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OPTIMAL INVERSION OF THE NOISY RADON TRANSFORM ON CLASSES DEFINED BY A DEGREE OF THE LAPLACE OPERATOR

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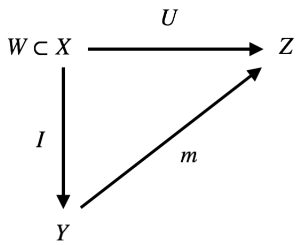
ABSTRACT. A general optimal recovery problem is to approximate a value of a linear operator on a subset (class) in linear space from a value of another linear operator (called information), measured with an error in given metric. We use this formulation to investigate the classical computerized tomography problem of inversion of the noisy Radon transform.

1. INTRODUCTION

In many applied and theoretical problems one needs to recover a function (functional or operator) from the information, which can be incomplete or given with an error. Such problems are investigated in optimal recovery theory - a modern branch of approximation theory, which has its roots in works of A.N. Kolmogorov and notion of Kolmogorov widths. The problem of the optimal recovery first appeared in [1] and has been developed in [2–4] and lately in [5–7] and many other works. The general problem is to find the best approximation of a linear operator $U : X \to Z$ value on a given set $W \subset X$ from values of another linear operator $I : X \to Y$ (called information) given with an error.



On the diagram above X is a linear space and Y,Z are normed linear spaces. W is a subset of X, U is a linear operator from X into Z (the *feature* operator) and I is a linear operator from X into Y (the *information* operator). An arbitrary mapping $m : Y \to Z$ is called a *method of*

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recovery. Suppose, that instead of the element Ix we are given its approximation $y \in Y$ with an error δ, i.e. $\|Ix - y\| \le \delta$. Each method of recovery m produces an *error of the method*

$$e(\delta, m) = \sup\{\|Ux - my\| : x \in W, \|Ix - y\| \le \delta\}$$

and the exact lower bound of these errors

$$E(\delta) = \inf_m e(\delta, m)$$

is called *the error of the optimal recovery*. The method of recovery m is *optimal* if the error of the optimal recovery $E(\delta)$ is attained by the error $e(\delta, m)$ of m, i.e. $e(\delta, m) = E(\delta)$. Particular cases usually use information in a form of a linear functional or an operator that maps function to the set of it's values in a number of nodes, it's Fourier transform, Fourier coefficients or the function itself. This covers a large set of numerical methods, such as quadrature formulas [8,9], optimal interpolation and approximation [5, 6], differential equations solutions recovery, signal reconstruction [7], optimal state estimation and others. In the present paper we consider the widely used in the computerized tomography theory Radon transform - an operator, that maps a function on R^d to the set of it's integrals over all hyperplanes. For the particular classes of functions there exist different inversion formulas that allow to produce an exact reconstruction (see [10]). Following the optimal recovery approach we consider the case when the Radon transform is measured inaccurately, with a known error δ in the mean square metric. In the optimal recovery theory operators of this kind have been introduced in [11] (example 3.2), where for a function on R^2 the information is the Radon transform being measured in a finite number of directions, and in papers [12, 13] where the radial integration operator is considered on the classes of analytic and harmonic functions. In paper [14] we have considered the optimal recovery of the Radon transform acting in the Hardy space of harmonic functions in a unit ball. Paper [15] has generalized this problem to the Laplacian degree recovery on classes defined by another degree of the same operator from inaccurately measured k-plane transform. Here we investigate the particular case of this problem where information is given by the Radon transform ($k = d - 1$) and we look to reconstruct the function itself. We present explicit formulas for the optimal methods of recovery and complete proofs of the main results.

