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SOME FIXED POINT THEOREMS IN CONNECTION WITH TWO WEAKLY COMPATIBLE MAPPINGS IN BICOMPLEX VALUED METRIC SPACES

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Abstract. In this paper, we aim to prove certain common fixed point theorems for a pair of weakly compatible mappings satisfying (CLRg) (or (E.A)) property in the *bicomplex valued metric spaces*. We also provide some examples which support the main results here.

1. Introduction

Bicomplex numbers have been studied for quite a long time, which probably began with the works [15, 16, 17, 18]. Their interest arose from the fact that such numbers offer a commutative alternative to the skew field of quaternions (both sets are real four dimensional spaces) and that, in many ways, they generalize complex numbers more closely and accurately than quaternions do. In recent years there has been a significant impulse to investigate bicomplex holomorphy. For the most comprehensive study of analysis in the bicomplex setting, we refer the reader to the book [13]. During the last several years the ideas of bicomplex functional analysis have been brought from different aspects and many important results have been gained (see [2]-[12]). This new research subject will be widely applied in physics, electric circuit theory, power system load frequency control, control engineering, communication, signal analysis and design, system analysis and solving differential

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equations. It is well known that the fixed point theory plays a very important role in theory and applications, in particular, whose importance comes from finding roots of algebraic equation and numerical analysis.

Recently, Azam *et al.* [1] introduced the notion of complex-valued metric space which is a generalization of classical metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The idea of complex-valued metric spaces can be exploited to define complex-valued normed spaces and complex-valued Hilbert spaces; Additionally it offers numerous research activities in mathematical analysis.

In this paper, we aim to prove certain common fixed point theorems for a pair of weakly compatible mappings satisfying (CLRg) (or (E.A)) property in the *bicomplex valued metric spaces.* We also provide some examples which support the main results here.

2. Definitions and Notations

We recall some notations and definitions which will be required in the subsequent sections (see, e.g., [7]).

Here and in the following, let $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{C}$, and $\mathbb{N}$ be the sets of real numbers, nonnegative real numbers, complex numbers, and positive integers, respectively. The set $\mathbb{C}$ is given as

$$\mathbb{C} := \left\{ z = x + iy \,|\, x,\, y \in \mathbb{R} \text{ and } i^2 = -1 \right\}.$$

Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows (see, e.g., [1]): For $z_1,\, z_2 \in \mathbb{C}$,

(1) $\quad z_1 \precsim z_2$ if and only if $\Re(z_1) \leq \Re(z_2)$ and $\Im(z_1) \leq \Im(z_2)$.

Thus $z_1 \precsim z_2$ if any one of the following statements holds:

(o_1) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$;
(o_2) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$;
(o_3) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$;
(o_4) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$, i.e., any one of (o_2), (o_3) and (o_4) is satisfied, and we write $z_1 \prec z_2$ if only (o_4) is satisfied. Considering (o_1)-(o_4), the following properties for the partial order $\precsim$ on $\mathbb{C}$ hold true:

(p_1) $0 \precsim z_1 \precsim z_2 \Longrightarrow |z_1| \leq |z_2|$;

(p_2) $z_1 \precsim z_2$ and $z_2 \precsim z_3 \Longrightarrow z_1 \precsim z_3$;

(p_3) $z_1 \precsim z_2$ and $\lambda > 0$ $(\lambda \in \mathbb{R}) \Longrightarrow \lambda\, z_1 \precsim \lambda\, z_2$.

The set of bicomplex numbers denoted by $\mathbb{C}_2$ is the first setting in an infinite sequence of multicomplex sets which are generalizations of the set of complex numbers $\mathbb{C}$. Here we recall the set of bicomplex numbers $\mathbb{C}_2$ (see, e.g., [8, 14]):

$$\mathbb{C}_2 = \left\{ w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3 \,\middle|\, p_k \in \mathbb{R} \ (k = 0, \ldots, 3) \right\}.$$

Since each element w in $\mathbb{C}_2$ can be written as

$$w = p_0 + i_1 p_1 + i_2 (p_2 + i_1 p_3)$$

or

$$w = z_1 + i_2 z_2 \quad (z_1,\, z_2 \in \mathbb{C}),$$

we can also express $\mathbb{C}_2$ as

$$\mathbb{C}_2 = \{ w = z_1 + i_2 z_2 \mid z_1,\, z_2 \in \mathbb{C} \},$$

where $z_1 = p_0 + i_1 p_1$, $z_2 = p_2 + i_1 p_3$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The products of all units are commutative and satisfy

$$i_1 i_2 = j, \quad i_1 j = -i_2, \quad i_2 j = -i_1.$$

Let $u = u_1 + i_2 u_2 \in \mathbb{C}_2$ and $v = v_1 + i_2 v_2 \in \mathbb{C}_2$. Define a partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows:

$$u \precsim_{i_2} v \text{ if and only if } u_1 \precsim v_1 \text{ and } u_2 \precsim v_2, \tag{2}$$

where the partial order $\precsim$ in the right-hand side is given as in (1). We find that $u \precsim_{i_2} v$ if any one of the following properties holds:

