

**THE COMBINED MODIFIED LAPLACE WITH ADOMIAN DECOMPOSITION  
METHOD FOR SOLVING THE NONLINEAR VOLTERRA-FREDHOLM  
INTEGRO DIFFERENTIAL EQUATIONS**

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**ABSTRACT.** A combined form of the modified Laplace Adomian decomposition method (LADM) is developed for the analytic treatment of the nonlinear Volterra-Fredholm integro differential equations. This method is effectively used to handle nonlinear integro differential equations of the first and the second kind. Finally, some examples will be examined to support the proposed analysis.

## 1. INTRODUCTION

The topic of Volterra-Fredholm integro differential equations which have attracted growing interest in recent years, the nonlinear Volterra-Fredholm integro differential equations as follows [1]:

$$y^{(j)}(x) = f(x) + \int_a^x K_1(x, t)G_1(y(t))dt + \int_a^b K_2(x, t)G_2(y(t))dt \quad (1.1)$$

with initial conditions

$$y^{(r)}(a) = b_r, r = 1, 2, 3, \dots, j - 1 \quad (1.2)$$

where  $y^{(j)}(x)$  is the  $j^{th}$  derivative of the unknown function  $y(x)$  that will be determined,  $K_i(x, t)$ ,  $i = 1, 2$ , be the kernels of the integro differential equation,  $f(x)$  is an analytic function,  $G_1(y)$  and  $G_2(y)$  are nonlinear functions of  $y$ . This paper deals with one of the most applied problems in the engineering sciences. It is concerned with the integro differential equations where both differential and integral operators will appear in the same equation. This type of equations was introduced by Volterra for the first time in early 1900. Volterra investigated the population growth, focussing his study on the hereditary influences, where through

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his research work the topic of integro differential equations was established [1,2]. More details about the sources where these equations arise can be found in physics, biology, and engineering applications as well as in advanced integral equations Some works based on an iterative scheme have been focusing on the development of more advanced and efficient methods for integral equations and integro differential equations such as the variational iteration method (VIM) which is a simple and Adomian decomposition method (ADM) [1–4], and the modified decomposition method (MDM) for solving Volterra-Fredholm integral and integro differential equations which is a simple and powerful method for solving a wide class of nonlinear problems [1, 5]. A variety of powerful methods has been presented, such as the homotopy analysis method [1], homotopy perturbation method [6], the triangular-function method [7], variational iteration method [1,3,8,9] and the Adomian decomposition method [1, 10, 11], and many methods for solving integro differential equations [10, 12–16]. By using the LADM we obtain analytical solutions for the integro-differential equations. Some fundamental works on various aspects of modifications of the Adomian’s decomposition method are given by Araghi [10]. The modified form of Laplace decomposition method has been introduced by Manafianheris [17]. Babolian et. al, [18], applied the new direct method to solve nonlinear Volterra-Fredholm integral and integro differential equation using operational matrix with block-pulse functions. The Laplace transform method with the Adomian decomposition method to establish exact solutions or approximations of the nonlinear Volterra integro differential equations, Wazwaz [4]. Elgasery [19], applied the Laplace decomposition method for the solution of Falkner Skan equation. Hussain and Khan in [14], the modified Laplace decomposition method have applied for solving some PDEs. Recently, the authors have used several methods for the numerical or the analytical solution of linear and nonlinear Fredholm and Volterra integral and integro differential equations of the second kind [1,2]. This numerical technique basically illustrates how the Laplace transform may be used to approximate the solutions of the nonlinear integro differential equations by manipulating the decomposition method. Our aim in this paper is to obtain the analytical solutions by using the modified Laplace Adomian decomposition method. The remainder of the paper is organized as follows: In Section 2, a brief discussion for the modified Laplace Adomian decomposition method is presented. In Section 3, We present and describe the basic formulation of the Leibnitz rule for differentiation of integrals. In Section 4, a brief discussion of convert nonlinear Volterra-Fredholm integro differential equations (VFIDs) of the first kind to nonlinear Volterra-Fredholm integro differential equations (VFIDs) of the second kind. In Section 5, applications of this method and the exact solutions for some examples are obtained. Finally, we will give report on our paper and a brief conclusion is given in Section 6.

