n_1}\frac{1}{\beta_{11}} + (-1+(\frac{t_i}{\alpha_1})^{-\beta_{11}})\mathrm{Log}(\frac{t_i}{\alpha_1}) + \sum_{i=1}^{n_2}(\frac{t_i}{\alpha_1})^{-\beta_{11}}\mathrm{Log}(\frac{t_i}{\alpha_1}) + \sum_{i=1}^{n_{12}}\frac{(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}} + [-(-1+(\frac{t_i}{\alpha_1})^{-\beta_{11}})(\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11} + (\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12}]\mathrm{Log}(\frac{t_i}{\alpha_1})}{(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{11} + (\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12}} \tag{A.1}$$

$$\partial L/\partial\beta_2 = \sum_{i=1}^{n_1}\frac{1}{\beta_{12}} + (-1+(\frac{t_i}{\alpha_1})^{-\beta_{12}})\mathrm{Log}(\frac{t_i}{\alpha_1}) + \sum_{i=1}^{n_2}(\frac{t_i}{\alpha_1})^{-\beta_{12}}\mathrm{Log}(\frac{t_i}{\alpha_1}) + \sum_{i=1}^{n_{12}}\frac{(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}} + [-(-1+(\frac{t_i}{\alpha_1})^{-\beta_{12}})(\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12} + (\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11}]\mathrm{Log}(\frac{t_i}{\alpha_1})}{(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{12} + (\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11}} \tag{A.2}$$

$$\partial L/\partial\gamma_1 = \sum_{i=1}^{n_1}\left(m_1(t_i) + m_2(t_i)\cdot\ m_3(t_i)\right) + \sum_{i=1}^{n_2}\left(m_4(t_i) + m_5(t_i)\cdot m_6(t_i)\right) + \sum_{i=1}^{n_{12}}\left(m_7(t_i) + m_8(t_i) + m_9(t_i)\cdot m_{10}(t_i)\right) \tag{A.3}$$

$$\partial L/\partial\gamma_0 = \sum_{i=1}^{n_1}m_{11}(t_i) + \sum_{i=1}^{n_2}\left(m_{11}(t_i)\right) + \sum_{i=1}^{n_{12}}m_{13}(t_i)\cdot m_{14}(t_i) \tag{A.4}$$

The likelihood equations are obtained by setting (A.1) – (A.4) to zero. The parameter values that solve "these equations summed over all test units" are the Maximum Likelihood estimates. As the system of likelihood equations has no closed form solution in $\gamma_0, \gamma_1, \beta_1$ and β_2, therefore the maximum likelihood estimates $\gamma_0, \gamma_1, \beta_1$ and β_2 are obtained by maximizing (17) using NMaximize option of Mathematica 10.

$$\partial^2 L/\partial\beta_1^2 = \left(\sum_{i=1}^{n_1}m_{15}(t_i) + \sum_{i=1}^{n_2}m_{16}(t_i) + \sum_{i=1}^{n_{12}}\frac{m_{17}(t_i) - (m_{18}(t_i))}{(m_{19}(t_i))^2}\right)(1+\gamma_1)^2, \tag{A.5}$$

$$\partial^2 L/\partial\beta_2^2 = \sum_{i=1}^{n_1} -\frac{1}{\beta_{12}} - (\frac{t_i}{\alpha_1})^{-\beta_{12}} \mathrm{Log}(\frac{t_i}{\alpha_1})^2 \quad + \quad \sum_{i=1}^{n_2}\left(-(\frac{t_i}{\alpha_1})^{-\beta_{12}} \mathrm{Log}(\frac{t_i}{\alpha_1})^2\right)$$
$$+\sum_{i=1}^{n_{12}} \frac{(\frac{t_i}{\alpha_1})^{-\beta_{12}}(-(\frac{t_i}{\alpha_1})^{\beta_{12}+2\beta_{11}} - 2(\frac{t_i}{\alpha_1})^{2\beta_{12}+\beta_{11}}\beta_{11}\mathrm{Log}(\frac{t_i}{\alpha_1})\mathrm{Log}(\frac{t_i}{\alpha_1})^2 - ((\frac{t_i}{\alpha_1})^{2\beta_{12}}\beta_{12}{}^2}{((\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{12} + (\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11})^2} \tag{A.6}$$
$$+\sum_{i=1}^{n_{12}} \frac{-(-2+(\frac{t_i}{\alpha_1})^{\beta_{12}})(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{11}\beta_{12} + (\frac{t_i}{\alpha_1})^{2\beta_{12}}\beta_{11}^2)\mathrm{Log}(\frac{t_i}{\alpha_1})^2}{((\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{12} + (\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11})^2} \quad ,$$