2. Main results

We define the Radon transform in a standard way as an operator of integration along the hyperplanes in space

$$Rf(\theta, s) = \int_{x\theta = s} f(x)dx,$$

where $s \in \mathbb{R}$, $\theta \in \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d$. The Hilbert space $L_2(Z)$ of square integrable functions on cylinder $Z = \mathbb{R} \times \mathbb{S}^{d-1}$ is produced by a scalar product $(g, h)_{L_2(Z)} = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} g(\theta, s)\bar{h}(\theta, s) ds d\theta$. The Fourier transfrom is defined by

$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx$$

and when applied to $g(\theta, s)$ (we use notation $g_\theta(s) = g(\theta, s)$) it acts on the second variable. The Radon transform and the Fourier transform are related by the so-called projection-slice theorem [10].

Theorem 2.1. *If $f \in L_1(\mathbb{R}^d)$, then*

$$\widehat{(R_\theta f)}(\sigma) = (2\pi)^{d-1/2}\widehat{f}(\sigma\theta), \quad \sigma \in \mathbb{R}, \quad R_\theta f(s) = Rf(\theta, s).$$

For $\alpha > 0$ define the degree of the Laplace operator by formula

$$\widehat{\big((-\Delta)^{\alpha/2} f\big)}(\xi) = |\xi|^\alpha \widehat{f}(\xi)$$

on the set of functions $f \in L_2(\mathbb{R}^d)$ that satisfy the condition $|\xi|^\alpha \widehat{f}(\xi) \in L_2(\mathbb{R}^d)$. Using a shorter notation $\Lambda = (-\Delta)^{1/2}$ we define the class

$$W = \{f \in L_2(\mathbb{R}^d) : \|\Lambda^\alpha f\|_{L_2(\mathbb{R}^d)} \leqslant 1; \quad Rf \in L_2(Z)\}.$$

As we work with the inaccurate information we assume that for a function Rf we know an approximation $g \in L_2(Z)$ such that

$$\|Rf - g\|_{L_2(Z)} \leqslant \delta, \quad \delta > 0.$$

On this information we want to recover function f as an element of $L_2(\mathbb{R}^d)$. We consider all possible methods or recovery – arbitrary maps $m : L_2(Z) \to L_2(\mathbb{R}^d)$. For every method of recovery m define its error $e(\delta, m)$ by

$$e(\delta, m) = \sup_{\substack{f \in W, g \in L_2(Z) \\ \|Rf - g\|_{L_2(Z)} \leqslant \delta}} ||f - m(g)||_{L_2(\mathbb{R}^d)}$$

and the error of the optimal recovery

$$E(\delta) = \inf_{m : L_2(Z) \to L_2(\mathbb{R}^d)} e(\delta, m). \tag{2.1}$$

We're going to solve problem (2.1) both by finding the extremal value – the error of the optimal recovery, and extremal elements – the optimal methods of recovery. To formulate the main result we introduce the following definitions for functions $t(\sigma)$, $y(\sigma)$ and constants $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$:

$$t(\sigma) = (2\pi)^{1-d}\sigma^{2\alpha+d-1}\chi_{[0,\infty)}(\sigma), \quad y(\sigma) = (2\pi)^{1-d}\sigma^{d-1}\chi_{[0,\infty)}(\sigma), \quad \sigma \in \mathbb{R}, \tag{2.2}$$

$$\widetilde{\lambda}_1 = (2\pi)^{\frac{2\alpha(1-d)}{2\alpha+d-1}} \frac{d-1}{2\alpha+d-1} \delta^{\frac{4\alpha}{2\alpha+d-1}}, \quad \widetilde{\lambda}_2 = (2\pi)^{\frac{2\alpha(1-d)}{2\alpha+d-1}} \frac{2\alpha}{2\alpha+d-1} \delta^{\frac{2(1-d)}{2\alpha+d-1}}. \tag{2.3}$$