## 2. THE MODIFIED LAPLACE ADOMIAN DECOMPOSITION METHOD

The nonlinear Volterra-Fredholm integro differential equation with difference kernels as follows:

$$y^{(j)}(x) = f(x) + \int_a^x K_1(x-t)G_1(y(t))dt + \int_a^b K_2(x-t)G_2(y(t))dt \quad (2.1)$$

To solve the nonlinear Volterra-Fredholm integro differential Eq. (2.1) by using the Laplace transform method, we recall that the Laplace transforms of the derivatives of  $y(x)$  are defined by

$$\mathcal{L}\{y^{(j)}(x)\} = s^j \mathcal{L}\{y(x)\} - s^{j-1}y(0) - s^{j-2}y'(0) - \dots - y^{(j-1)}(0), \quad (2.2)$$

Applying the Laplace transform to both sides of Eq. (2.1) gives:

$$\begin{aligned} s^j \mathcal{L}\{y(x)\} - s^{j-1}y(0) - s^{j-2}y'(0) - \dots - y^{(j-1)}(0) &= \mathcal{L}\{f(x)\} \\ &+ \mathcal{L}\{K_1(x-t)\} \mathcal{L}\{G_1(y(t))\} + \mathcal{L}\{K_2(x-t)\} \mathcal{L}\{G_2(y(t))\} \end{aligned} \quad (2.3)$$

This can be reduced to

$$\begin{aligned} \mathcal{L}\{y(x)\} &= \frac{1}{s}y(0) + \frac{1}{s^2}y'(0) + \dots + \frac{1}{s^i}y^{(i-1)}(0) + \frac{1}{s^i} \mathcal{L}\{f(x)\} + \frac{1}{s^i} \mathcal{L}\{K_1(x-t)\} \\ &\mathcal{L}\{G_1(y(t))\} + \frac{1}{s^i} \mathcal{L}\{K_2(x-t)\} \mathcal{L}\{G_2(y(t))\} \end{aligned} \quad (2.4)$$

The Adomian decomposition method and the Adomian polynomials can be used to handle Eq. (2.4) and to address the nonlinear term  $G(y(x))$ . We first represent the linear term  $y(x)$  at the left side by an infinite series of components given by

$$y = \sum_{m=0}^{\infty} y_m(x) \quad (2.5)$$

where the components  $y_m(x)$ ,  $m \geq 0$  will be determined recursively. However, the nonlinear terms  $G_1(y(x))$  and  $G_2(y(x))$  at the right side of Eq. (2.4) will be represented by an infinite series of the Adomian polynomials  $A_m$  and  $B_m$  in the form:

$$G_1(y(x)) = \sum_{m=0}^{\infty} A_m(x), \quad G_2(y(x)) = \sum_{m=0}^{\infty} B_m(x) \quad (2.6)$$

where  $A_m$  and  $B_m$ ,  $m \geq 0$  are defined by

$$A_m = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \left[ G_1 \left( \sum_{i=0}^m \lambda^i y_i \right) \right] \right]_{\lambda=0} \quad (2.7)$$

$$B_m = \frac{1}{m!} \left[ \frac{d^m}{d\mu^m} \left[ G_2 \left( \sum_{i=0}^m \mu^i y_i \right) \right] \right]_{\mu=0} \quad (2.8)$$

where the so-called Adomian polynomials  $A_m$  can be evaluated for all forms of nonlinearity. In other words, assuming that the nonlinear function is  $G_1(y(x))$  therefore the Adomian polynomials are given by:

$$\begin{aligned} A_0 &= G_1(y_0), \\ A_1 &= y_1 G_1'(y_0), \\ A_2 &= y_2 G_1'(y_0) + \frac{1}{2!} y_1^2 G_1''(y_0), \end{aligned}$$