$$\partial^2 L/\partial\gamma_1^2 = \sum_{i=1}^{n_1}\left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t\alpha_1^{-1})(-1+\beta_{11}\mathrm{Log}(t_i\alpha_1^{-1}))\right)$$
$$+\sum_{i=1}^{n_2}\left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})(-1+(t_i\alpha_1^{-1}))^{\beta_{12}} + \beta_{12}\mathrm{Log}(t_i\alpha_1^{-1})\right)$$

$$+\sum_{i=1}^{n_{12}} \frac{(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1}))\cdot(-\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}$$
$$+\sum_{i=1}^{n_{12}} \frac{(1+(t_i\alpha_1^{-1})^{\beta_{11}}) + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}}) + 2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}})\beta_{12}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}$$
$$+\sum_{i=1}^{n_{12}} \frac{\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} * (1+(t_i\alpha_1^{-1})^{\beta_{11}}) - \beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{12}})\mathrm{Log}(t_i\alpha_1^{-1})}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}, \tag{A.7}$$

$$\partial^2 L/\partial\gamma_0^2 = \sum_{i=1}^{n_1}(m_{24}(t_i)(m_{25}(t_i))^2 + (\frac{\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}}{\alpha_1})\cdot g(t) +$$
$$+ \sum_{i=1}^{n_2}(m_{26}(t_i)((e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(m_{25}(t_i))^2 + (\frac{\beta_{12} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}}{\alpha_1})\cdot g(t) \tag{A.8}$$
$$+\sum_{i=1}^{n_{12}}(m_{27}(t_i)((\frac{t_i}{\alpha_1})^{-\beta_{11}-\beta_{12}}(m_{28}(t_i) + \beta_{11}\beta_{12}^2(m_{29}(t_i)) - m_{31}(t_i)m_{25}(t_i) + m_{32}(t_i),$$

$$\partial^2 L/\partial\beta_2\gamma_0 = \sum_{i=1}^{n_1}\Big((t_i\alpha_1^{-1})^{-\beta_{12}}(-1+\beta_{12}\mathrm{Log}(t_i\alpha_1^{-1})\Big) + \sum_{i=1}^{n_2}\Big((t_i\alpha_1^{-1})^{-\beta_{12}}(-1+t_i\alpha_1^{-1})^{\beta_{12}} + \beta_{12}\mathrm{Log}(t_i\alpha_1^{-1})\Big)$$
$$+\sum_{i=1}^{n_{12}}(t_i\alpha_1^{-1})^{-\beta_{12}}(\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}(1+(t_i\alpha_1^{-1})^{\beta_{11}}) + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}}) + 2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}$$
$$+\sum_{i=1}^{n_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}}) + \beta_{12}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}} + \beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} * (1+(t_i\alpha_1^{-1})^{\beta_{11}}) - \beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}$$
$$+\sum_{i=1}^{n_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{12}}))\mathrm{Log}(t_i\alpha_1^{-1})/(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2 \quad , \tag{A.9}$$

$$\partial^2 L/\partial\beta_1\gamma_0 = \sum_{i=1}^{n_1}\Big((t_i\alpha_1^{-1})^{-\beta_{11}}(-1+\beta_{11}\mathrm{Log}(t_i\alpha_1^{-1})\Big) + \sum_{i=1}^{n_2}\Big((t_i\alpha_1^{-1})^{-\beta_{11}}(-1+t_i\alpha_1^{-1})^{\beta_{11}} + \beta_{11}\mathrm{Log}(t_i\alpha_1^{-1})\Big)$$
$$+\sum_{i=1}^{n_{12}}(t_i\alpha_1^{-1})^{-\beta_{11}}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(1+(t_i\alpha_1^{-1})^{\beta_{12}}) + \beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{11}})$$
$$+\sum_{i=1}^{n_{12}}\frac{(t_i\alpha_1^{-1})^{-\beta_{11}}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(1+(t_i\alpha_1^{-1})^{\beta_{12}}) + \beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{11}})}{(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}$$
$$+\sum_{i=1}^{n_{12}}\frac{2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{11}})\mathrm{Log}(t_i\alpha_1^{-1})}{(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}$$
$$+\sum_{i=1}^{n_{12}}\frac{\beta_{11}(\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(1+(t_i\alpha_1^{-1})^{\beta_{12}}) - \beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{11}}))\mathrm{Log}(t_i\alpha_1^{-1})}{(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}, \tag{A.10}$$