Theorem 2.2. *The error of the optimal recovery is given by*

$$E(\delta) = \sqrt{\widetilde{\lambda}_1 + \widetilde{\lambda}_2\delta^2} = (2\pi)^{\frac{\alpha(1-d)}{2\alpha+d-1}} \delta^{\frac{2\alpha}{2\alpha+d-1}}$$

and the following methods are optimal:

$$\widehat{m_a(g)}(\sigma\theta) = (2\pi)^{(1-d)/2} a(\sigma)\widehat{g_\theta}(\sigma), \quad \sigma \in [0, \infty), \quad \theta \in \mathbb{S}^{d-1}, \tag{2.4}$$

where

$$a(\sigma) = \left(\frac{\widetilde{\lambda}_2}{\widetilde{\lambda}_1 t(\sigma) + \widetilde{\lambda}_2} + \varepsilon(\sigma) \frac{\sigma^\alpha \sqrt{\widetilde{\lambda}_1 \widetilde{\lambda}_2}}{\widetilde{\lambda}_1 t(\sigma) + \widetilde{\lambda}_2} \sqrt{\widetilde{\lambda}_1 t(\sigma) + \widetilde{\lambda}_2 - y(\sigma)} \right) \chi_{[0,\infty)}(\sigma), \qquad (2.5)$$

ε is an arbitrary function satisfying $\|\varepsilon\|_{L_\infty(\mathbb{R})} \leqslant 1$.

The optimal methods are designed as a set of filters $a(\sigma)$ (for all suitable functions ε) that are applied to the Fourier transform of the information g. A reconstructed function f is then provided by application of the inverse Fourier transform. In the case of an accurate information ($\delta = 0$) the exact reconstruction method is given by the projection-slice Theorem 2.1, i.e. $a(\sigma) = 1$. Filter $a(\sigma)$ is used to suppress certain frequencies and in this sense defines the amount of useful information for the optimal recovery when the information is measured inaccurately. Particularly when $a(\sigma)$ can be chosen equal to 0 the corresponding volume of information is unnecessary as the optimal methods may not use it. On the other hand when $a(\sigma)$ can be equal to 1 the information doesn't need to be filtered. The following corollary shows that for sufficiently small σ information $\widehat{g_\theta}(\sigma)$ doesn't need to be filtered and, on the contrary, for large σ the information is useless, as it has no effect on the error of the optimal recovery.

Corollary 2.3. *In the conditions of Theorem 2.2 the following filters produce optimal methods of recovery*

$$a(\sigma) = \begin{cases} 1 & , 0 \leqslant \sigma \leqslant (2\pi)\widetilde{\lambda}_2^{\frac{1}{d-1}}, \\ \frac{\widetilde{\lambda}_2}{\widetilde{\lambda}_1 t(\sigma) + \widetilde{\lambda}_2} + \varepsilon(\sigma) \frac{\sqrt{\widetilde{\lambda}_1 \widetilde{\lambda}_2}\sigma^\alpha}{\widetilde{\lambda}_1 t(\sigma) + \widetilde{\lambda}_2} \sqrt{t(\sigma)\widetilde{\lambda}_1 + \widetilde{\lambda}_2 - y(\sigma)} & , (2\pi)\widetilde{\lambda}_2^{\frac{1}{d-1}} < \sigma < \widetilde{\lambda}_1^{\frac{-1}{2\alpha}}, \\ 0 & , \sigma \geqslant \widetilde{\lambda}_1^{\frac{-1}{2\alpha}}, \end{cases}$$

ε is an arbitrary function satisfying $\|\varepsilon\|_{L_\infty(\mathbb{R})} \leqslant 1$.

An obvious observation here is that the methods from Corollary 2.3 provide the reconstruction as a bandlimited function. Another application of Theorem 2.2 is a new inequality for the norm of a function and the norms of the Radon transform and the Laplacian degree.