$$\begin{aligned}
A_3 &= y_3 G_1'(y_0) + y_1 y_2 G_1''(y_0) + \frac{1}{3!} y_1^3 G_1'''(y_0), \\
A_4 &= y_4 G_1'(y_0) + \left( \frac{1}{2!} y_2^2 + y_1 y_3 \right) G_1''(y_0) + \frac{1}{2!} y_1^2 y_2 G_1'''(y_0) + \frac{1}{4!} y_1^4 G_1^{(iv)}(y_0), \quad (2.9)
\end{aligned}$$

similarly, Adomian polynomials  $B_m$  can be evaluated for all forms of nonlinearity. In other words, assuming that the nonlinear function is  $G_2(y(x))$ , therefore the Adomian polynomials are given by

$$\begin{aligned}
B_0 &= G_2(y_0), \\
B_1 &= y_1 G_2'(y_0), \\
B_2 &= y_2 G_2'(y_0) + \frac{1}{2!} y_1^2 G_2''(y_0), \\
B_3 &= y_3 G_2'(y_0) + y_1 y_2 G_2''(y_0) + \frac{1}{3!} y_1^3 G_2'''(y_0), \\
B_4 &= y_4 G_2'(y_0) + \left( \frac{1}{2!} y_2^2 + y_1 y_3 \right) G_2''(y_0) + \frac{1}{2!} y_1^2 y_2 G_2'''(y_0) + \frac{1}{4!} y_1^4 G_2^{(iv)}(y_0), \quad (2.10)
\end{aligned}$$

Substituting Eq. (2.5) and Eq. (2.6) into Eq. (2.4) leads to

$$\begin{aligned}
\mathcal{L}\left\{\sum_{m=0}^{\infty} y_m(x)\right\} &= \frac{1}{s} y(0) + \frac{1}{s^2} y'(0) + \cdots + \frac{1}{s^i} y^{(m-1)}(0) + \frac{1}{s^i} \mathcal{L}\{f(x)\} + \frac{1}{s^i} \mathcal{L}\{K_1(x-t)\} \\
&\quad + \mathcal{L}\left\{\sum_{m=0}^{\infty} A_m(y(t))\right\} + \frac{1}{s^i} \mathcal{L}\{K_2(x-t)\} \mathcal{L}\left\{\sum_{m=0}^{\infty} B_m(y(t))\right\} \quad (2.11)
\end{aligned}$$

The Adomian decomposition method presents the recursive relation

$$\begin{aligned}
\mathcal{L}\{y_0(x)\} &= \frac{1}{s} y(0) + \frac{1}{s^2} y'(0) + \cdots + \frac{1}{s^i} y^{(m-1)}(0) + \frac{1}{s^i} \mathcal{L}\{f(x)\}, \\
\mathcal{L}\{y_{k+1}(x)\} &= \frac{1}{s^i} (\mathcal{L}\{K_1(x-t)\} \mathcal{L}\{A_k(y(t))\} + \mathcal{L}\{K_2(x-t)\} \mathcal{L}\{B_k(y(t))\}). \quad (2.12)
\end{aligned}$$

Applying the inverse Laplace transform to the first part of Eq. (2.12) with  $k \geq 0$  gives  $y_0(x)$ , that will define  $A_0(x)$  and  $B_0(x)$ . Using  $A_0(x)$  and  $B_0(x)$  will enable us to evaluate  $y_1(x)$ . The determination of  $y_0(x)$  and  $y_1(x)$  leads to the determination of  $A_1(x)$  and  $B_1(x)$  that will allow us to determine  $y_2(x)$ , and so on. This in turn will lead to the complete determination of the components of  $y_k(x)$ ,  $k \geq 0$  upon using the second part of Eq. (2.12). The series solution follows immediately after using Eq. (2.5). The obtained series solution may converge to an exact solution if such a solution exists. Otherwise, the series solution can be used for numerical purposes. The combined modified Laplace Adomian decomposition method for solving nonlinear Volterra-Fredholm integro differential equations of the second kind is illustrated by studying the following examples in Section 5.