$$\partial^2 L/\partial\beta_2\gamma_1 = \sum_{i=1}^{n_1}\left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t\alpha_1^{-1})(-1+\beta_{11}\mathrm{Log}(t_i\alpha_1^{-1}))\right)$$
$$+\sum_{i=1}^{n_2}\left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})(-1+(t_i\alpha_1^{-1}))^{\beta_{12}} + \beta_{12}\mathrm{Log}(t_i\alpha_1^{-1})\right)$$

$$+\sum_{i=1}^{n_{12}}\frac{(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1}))\cdot(-\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}$$
$$+\sum_{i=1}^{n_{12}}\frac{(1+(t_i\alpha_1^{-1})^{\beta_{11}}) + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}}) + 2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}})\beta_{12}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}$$
$$+\sum_{i=1}^{n_{12}}\frac{\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} * (1+(t_i\alpha_1^{-1})^{\beta_{11}}) - \beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{12}})\mathrm{Log}(t_i\alpha_1^{-1})}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}, \tag{A.11}$$

$$\partial^2 L/\partial\beta_1\gamma_1 = \sum_{i=1}^{n_1}\left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{11}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})(-1+\beta_{12}\mathrm{Log}(t_i\alpha_1^{-1}))\right)$$
$$+\sum_{i=1}^{n_2}\left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{11}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})(-1+(t_i\alpha_1^{-1}))^{\beta_{11}} + \beta_{11}\mathrm{Log}(t_i\alpha_1^{-1})\right)$$
$$+\sum_{i=1}^{n_{12}}\frac{(t_i\alpha_1^{-1})^{\beta_{11}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1}))}{((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}$$
$$+\sum_{i=1}^{n_{12}}\frac{2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{11}})\beta_{11}(\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}} * (1+(t_i\alpha_1^{-1})^{\beta_{12}})}{((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}$$
$$-\sum_{i=1}^{n_{12}}\frac{\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{11}})\mathrm{Log}(t_i t\alpha_1^{-1}) / ((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}{((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}, \tag{A.12}$$

$$\partial^2 L/\partial\gamma_0\gamma_1 = \sum_{i=1}^{n_1}\left(\begin{array}{l}\frac{-1}{1+\gamma_1}(t_i\alpha_1^{-1})^{-(\beta_{11}+\beta_{12})}(\beta_1^2(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2^2(t\alpha_1^{-1})^{\beta_{11}})(1+(1+\gamma_1)\log(\frac{S_0}{k}) \\ -\log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})\end{array}\right)$$
$$+\sum_{i=1}^{n_2}\left(\begin{array}{l}\frac{-1}{1+\gamma_1}(t_i\alpha_1^{-1})^{-(\beta_{11}+\beta_{12})}(\beta_1^2(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2^2(t_i\alpha_1^{-1})^{\beta_{11}})(1+(1+\gamma_1)\log(\frac{S_0}{k}) \\ -\log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})\end{array}\right) \tag{A.13}$$
$$+\sum_{i=1}^{n_{12}}\left(m_{20}(t_i)(m_{21}(t_i)(m_{22}(t_i)m_{23}(t_i))) / ((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2)\right),$$

$$\partial^2 L/\partial\beta_1\beta_2 = \sum_{i=1}^{n_{12}}\frac{(-(\frac{t_i}{\alpha_1})^{\beta_{11}+2\beta_{12}}(-1+\beta_{11}(\frac{t_i}{\alpha_1})^{\beta_{11}})(-1+\beta_{12}(\frac{t_i}{\alpha_1})^{\beta_{12}}))}{(\beta_1(\frac{t_i}{\alpha_1})^{\beta_{12}} + \beta_2(\frac{t_i}{\alpha_1})^{\beta_{11}})^2} \tag{A.14}$$