Corollary 2.4. *The following exact inequality takes place for a function $f \in L_2(\mathbb{R}^d)$ such that $|\xi|^\alpha \widehat{f}(\xi) \in L_2(\mathbb{R}^d)$, $\alpha > 0$, $Rf \in L_2(Z)$:*

$$\|f\|_{L_2(\mathbb{R}^d)} \leqslant (2\pi)^{\frac{\alpha(1-d)}{2\alpha+d-1}} \|Rf\|_{L_2(Z)}^{\frac{2\alpha}{2\alpha+d-1}} \|\Lambda^\alpha f\|_{L_2(\mathbb{R}^d)}^{\frac{d-1}{2\alpha+d-1}}.$$

3. Proofs

3.1. **Proof of Theorem 2.2.** Consider the extremal problem

$$\|f\|^2_{L_2(\mathbb{R}^d)} \to \sup, \quad \|\Lambda^\alpha f\|^2_{L_2(\mathbb{R}^d)} \leqslant 1, \quad \|Rf\|^2_{L_2(Z)} \leqslant \delta^2,$$

which is usually called the dual problem to (2.1) in the optiml recovery theory. Its solution gives the lower bound for $E(\delta)$. Indeed, for an arbitrary method m:

$$
\begin{aligned}
e(\delta, m) = & \sup_{\substack{f\in W, g\in L_2(Z)\\ \|Rf-g\|_{L_2(Z)}\leqslant\delta}} \|f-m(g)\|_{L_2(\mathbb{R}^d)} \\
& \geqslant \sup_{\substack{f\in W\\ \|Rf\|_{L_2(Z)}\leqslant\delta}} \|f-m(0)\|_{L_2(\mathbb{R}^d)} \\
& \geqslant \sup_{\substack{f\in W\\ \|Rf\|_{L_2(Z)}\leqslant\delta}} \frac{\|f-m(0)\|_{L_2(\mathbb{R}^d)} + \|-f-m(0)\|_{L_2(\mathbb{R}^d)}}{2} \\
& \geqslant \sup_{\substack{f\in W\\ \|Rf\|_{L_2(Z)}\leqslant\delta}} \|f\|_{L_2(\mathbb{R}^d)}.
\end{aligned}
$$

The inequalities above are true due to the central symmetry of the set W. Hence

$$
E(\delta) \geqslant \sup_{\substack{f\in W\\ \|Rf\|_{L_2(Z)}\leqslant\delta}} \|f\|_{L_2(\mathbb{R}^d)}.
$$

We use Theorem 2.1 to transform the functional and the constraints in the dual problem as follows:

$$
\begin{aligned}
\|f\|^2_{L_2(\mathbb{R}^d)} &= \|\widehat{f}\|^2_{L_2(\mathbb{R}^d)} = \int_0^\infty \sigma^{d-1}\int_{\mathbb{S}^{d-1}} |\widehat{f}(\sigma\theta)|^2 d\theta d\sigma, \\
\|\Lambda^\alpha f\|^2_{L_2(\mathbb{R}^d)} &= \|\widehat{\Lambda^\alpha f}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2\alpha}|\widehat{f}(\xi)|^2 d\xi = \int_0^\infty \sigma^{2\alpha+d-1}\int_{\mathbb{S}^{d-1}} |\widehat{f}(\sigma\theta)|^2 d\theta d\sigma, \\
\|Rf\|^2_{L_2(Z)} &= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}} |Rf(\theta,s)|^2 ds d\theta = \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}} |\widehat{(R_\theta f)}(\sigma)|^2 d\sigma d\theta \\
&= (2\pi)^{d-1}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}} |\widehat{f}(\sigma\theta)|^2 d\sigma d\theta = (2\pi)^{d-1}\int_{\mathbb{R}}\int_{\mathbb{S}^{d-1}} |\widehat{f}(\sigma\theta)|^2 d\theta d\sigma.
\end{aligned}
$$

If we denote $\int_{\mathbb{S}^{d-1}} |\widehat{f}(\sigma\theta)|^2 d\theta d\sigma = d\mu(\sigma)$ the dual problem can be presented as

$$
\int_0^\infty \sigma^{d-1} d\mu \to \sup, \quad \int_0^\infty \sigma^{2\alpha+d-1} d\mu \leqslant 1, \quad (2\pi)^{d-1}\int_{\mathbb{R}} d\mu \leqslant \delta^2. \tag{3.1}
$$