### 3. LEIBNITZ RULE FOR DIFFERENTIATION OF INTEGRALS

One of the methods that will be used to solve integral equations is the conversion of the integral equation to an equivalent differential equation. The conversion is achieved by using the well-known Leibnitz rule for differentiation of integrals [1]. Let  $f(x, t)$  be continuous and  $\frac{\partial f}{\partial t}$  be continuous in a domain of the  $(x - t)$  plane that includes the rectangle  $a \leq x \leq b, t_0 \leq t \leq t_1$ , and let

$$F(x) = \int_{g(x)}^{h(x)} f(x, t) dt, \quad (3.1)$$

then differentiation of the integral in Eq. (3.1) exists and is given by

$$\begin{aligned} F'(x) &= \frac{dF}{dx} \\ &= f(x, h(x)) \frac{dh(x)}{dx} - f(x, g(x)) \frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial}{\partial x} f(x, t) dt. \end{aligned} \quad (3.2)$$

If  $g(x) = a$  and  $h(x) = b$  where  $a$  and  $b$  are constants, then the Leibnitz rule Eq. (3.2) reduces to

$$F'(x) = \int_a^b \frac{\partial}{\partial x} f(x, t) dt. \quad (3.3)$$

### 4. CONVERSION VFIDE FIRST KIND TO VFIDE OF THE SECOND KIND

In this section, we will present a method that will convert nonlinear Volterra-Fredholm integro differential equations (VFIDE) of the first kind to nonlinear Volterra-Fredholm integro differential equations (VFIDE) of the second kind. The conversion technique works effectively only if  $K_2(x, x) \neq 0$  and  $L_2(x, x) \neq 0$ . The VFIDE of the first kind as:

$$\begin{aligned} f(x) &= \int_a^x K_1(x, t) G_1(y(t)) dt + \int_a^x K_2(x, t) y^{(i)}(t) dt + \int_a^b L_1(x, t) G_2(y(t)) dt \\ &+ \int_a^b L_2(x, t) y^{(i)}(t) dt. \end{aligned} \quad (4.1)$$

We will study the case of the second order derivatives ( $i = 2$ ). Integrating the second and fourth integrals by parts and using Leibnitz rule Eq. (3.2), we find

$$\begin{aligned} f(x) &= \int_a^x K_1(x, t) G_1(y(t)) dt + K_2(x, x) y'(x) - K_2(x, a) y'(a) - \int_a^x \frac{\partial}{\partial t} K_2(x, t) y'(t) dt \\ &+ \int_a^b L_1(x, t) G_2(y(t)) dt - \int_a^b \frac{\partial}{\partial t} L_2(x, t) y'(t) dt, \end{aligned} \quad (4.2)$$

or equivalently

$$y'(t) = \frac{f(x)}{K_2(x, x)} + \frac{K_2(x, a)}{K_2(x, x)} y'(a) - \frac{1}{K_2(x, x)} \int_a^x K_1(x, t) G_1(y(t)) dt$$

$$\begin{aligned}
& + \frac{1}{K_2(x, x)} \int_a^x \frac{\partial}{\partial t} K_2(x, t) y'(t) dt - \frac{1}{K_2(x, x)} \int_a^b L_1(x, t) G_2(y(t)) dt \\
& + \frac{1}{K_2(x, x)} \int_a^b \frac{\partial}{\partial t} L_2(x, t) y'(t) dt,
\end{aligned}$$

It is important to notice that if the nonlinear Volterra-Fredholm integro differential equations of the first kind contains the second derivative of  $y(x)$ , then the conversion process will give a nonlinear Volterra-Fredholm integro differential equations of the second kind. Other methods can be used as well.

## 5. EXAMPLES

In order to elucidate the solution procedure of the modified Laplace Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro differential equations is illustrated in the three examples in this section which shows the effectiveness and generalization of our proposed method given above.