The elements of the Fisher information matrix for an observation are the negative expectations of these second partial derivatives:

$$E[\partial^2 L/\partial\beta_1^2] = \left(E[n_1]\int_0^\infty m_{15}(t_i) + E[n_2]\int_0^\infty m_{16}(t_i) + E[n_{12}]\int_0^\infty \frac{m_{17}(t_i) - (m_{18}(t_i))}{(m_{19}(t_i))^2}\right)(1+\gamma_1)^2, \tag{A.15}$$

$$E[\partial^2 L/\partial\beta_2^2] = E[n_1]\int_0^\infty -\frac{1}{\beta_{12}} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\,\mathrm{Log}(\frac{t_i}{\alpha_1})^2 + \quad E[n_2]\int_0^\infty\left(-(\frac{t_i}{\alpha_1})^{-\beta_{12}}\,\mathrm{Log}(\frac{t_i}{\alpha_1})^2\right)$$

$$E[n_{12}]\int_0^\infty \frac{(\frac{t_i}{\alpha_1})^{-\beta_{12}}(-(\frac{t_i}{\alpha_1})^{\beta_{12}+2\beta_{11}} - 2(\frac{t_i}{\alpha_1})^{2\beta_{12}+\beta_{11}}\beta_{11}\mathrm{Log}(\frac{t_i}{\alpha_1})\mathrm{Log}(\frac{t_i}{\alpha_1})^2 - ((\frac{t_i}{\alpha_1})^{2\beta_{12}}\beta_{12}{}^2}{((\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{12} + (\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11})^2} \quad \text{(A.16)}$$

$$E[n_{12}]\int_0^\infty \frac{-(-2+(\frac{t_i}{\alpha_1})^{\beta_{12}})(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{11}\beta_{12} + (\frac{t_i}{\alpha_1})^{2\beta_{12}}\beta_{11}^2)\mathrm{Log}(\frac{t_i}{\alpha_1})^2}{((\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{12} + (\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11})^2},$$

$$E[\partial^2 L/\partial\gamma_0^2] = E[n_1]\int_0^\infty \sum_{i=1}^{n_1}(m_{24}(t_i)(m_{25}(t_i))^2 + (\frac{\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}}{\alpha_1})\cdot g(t)dt_i$$

$$+ \quad E[n_2]\int_0^\infty (m_{26}(t_i)e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(m_{25}(t_i))^2 + (\frac{\beta_{12} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}}{\alpha_1})\cdot g(t)dt_i$$

$$+ E[n_{12}]\int_0^\infty \sum_{i=1}^{n_{12}}(m_{27}(t_i)((\frac{t_i}{\alpha_1})^{-\beta_{11}-\beta_{12}}(m_{28}(t_i) + \beta_{11}\beta_{12}^2(m_{29}(t_i)) - m_{31}(t_i)m_{25}(t_i) + m_{32}(t_i)dt_i,$$

(A.17)

$$E[\partial^2 L/\partial\beta_2\gamma_0] = E[n_1]\int_0^\infty (t_i\alpha_1^{-1})^{-\beta_{12}}(-1+\beta_{12}\mathrm{Log}(t_i\alpha_1^{-1}) + E[n_2]\int_0^\infty\left((t_i\alpha_1^{-1})^{-\beta_{12}}(-1+t_i\alpha_1^{-1})^{\beta_{12}} + \beta_{12}\mathrm{Log}(t_i\alpha_1^{-1})\right)$$

$$+ E[n_{12}]\int_0^\infty t_i\alpha_1^{-1})^{-\beta_{12}}(\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}(1+(t_i\alpha_1^{-1})^{\beta_{11}}) + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}}) + 2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}$$

$$+ E[n_{12}]\int_0^\infty (-1+(t_i\alpha_1^{-1})^{\beta_{12}}) + \beta_{12}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}} + \beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} * (1+(t_i\alpha_1^{-1})^{\beta_{11}}) - \beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}$$

$$+ E[n_{12}]\int_0^\infty (-2+(t_i\alpha_1^{-1})^{\beta_{12}}))\mathrm{Log}(t_i\alpha_1^{-1}) / (\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2,$$