Now we consider (3.1) to be a new extremal problem, where $d\mu(\sigma)$ is an arbitrary measure. Obviously its solution isn't less than the solution of the original dual problem. To solve the dual problem we will present the solution of (3.1) and the sequence of admissible functions, that bring the same value in the dual problem. Consider the Lagrange function of (3.1):

$$
L(d\mu, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2\delta^2 + (2\pi)^{d-1}\int_{\mathbb{R}} \Big(\lambda_1 t(\sigma) + \lambda_2 - y(\sigma)\Big) d\mu.
$$

If there exist the Lagrange multipliers $\widetilde{\lambda}_1,\widetilde{\lambda}_2 \geqslant 0$ and measure $\widetilde{d\mu}$, admissible in (3.1), that minimizes the Lagrange function, i.e.

$$\min_{d\mu\geqslant 0} L(d\mu,\widetilde{\lambda}_1,\widetilde{\lambda}_2) = L(\widetilde{d\mu},\widetilde{\lambda}_1,\widetilde{\lambda}_2),$$

and satisfies

$$\widetilde{\lambda}_1\left(\int_0^\infty \sigma^{2\alpha+d-1}\widetilde{d\mu}-1\right)+\widetilde{\lambda}_2\left((2\pi)^{d-1}\int_{\mathbb{R}}\widetilde{d\mu}-\delta^2\right)=0$$

(complementary slackness condition), then $\widetilde{d\mu}$ brings maximum to (3.1). That statement easily follows from

$$-\int_0^\infty \sigma^{d-1}\widetilde{d\mu} = L(\widetilde{d\mu},\widetilde{\lambda}_1,\widetilde{\lambda}_2) = \min_{d\mu\geqslant 0} L(d\mu,\widetilde{\lambda}_1,\widetilde{\lambda}_2) \leqslant \min_{\substack{d\mu\geqslant 0\\ \int_0^\infty \sigma^{2\alpha+d-1}d\mu\leqslant 1\\ (2\pi)^{d-1}\int_{\mathbb{R}} d\mu\leqslant\delta^2}} -\int_0^\infty \sigma^{d-1}d\mu.$$

Where the last inequality holds due to

$$L(d\mu,\widetilde{\lambda}_1,\widetilde{\lambda}_2)\leqslant -\int_0^\infty \sigma^{d-1}d\mu \quad \text{as} \quad \widetilde{\lambda}_1,\widetilde{\lambda}_2\geqslant 0.$$

We shall present such $\widetilde{\lambda}_1,\widetilde{\lambda}_2$ and $\widetilde{d\mu}$. Equations (2.2) define function $y(t)$ by

$$y(t)=(2\pi)^{\frac{2\alpha(1-d)}{2\alpha+d-1}}t^{\frac{d-1}{2\alpha+d-1}},\quad t\geqslant 0$$

which is concave for $\alpha>0$. The equation of the tangent line to $y(t)$ at a point $1/\delta^2$ (the corresponding value of σ is $\sigma^*=[(2\pi)^{d-1}\delta^{-2}]^{1/(2\alpha+d-1)}$) is $y=\widetilde{\lambda}_1 t+\widetilde{\lambda}_2$, where $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ are defined in (2.3). Thus, we have $\widetilde{\lambda}_1 t(\sigma)+\widetilde{\lambda}_2-y(\sigma)\geqslant 0$ and $L(d\mu,\widetilde{\lambda}_1,\widetilde{\lambda}_2)\geqslant-\widetilde{\lambda}_1-\widetilde{\lambda}_2\delta^2$.