**Example 5.1.** Consider the nonlinear integro differential equation of the second kind with:

$$f(x) = 2e^{2x} - \frac{1}{24}e^x, \quad k_1 = 0, \quad k_2 = e^{x-4t}, \quad y(0) = 1, \quad j = 1.$$

We can write Eq. (1.1)

$$y'(x) = 2e^{2x} - \frac{1}{24}e^x + \frac{1}{24} \int_0^1 e^{x-4t} y^2(t) dt. \quad (5.1)$$

Taking Laplace transform of both sides of Eq. (5.1) gives

$$\mathcal{L}\{y'(x)\} = \mathcal{L}\{2e^{2x} - \frac{1}{24}e^x\} + \frac{1}{24} \mathcal{L}\{e^{x-4t} * y^2(x)\},$$

so that

$$sY(s) - y(0) = \frac{2}{s-2} - \frac{1}{24(s-1)} + \frac{1}{24(s-1)} \mathcal{L}\{y^2(x)\},$$

or equivalently

$$Y(s) = \frac{1}{s} + \frac{2}{s(s-2)} - \frac{1}{24s(s-1)} + \frac{1}{24s(s-1)} \mathcal{L}\{y^2(x)\}. \quad (5.2)$$

Substituting the series assumption for  $Y(s)$  and the Adomian polynomials for  $y^2(x)$  as given above in Eq. (2.5) and Eq. (2.6) respectively, and using the recursive relation Eq. (2.12) we obtain

$$\begin{aligned}
Y_0(s) &= \frac{1}{s} + \frac{2}{s(s-2)} - \frac{1}{24s(s-1)} \\
\mathcal{L}\{y_{k+1}(x)\} &= \frac{1}{24s(s-1)} \mathcal{L}\{B_k(x)\}, \quad k \geq 0,
\end{aligned} \quad (5.3)$$

where  $B_k(x)$  are the Adomian polynomials for the nonlinear term  $y^2(x)$ . The Adomian polynomials for  $G_2(y(x)) = y^2(x)$  are given by

$$\begin{aligned} B_0 &= y_0^2, \\ B_1 &= 2y_1y_0, \\ B_2 &= 2y_2y_0 + y_1^2, \\ B_3 &= 2y_3y_0 + 2y_1y_2. \end{aligned} \quad (5.4)$$

Taking the inverse Laplace transform of both sides of the first part of Eq. (5.3), and using the recursive relation Eq. (5.3) gives

$$\begin{aligned} y_0 &= 1 + (2 - \frac{1}{24})x + (4 - \frac{1}{24})\frac{x^2}{2!} + (8 - \frac{1}{24})\frac{x^3}{3!} + \dots, \\ y_1 &= \frac{1}{24}(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots), \end{aligned} \quad (5.5)$$

and so on for other components. Using Eq. (2.5), the series solution is therefore given by

$$y(x) = 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \dots, \quad (5.6)$$

that converges to the exact solution

$$y(x) = e^{2x}.$$

**Example 5.2.** Consider the nonlinear Volterra-Fredholm integro differential equation of the second kind with:

$$f(x) = -xe^x, \quad y(0) = 1, \quad j = 1, \quad k_1 = e^{x-3t}, \quad k_2 = e^{x-2t},$$

we can write Eq. (1.1)

$$y'(x) = -xe^x + \int_0^x e^{x-3t}y^3(t)dt + \int_0^1 e^{x-2t}y^2(t)dt, \quad y(0) = 1. \quad (5.7)$$

Taking Laplace transform of both sides of Eq. (5.7) gives

$$\mathcal{L}\{y'(x)\} = \mathcal{L}\{-xe^x\} + \mathcal{L}\{e^{x-3t} * y^3(x)\} + \mathcal{L}\{e^{x-2t} * y^2(x)\},$$

so that

$$sY(s) - y(0) = \frac{-1}{(s-1)^2} + \frac{1}{(s-1)}\mathcal{L}\{y^3(x)\} + \frac{1}{(s-1)}\mathcal{L}\{y^2(x)\},$$

or equivalently

$$Y(s) = \frac{1}{s} - \frac{1}{s(s-1)^2} + \frac{1}{s(s-1)}[\mathcal{L}\{y^3(x)\} + \mathcal{L}\{y^2(x)\}].$$