(A.18)

$$E[\partial^2 L/\partial\beta_1\gamma_0] = E[n_1]\int_0^\infty\left((t_i\alpha_1^{-1})^{-\beta_{11}}(-1+\beta_{11}\mathrm{Log}(t_i\alpha_1^{-1})\right) + E[n_2]\int_0^\infty\sum_{i=1}^{n_2}\left((t_i\alpha_1^{-1})^{-\beta_{11}}(-1+t_i\alpha_1^{-1})^{\beta_{11}} + \beta_{11}\mathrm{Log}(t_i\alpha_1^{-1})\right)$$

$$+ E[n_{12}]\int_0^\infty (t_i\alpha_1^{-1})^{-\beta_{11}}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(1+(t_i\alpha_1^{-1})^{\beta_{12}}) + \beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{11}})$$

$$+ E[n_{12}]\int_0^\infty \frac{(t_i\alpha_1^{-1})^{-\beta_{11}}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(1+(t_i\alpha_1^{-1})^{\beta_{12}}) + \beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{11}})}{(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}$$

$$+ E[n_{12}]\int_0^\infty \frac{2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{11}})\mathrm{Log}(t_i\alpha_1^{-1})}{(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}$$

$$+ E[n_{12}]\int_0^\infty \frac{\beta_{11}(\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(1+(t_i\alpha_1^{-1})^{\beta_{12}}) - \beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{11}}))\mathrm{Log}(t_i\alpha_1^{-1})}{(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2},$$

(A.19)

$$
\begin{aligned}
E[\partial^2 L/\partial\beta_2\gamma_1] = {} & E[n_1]\int_0^\infty \left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t\alpha_1^{-1})(-1+\beta_{11}\mathrm{Log}(t_i\alpha_1^{-1}))\right)dt_i \\
& + E[n_2]\int_0^\infty \left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})(-1+(t_i\alpha_1^{-1}))^{\beta_{12}} + \beta_{12}\mathrm{Log}(t_i\alpha_1^{-1})\right)dt_i \\
& + E[n_{12}]\int_0^\infty \frac{(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1}))\cdot(-\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}dt_i \\
& + E[n_{12}]\int_0^\infty \frac{(1+(t_i\alpha_1^{-1})^{\beta_{11}}) + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}}) + 2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}})\beta_{12}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}dt_i \\
& + E[n_{12}]\int_0^\infty \frac{\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} * (1+(t_i\alpha_1^{-1})^{\beta_{11}}) - \beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{12}})\mathrm{Log}(t_i\alpha_1^{-1})}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}} + \beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}dt_i,
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
E[\partial^2 L/\partial\beta_1\gamma_1] = {} & E[n_1]\int_0^\infty \left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{11}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})(-1+\beta_{12}\mathrm{Log}(t_i\alpha_1^{-1}))\right)dt_i \\
& + E[n_2]\int_0^\infty \left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{11}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1})(-1+(t_i\alpha_1^{-1}))^{\beta_{11}} + \beta_{11}\mathrm{Log}(t_i\alpha_1^{-1})\right)dt_i \\
& + E[n_{12}]\int_0^\infty \frac{(t_i\alpha_1^{-1})^{\beta_{11}}(1+(1+\gamma_1)\log(\frac{S_0}{k}) - \log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1}))}{((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}dt_i \\
& + E[n_{12}]\int_0^\infty \frac{2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{11}})\beta_{11}(\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}} + \beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}} * (1+(t_i\alpha_1^{-1})^{\beta_{12}})}{((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}dt_i \\
& - E[n_{12}]\int_0^\infty \frac{\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{11}})\mathrm{Log}(t_i t\alpha_1^{-1}) / ((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}{((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2}dt_i,
\end{aligned}
\tag{A.21}
$$

$$
\begin{aligned}
E[\partial^2 L/\partial\gamma_0\gamma_1] = {} & E[n_1]\int_0^\infty \begin{pmatrix} \frac{-1}{1+\gamma_1}(t_i\alpha_1^{-1})^{-(\beta_{11}+\beta_{12})}(\beta_1^2(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2^2(t\alpha_1^{-1})^{\beta_{11}})(1+(1+\gamma_1)\log(\frac{S_0}{k}) \\ -\log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1}) \end{pmatrix}dt_i \\
& + E[n_2]\int_0^\infty \begin{pmatrix} \frac{-1}{1+\gamma_1}(t_i\alpha_1^{-1})^{-(\beta_{11}+\beta_{12})}(\beta_1^2(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2^2(t_i\alpha_1^{-1})^{\beta_{11}})(1+(1+\gamma_1)\log(\frac{S_0}{k}) \\ -\log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1)) - \log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1}) \end{pmatrix}dt_i \\
& + E[n_{12}]\int_0^\infty \left(m_{20}(t_i)(m_{21}(t_i)(m_{22}(t_i)m_{23}(t_i))) / ((1+\gamma_1)(\beta_1(t_i\alpha_1^{-1})^{\beta_{12}} + \beta_2(t_i\alpha_1^{-1})^{\beta_{11}})^2)\right)dt_i,
\end{aligned}
\tag{A.22}
$$