Consider a measure supported at σ^* (i.e. the δ-function at this point)

$$\widetilde{d\mu}=\frac{\delta^2}{(2\pi)^{d-1}}\delta(\sigma-\sigma^*).$$

It's admissible in (3.1), satisfies the complementary slackness condition and minimizes the Lagrange function, as $L(\widetilde{d\mu},\widetilde{\lambda}_1,\widetilde{\lambda}_2)=-\widetilde{\lambda}_1-\widetilde{\lambda}_2\delta^2$. Thus, it brings the extremum in problem (3.1), which solution is equal to $\widetilde{\lambda}_1+\widetilde{\lambda}_2\delta^2$. By a standard approximation of the δ-function it's easy to show that the solution of the dual problem is the same as in (3.1). And we obtain a lower bound for the error of the optimal recovery $E(\delta)\geqslant\sqrt{\widetilde{\lambda}_1+\widetilde{\lambda}_2\delta^2}$.

Now we show, that the error of the methods (2.4) is equal to the achieved estimate. We have

$$\begin{aligned}\|f-m_a(g)\|^2_{L_2(\mathbb{R}^d)} &= \|\widehat{f}-\widehat{m_a(g)}\|^2_{L_2(\mathbb{R}^d)}\\ &= \int_{\mathbb{S}^{d-1}}\int_0^\infty \sigma^{d-1}|\widehat{f}(\sigma\theta)-(2\pi)^{(1-d)/2}a(\sigma)\widehat{g}_\theta(\sigma)|^2 d\sigma d\theta\\ &= \int_{\mathbb{S}^{d-1}}\int_0^\infty \sigma^{d-1}|a(\sigma)(2\pi)^{\frac{1-d}{2}}\left(\widehat{g}_\theta(\sigma)-(2\pi)^{\frac{d-1}{2}}\widehat{f}(\sigma\theta)\right)\\ &+\widehat{f}(\sigma\theta)(a(\sigma)-1)|^2 d\sigma d\theta.\end{aligned}$$

Transform this expression using the Cauchy-Schwarz inequality $|xy|\leqslant|x||y|$ applied to vectors

$$x=\left((2\pi)^{\frac{1-d}{2}}\frac{a(\sigma)}{\sqrt{\widetilde{\lambda}_2}},\sigma^{\frac{1-d-2\alpha}{2}}\frac{(a(\sigma)-1)}{\sqrt{\widetilde{\lambda}_1}}\right),$$

$$y=\left(\left(\widehat{g}_\theta(\sigma)-(2\pi)^{(d-1)/2}\widehat{f}(\sigma\theta)\right)\sqrt{\widetilde{\lambda}_2},\sigma^{\frac{2\alpha+d-1}{2}}\sqrt{\widetilde{\lambda}_1}\widehat{f}(\sigma\theta)\right).$$

We obtain

$$\begin{aligned}&\|f-m_a(g)\|^2_{L_2(\mathbb{R}^d)}\\ &\quad\leqslant\int_{\mathbb{S}^{d-1}}\int_0^\infty A(\sigma)\left(\sigma^{2\alpha+d-1}\widetilde{\lambda}_1|\widehat{f}(\sigma\theta)|^2+|\widehat{g}_\theta(\sigma)-(2\pi)^{(d-1)/2}\widehat{f}(\sigma\theta)|^2\widetilde{\lambda}_2\right)d\sigma d\theta,\end{aligned}$$

where

$$A(\sigma)=\sigma^{d-1}\left((2\pi)^{1-d}\frac{a^2(\sigma)}{\widetilde{\lambda}_2}+\sigma^{2\alpha+d-1}\frac{(a(\sigma)-1)^2}{\widetilde{\lambda}_1}\right).$$

Condition (2.5) is equivalent to $A(\sigma)\leqslant 1$, $\sigma\in[0,\infty)$ and other terms are estimated by the constraints of the class W, which leads to $\|f-m_a(g)\|^2_{L_2(\mathbb{R}^d)}\leqslant\widetilde{\lambda}_1+\widetilde{\lambda}_2\delta^2$. Thus we end with the proof.