Substituting the series assumption for  $Y(s)$  and the Adomian polynomials for  $y^3(x)$  as given above in Eq. (2.5) and Eq. (2.6) respectively, and using the recursive relation Eq. (2.12) we obtain

$$\begin{aligned} Y_0(s) &= \frac{1}{s} - \frac{1}{s(s-1)^2} \\ \mathcal{L}\{y_{k+1}(x)\} &= \frac{1}{s(s-1)}[\mathcal{L}\{A_k(x)\} + \mathcal{L}\{B_k(x)\}], \quad k \geq 0, \end{aligned} \quad (5.8)$$

where  $A_k(x)$  and  $B_k(x)$  are the Adomian polynomials for the nonlinear term  $y^3(x)$  and  $y^2(x)$  respectively. The Adomian polynomials for  $G_1(y(x)) = y^3(x)$  and  $G_2(y(x)) = y^2(x)$  are given by

$$\begin{aligned} A_0 &= y_0^3, \\ A_1 &= 3y_1y_0^2, \\ A_2 &= 3y_2y_0^2 + 3y_1^2y_0, \\ A_3 &= 3y_3y_0^2 + 6y_0y_1y_2 + y_1^3, \end{aligned}$$

and

$$\begin{aligned} B_0 &= y_0^2, \\ B_1 &= 2y_1y_0, \\ B_2 &= 2y_2y_0 + y_1^2, \\ B_3 &= 2y_3y_0 + 2y_1y_2, \end{aligned} \quad (5.9)$$

Taking the inverse Laplace transform of both sides of the first part of Eq. (5.8), and using the recursive relation Eq. (5.8) gives

$$\begin{aligned} y_0 &= e^x - xe^x, \\ &= 1 - \frac{1}{2!}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \dots, \\ y_1 &= 2 \left[ \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots \right], \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

that converges to the exact solution

$$y(x) = e^x.$$

**Example 5.3.** Consider the nonlinear integro differential equation of the second kind with:

$$f(x) = \frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x}, \quad k_1 = (x-t), \quad k_2 = 0,$$



we can write Eq. (1.1)

$$y'(x) = \frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x} + \int_0^x (x-t)y^2(t)dt, \quad y(0) = 2. \quad (5.10)$$

Taking Laplace transform of both sides of Eq. (5.10) gives

$$\mathcal{L}\{y'(x)\} = \mathcal{L}\left\{\frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x}\right\} + \mathcal{L}\{(x-t) * y^2(x)\},$$

so that

$$sY(s) - y(0) = \frac{9}{4s} - \frac{5}{2s^2} - \frac{1}{s^3} - \frac{3}{(s+1)} - \frac{1}{4(s+2)} + \frac{1}{s^2}\mathcal{L}\{y^2(x)\},$$

or equivalently

$$Y(s) = \frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)} + \frac{1}{s^3}\mathcal{L}\{y^2(x)\}. \quad (5.11)$$

Substituting the series assumption for  $Y(s)$  and the Adomian polynomials for  $y^2(x)$  as given above in Eq. (2.5) and Eq. (2.6) respectively, and using the recursive relation Eq. (2.12) we obtain

$$\begin{aligned} Y_0(s) &= \frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)} \\ \mathcal{L}\{y_{k+1}(x)\} &= \frac{1}{s^3}\mathcal{L}\{A_k(x)\}, \quad k \geq 0. \end{aligned} \quad (5.12)$$

where  $A_k(x)$  are the Adomian polynomials for the nonlinear term  $y^2(x)$ . The Adomian polynomials for  $G_1(y(x)) = y^2(x)$  are given by

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_1y_0, \\ A_2 &= 2y_2y_0 + y_1^2, \\ A_3 &= 2y_3y_0 + 2y_1y_2. \end{aligned} \quad (5.13)$$