$$E[\partial^2 L/\partial\beta_1\beta_2] = E[n_{12}]\int_0^\infty \frac{(-(\frac{t_i}{\alpha_1})^{\beta_{11}+2\beta_{12}}(-1+\beta_{11}(\frac{t_i}{\alpha_1})^{\beta_{11}})(-1+\beta_{12}(\frac{t_i}{\alpha_1})^{\beta_{12}}))}{(\beta_1(\frac{t_i}{\alpha_1})^{\beta_{12}}+\beta_2(\frac{t_i}{\alpha_1})^{\beta_{11}})^2}dt_i \quad , \quad (A.23)$$

$$\begin{aligned}
E[\partial^2 L/\partial\gamma_1^2] = {} & E[n_1]\int_0^\infty\left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k})-\log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1))-\log(t_i\alpha_1^{-1})-\gamma_1\log(t\alpha_1^{-1})(-1+\beta_{11}\mathrm{Log}(t_i\alpha_1^{-1}))\right)dt_i \\
& + E[n_2]\int_0^\infty\left(\frac{1}{1+\gamma_1}(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k})-\log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1))-\log(t_i\alpha_1^{-1})-\gamma_1\log(t_i\alpha_1^{-1})(-1+(t_i\alpha_1^{-1}))^{\beta_{12}}+\beta_{12}\mathrm{Log}(t_i\alpha_1^{-1})\right)dt_i \\
& + E[n_{12}]\int_0^\infty \frac{(t_i\alpha_1^{-1})^{\beta_{12}}(1+(1+\gamma_1)\log(\frac{S_0}{k})-\log(e^{\gamma_0}(\frac{S_0}{k})^{\gamma_1}(1+\gamma_1))-\log(t_i\alpha_1^{-1})-\gamma_1\log(t_i\alpha_1^{-1}))\cdot(-\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}}+\beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}dt_i \\
& + E[n_{12}]\int_0^\infty \frac{(1+(t_i\alpha_1^{-1})^{\beta_{11}})+\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}})+2\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-1+(t_i\alpha_1^{-1})^{\beta_{12}})\beta_{12}(\beta_2^2(t_i\alpha_1^{-1})^{2\beta_{11}}}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}}+\beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}dt_i \\
& + E[n_{12}]\int_0^\infty \frac{\beta_1^2(t_i\alpha_1^{-1})^{2\beta_{12}}*(1+(t_i\alpha_1^{-1})^{\beta_{11}})-\beta_1\beta_2(t_i\alpha_1^{-1})^{\beta_{11}+\beta_{12}}(-2+(t_i\alpha_1^{-1})^{\beta_{12}})\mathrm{Log}(t_i\alpha_1^{-1})}{((1+\gamma_1)(\beta_2(t_i\alpha_1^{-1})^{\beta_{11}}+\beta_1(t_i\alpha_1^{-1})^{\beta_{12}})^2}dt_i,
\end{aligned}$$

(A.24)

$$\text{where}: m_1(t_i) = (\frac{1}{\beta_{11}} + (-1 + (\frac{t_i}{\alpha_1})^{-\beta_{11}})\text{Log}(\frac{t_i}{\alpha_1}))\beta_1 + ((\frac{t_i}{\alpha_1})^{-\beta_{12}}\text{Log}(\frac{t_i}{\alpha_1}))\beta_2,$$

$$m_2(t_i) = (\frac{\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}}{\alpha_1}) \times (e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{\frac{1}{(1+\gamma_1)}}$$

$$m_3(t_i) = (1+(1+\gamma_1)\text{Log}(\frac{s_0}{k}) + \text{Log}(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))),$$