3.2. **Proof of Corollary 2.3.** As we've seen in the proof of the Theorem 2.2 the condition on $a(\sigma)$ for the method $m_a(g)$ to be optimal is $A(\sigma)\leqslant 1$. Put $a(\sigma)=1$ to this inequality and solve it for σ to obtain $\sigma\leqslant 2\pi\widetilde{\lambda}_2^{\frac{1}{d-1}}$. By the analogue put $a(\sigma)=0$, then $A(\sigma)\leqslant 1$ is true when $\sigma\geqslant\widetilde{\lambda}_1^{\frac{-1}{2(\alpha)}}$.

3.3. **Proof of Corollary 2.4.** From the solution of the dual problem in Theorem 2.2 it follows, that $\|v\|_{L_2(\mathbb{R}^d)}\leqslant E(\delta)=(2\pi)^{\frac{\alpha(1-d)}{2\alpha+d-1}}\delta^{\frac{2\alpha}{2\alpha+d-1}}$, when the following constraints are satisfied: $\|Rv\|_{L_2(Z)}=\delta$ and $\|\Lambda^\alpha v\|_{L_2(\mathbb{R}^d)}=1$. So the expression can be presented as $\|v\|_{L_2(\mathbb{R}^d)}\leqslant(2\pi)^{\frac{\alpha(1-d)}{2\alpha+d-1}}\delta^{\frac{2\alpha}{2\alpha+d-1}}\|Rv\|^{\frac{2\alpha}{2\alpha+d-1}}_{L_2(Z)}$. Now we put $v(x)=\frac{f(x)}{\|\Lambda^\alpha f\|_{L_2(\mathbb{R}^d)}}$, $f\neq 0$ to obtain

$$\|f\|_{L_2(\mathbb{R}^d)}\leqslant(2\pi)^{\frac{\alpha(1-d)}{2\alpha+d-1}}\|Rf\|^{\frac{2(\alpha)}{2\alpha+d-1}}_{L_2(Z)}\|\Lambda^\alpha f\|^{\frac{d-1}{2\alpha+d-1}}_{L_2(\mathbb{R}^d)}.$$

4. NUMERICAL EXPERIMENT

To emphasize the practical applications of the achieved results, we bring a numerical example. We use standard Shepp-Logan phantom smoothed with Gaussian kernel and approximate its Radon transform with $g(\theta, s) = Rf(\theta, s) + \delta \cdot e(\theta, s)/\|e\|_{L_2(Z)}$, where $e(\theta, s)$ is Gaussian distributed noise with zero mean and standard deviation 1 and $\delta =$ noise level $* \|Rf\|_{L_\infty(Z)}$. We apply the optimal method ($\varepsilon(\sigma) = 1$) on noise level of 0.05 and compare it to the recovery by standard FBP algorithm (Ram-Lack filter) and FBP with Hamming filter. Recovery results are presented in Fig. 1.

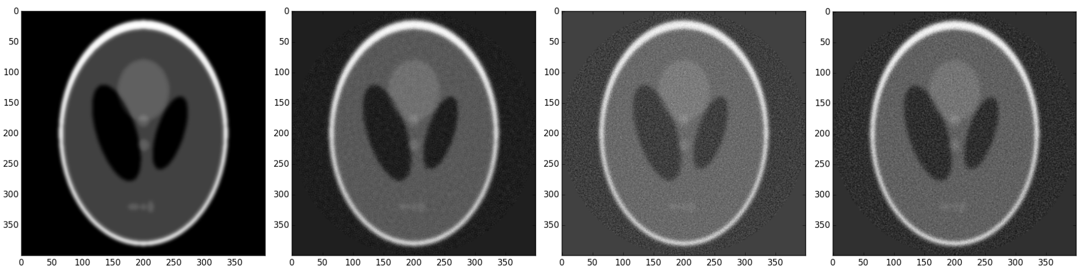

FIGURE 1. From left to right: original phantom, reconstruction by the optimal method, reconstruction by FBP (Ram-Lak filter), reconstruction by FBP with Hamming filter.

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