Taking the inverse Laplace transform of both sides of the first part of Eq. (5.12), and using the recursive relation Eq. (5.12) gives

$$\begin{aligned} y_0 &= 2 - x + \frac{1}{2!}x^2 - \frac{5}{3!}x^3 + \frac{5}{4!}x^4 - \dots, \\ y_1 &= \frac{2}{3}x^3 - \frac{1}{3!}x^4 + \frac{1}{20}x^5 + \dots, \end{aligned} \quad (5.14)$$

and so on for other components. Using Eq. (2.5), the series solution is therefore given by

$$y(x) = 2 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots,$$

that converges to the exact solution

$$y(x) = 1 + e^{-x}.$$

**Example 5.4.** Consider the following nonlinear Volterra-Fredholm integro differential equation of the first kind with:

$$f(x) = -\frac{9}{5} - \frac{5}{2}x + \frac{1}{2}x^2 + 2e^x + \frac{1}{4}e^{2x} + xe^x, \quad k_1 = (x-t)^2, \quad k_2 = e^{x-t}, \quad y(0) = 2,$$

we can write Eq. (1.1)

$$-\frac{9}{5} - \frac{5}{2}x + \frac{1}{2}x^2 + 2e^x + \frac{1}{4}e^{2x} + xe^x = \int_0^x (x-t)^2 y^2(t) dt + \int_0^1 e^{x-t} y'(t) dt. \quad (5.15)$$

Taking Laplace transform of both sides of Eq. (5.15) gives

$$-\frac{9}{5s} - \frac{5}{2s^2} + \frac{1}{s^3} + \frac{2}{s-1} + \frac{1}{4(s-2)} + \frac{1}{(s-1)^2} = \frac{1}{s^2} \mathcal{L}\{y^2(s)\} + \frac{1}{s-1} (sY(s) - y(0))$$

so that

$$Y(s) = \frac{2}{s} + \frac{s-1}{s} \left( -\frac{9}{5s} - \frac{5}{2s^2} + \frac{1}{s^3} + \frac{2}{s-1} + \frac{1}{4(s-2)} + \frac{1}{(s-1)^2} \right) - \frac{s-1}{s^3} \mathcal{L}\{y^2(x)\} \quad (5.16)$$

Substituting the series assumption for  $Y(s)$  and the Adomian polynomials for  $y^2(x)$  as given above in Eq. (2.5) and Eq. (2.6) respectively, and using the recursive relation Eq. (2.12) we obtain

$$\begin{aligned} Y_0(s) &= \frac{2}{s} + \frac{s-1}{s} \left( -\frac{9}{5s} - \frac{5}{2s^2} + \frac{1}{s^3} + \frac{2}{s-1} + \frac{1}{4(s-2)} + \frac{1}{(s-1)^2} \right), \\ \mathcal{L}\{y_{k+1}(x)\} &= -\frac{s-1}{s^3} \mathcal{L}\{A_k(x)\}, \quad k \geq 0. \end{aligned} \quad (5.17)$$

where  $A_k(x)$  are the Adomian polynomials for the nonlinear term  $y^2(x)$ . The Adomian polynomials for  $G_1(y(x)) = y^2(x)$  are given by

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_1y_0, \\ A_2 &= 2y_2y_0 + y_1^2, \\ A_3 &= 2y_3y_0 + 2y_1y_2, \end{aligned} \quad (5.18)$$

Taking the inverse Laplace transform of both sides of the first part of Eqs.(5.17), and using the recursive relation Eq. (5.17) gives

$$\begin{aligned} y_0 &= 2 + x + \frac{5}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \dots, \\ y_1 &= -2x^2 - \frac{3}{4}x^4 - \frac{1}{10}x^5 + \dots, \end{aligned} \quad (5.19)$$

and so on for other components. Using Eq. (2.5), the series solution is therefore given by

$$y(x) = 2 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots, \quad (5.20)$$

that converges to the exact solution

$$y(x) = 1 + e^x$$

## 6. CONCLUSIONS

The main idea of this work was to give a simple method for solving the Volterra-Fredholm integro differential equations (VFIDEs). We carefully applied a reliable modification of Laplace Adomian decomposition method for VFIDEs. The main advantage of this method is the fact that it gives the analytical solution. Also, this method is combining of two powerful methods for obtaining exact solutions of nonlinear (VFIDEs). In the above examples we observed that the MLADM with the initial approximation obtained from initial conditions yields a good approximation to the exact solution only in a few iterations. It is also worth noting that the advantage of the decomposition methodology displays a fast convergence of the solutions.

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