$$m_4(t_i) = ((\frac{t_i}{\alpha_1})^{-\beta_{11}}\text{Log}(\frac{t_i}{\alpha_1}))\beta_1 + (\frac{1}{\beta_{12}} + (-1 + (\frac{t_i}{\alpha_1})^{-\beta_{12}})\text{Log}(\frac{t_i}{\alpha_1}))\beta_2,$$

$$m_5(t_i) = (\frac{\beta_{12} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}}{\alpha_1})(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{\frac{1}{(1+\gamma_1)}},$$

$$m_6(t_i) = (1+(1+\gamma_1)\text{Log}(\frac{s_0}{k}) + \text{Log}(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))),$$

$$m_7(t_i) = \frac{(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}} + (-(-1 + (\frac{t_i}{\alpha_1})^{-\beta_{11}})(\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11} + (\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12})\text{Log}(\frac{t_i}{\alpha_1})}{(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{11} + (\frac{t_i}{\alpha_1})^{2\beta_{11}}\beta_{12}}\beta_1,$$

$$m_8(t_i) = \frac{(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}} + (-(-1 + (\frac{t_i}{\alpha_1})^{-\beta_{12}})(\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12} + (\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11})\text{Log}(\frac{t_i}{\alpha_1})}{(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{12} + (\frac{t_i}{\alpha_1})^{2\beta_{12}}\beta_{11}}\beta_2,$$

$$m_9(t_i) = \frac{(-1 + (\frac{t_i}{\alpha_1})^{-\beta_{11}})(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{11}^2 - 2\beta_{11}\beta_{12} + (-1 + (\frac{t_i}{\alpha_1})^{-\beta_{12}})(\frac{t_i}{\alpha_1})^{\beta_{11}-\beta_{12}}\beta_{12}^2}{\alpha_1(\frac{t}{\alpha_1})^{\beta_{12}}\beta_{11} + (\frac{t}{\alpha_1})^{\beta_{11}}\beta_{12}},$$

$$m_{10}(t_i) = (e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{\frac{1}{(1+\gamma_1)}} \cdot (1+(1+\gamma_1)\text{Log}(\frac{s_0}{k}) + \text{Log}(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1)))$$

$$m_{11}(t_i) = (\frac{\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}}{\alpha_1})(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{-1+\frac{1}{1+\gamma_1}},$$

$$m_{12}(t_i) = (\frac{\beta_{12} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}}{\alpha_1})(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{-1+\frac{1}{1+\gamma_1}},$$

$$m_{13}(t_i) = (\frac{(-1 + (\frac{t_i}{\alpha_1})^{-\beta_{11}})(\frac{t_i}{\alpha_1})^{-\beta_{11}+\beta_{12}}\beta_{11}^2 - 2\beta_{11}\beta_{12} + (-1 + (\frac{t_i}{\alpha_1})^{\beta_{12}})(\frac{t_i}{\alpha_1})^{\beta_{11}-\beta_{12}}\beta_{12}^2}{\alpha_1((\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11} + (\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12})}),$$

$$m_{14}(t_i) = (e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{-1+\frac{1}{1+\gamma_1}}, m_{15}(t_i) = -\frac{1}{\beta_{11}} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\text{Log}(\frac{t_i}{\alpha_1})^2,$$

$$m_{16}(t_i) = -(\frac{t_i}{\alpha_1})^{-\beta_{11}}\text{Log}(\frac{t_i}{\alpha_1})^2,$$

$$m_{17}(t_i) = (\frac{t_i}{\alpha_1})^{-\beta_{11}}(-(\frac{t_i}{\alpha_1})^{\beta_{11}+2\beta_{12}} - 2(\frac{t_i}{\alpha_1})^{2\beta_{11}+\beta_{12}}\beta_{12}\mathrm{Log}(\frac{t_i}{\alpha_1}),$$

$$m_{18}(t_i) = (\frac{t_i}{\alpha_1})^{2\beta_{11}}\beta_{11}{}^2 - (-2 + (\frac{t_i}{\alpha_1})^{\beta_{11}})(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{11}\beta_{12} + (\frac{t_i}{\alpha_1})^{2\beta_{11}}\beta_{12}^2)\mathrm{Log}(\frac{t_i}{\alpha_1})^2,$$

$$m_{19}(t_i) = (\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{11} + (\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12},$$

$$m_{20}(t_i) = (t_i\alpha_1^{-1})^{-(\beta_{11}+\beta_{12})}, m_{21}(t_i) = \beta_2^4(t_i\alpha_1^{-1})^{3\beta_{11}} + \beta_1^4(t_i\alpha_1^{-1})^{3\beta_{12}} + \beta_1^2\beta_2^2,$$

$$m_{22}(t_i) = 2(t_i\alpha_1^{-1})^{2(\beta_{11}+\beta_{12})} + (t_i\alpha_1^{-1})^{2(\beta_{11}+\beta_{12})} + (t_i\alpha_1^{-1})^{(\beta_{11}+2\beta_{12})} - \beta_1^3\beta_2(t_i\alpha_1^{-1})^{(\beta_{11}+2\beta_{12})},$$

$$m_{23}(t_i) = (-2 + (t_i\alpha_1^{-1})^{\beta_{11}}) - \beta_2^3\beta_1(t_i\alpha_1^{-1})^{(\beta_{12}+2\beta_{11})}(-2 + (t_i\alpha_1^{-1})^{\beta_{12}}))(1 + (1+\gamma_1)\log(\frac{s_0}{k}) - \log(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))$$
$$-\log(t_i\alpha_1^{-1}) - \gamma_1\log(t_i\alpha_1^{-1}),$$

$$m_{24}(t_i) = \frac{\beta_{11} - (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} + (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11}^2 + (\frac{t_i}{\alpha_1})^{-\beta_{12}}(-1+\beta_{12})\beta_{12}}{\alpha_1^2},$$

$$m_{25}(t_i) = (e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(e^{\gamma_0}(\frac{s_0}{k})^{\gamma_1}(1+\gamma_1))^{-1+\frac{1}{1+\gamma_1}},$$

$$m_{26}(t_i) = \frac{-(\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11} + (\frac{t_i}{\alpha_1})^{-\beta_{11}}\beta_{11}^2 + (\frac{t_i}{\alpha_1})^{-\beta_{12}}\beta_{12}(-1 + (\frac{t_i}{\alpha_1})^{\beta_{12}} + \beta_{12})\beta_{12}}{\alpha_1^2},$$

$$m_{27}(t_i) = \frac{1}{\alpha_1^2((\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11} + (\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12})^2}((\frac{t_i}{\alpha_1})^{-\beta_{11}-\beta_{12}},$$

$$m_{28}(t_i) = (\frac{t_i}{\alpha_1})^{-3\beta_{12}}\beta_{11}^4 - (\frac{t_i}{\alpha_1})^{3\beta_{11}}\beta_{12}^3(-1 + (\frac{t_i}{\alpha_1})^{\beta_{12}} + \beta_{12}), m_{29}(t_i) = 3 - (\frac{t_i}{\alpha_1})^{\beta_{12}} + (-2 + (\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{12},$$

$$m_{30}(t_i) = (3 - (\frac{t_i}{\alpha_1})^{\beta_{12}})(\frac{t_i}{\alpha_1})^{\beta_{12}} + ((\frac{t_i}{\alpha_1})^{\beta_{11}} + (\frac{t_i}{\alpha_1})^{\beta_{12}} + 2(\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}})\beta_{12},$$

$$m_{31}(t_i) = (\frac{t_i}{\alpha_1})^{\beta_{11}+\beta_{12}}\beta_{12}\beta_{11}^2((+(\frac{t_i}{\alpha_1})^{2\beta_{12}}\beta_{11}^3(-(-1 + (\frac{t_i}{\alpha_1})^{\beta_{11}})(\frac{t_i}{\alpha_1})^{\beta_{12}}(-2 + (\frac{t_i}{\alpha_1})^{\beta_{11}})(\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12})))),$$

$$m_{32}(t_i) = (\frac{(-1 + (\frac{t_i}{\alpha_1})^{-\beta_{11}})(\frac{t_i}{\alpha_1})^{-\beta_{11}+\beta_{12}}\beta_{11}^2 - 2\beta_{11}\beta_{12} + (-1 + (\frac{t_i}{\alpha_1})^{\beta_{12}})(\frac{t_i}{\alpha_1})^{\beta_{11}-\beta_{12}}\beta_{12}^2}{\alpha_1((\frac{t_i}{\alpha_1})^{\beta_{12}}\beta_{11} + (\frac{t_i}{\alpha_1})^{\beta_{11}}\beta_{12})})(g(t)),$$