(bo_1) $u_1 = v_1$ and $u_2 = v_2$;
(bo_2) $u_1 \prec v_1$ and $u_2 = v_2$;
(bo_3) $u_1 = v_1$ and $u_2 \prec v_2$;
(bo_4) $u_1 \prec v_1$ and $u_2 \prec v_2$.

We write $u \precnsim_{i_2} v$ if $u \precsim_{i_2} v$ and $u \neq v$, i.e., one of (bo_2), (bo_3) and (bo_4) is satisfied and we write $u \prec_{i_2} v$ if only (bo_4) is satisfied.

A norm of a bicomplex number $w = z_1 + i_2 z_2$ denoted by $||w||$ is defined by

$$||w|| = \|z_1 + i_2 z_2\| = \left(|z_1|^2 + |z_2|^2 \right)^{\frac{1}{2}},$$

which, upon choosing $w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$ ($p_k \in \mathbb{R}\,(k = 0,\, 1,\, 2,\, 3)$), gives

$$\|w\| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2 \right)^{\frac{1}{2}}. \tag{3}$$

For any two bicomplex numbers u, $v \in \mathbb{C}_2$, one can easily verify that

(4) $0 \precsim_{i_2} u \precsim_{i_2} v \Rightarrow \|u\| \leq \|v\|\,;\ \|u+v\| \leq \|u\| + \|v\|\,;\ \|\alpha\, u\| = \alpha\ \|u\|$

where α is non-negative real number.

In parallel to the method Azam et al. [1] defined a complex-valued metric, we define a bicomplex-valued metric as follows: Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}_2$ is a bicomplex-valued metric on X if it satisfies the following properties: For x, y, $z \in X$,

(m_1) $0 \precsim_{i_2} d(x, y)$ for all x, $y \in X$;

(m_2) $d(x, y) = 0$ if and only if $x = y$;

(m_3) $d(x, y) = d(y, x)$ for all x, $y \in X$;

(m_4) $d(x, y) \precsim_{i_2} d(x, z) + d(z, y)$ for all x, y, $z \in X$.

Then (X, d) is called a bicomplex-valued metric space.

For example, let $X = \mathbb{R}$ and a mapping $d : X \times X \longrightarrow \mathbb{C}_2$ be defined by

$$d(x, y) := (1 + i_1 + i_2 + i_1 i_2)\,|x - y| \quad (x, y \in X)\,,$$

where $|\ |$ is the usual real modulus. One can easily check that (X, d) is a bicomplex-valued metric on $\mathbb{C}$.

A sequence in a nonempty set X is a function $x : \mathbb{N} \to X$, which is expressed by its range set $\{x_n\}$ where $x(n) := x_n$ $(n \in \mathbb{N})$. Let $\{x_n\}$ be a sequence in a bicomplex-valued metric space (X, d). The sequence $\{x_n\}$ is said to converge to $x \in X$ if and only if for any $0 \prec_{i_2} \varepsilon \in \mathbb{C}_2$, there exists $N \in \mathbb{N}$ depending on ε such that $d(x_n, x) \prec_{i_2} \varepsilon$ for all $n > N$. It is denoted by $x_n \to x$ as $n \to \infty$, or, $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in a bicomplex-valued metric space (X, d) is said to be a Cauchy sequence if and only if for any $0 \prec_{i_2} \varepsilon \in \mathbb{C}_2$, there exists $N \in \mathbb{N}$ depending on ε such that $d(x_m, x_n) \prec_{i_2} \varepsilon$ for all $m, n > N$. A bicomplex-valued metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges in X.

Let (X, d) be a metric space and S, $T : X \to X$ be two mappings. A point $x \in X$ is said to be a common fixed point of S and T if and only if $Sx = Tx = x$.

Let (X, d) be a metric space. The self maps S and T on X are said to be commuting if $STx = TSx$ for all $x \in X$. The self maps S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. The self-maps S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$, that is, they commute at their coincidence points. The self-maps S and T are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Suppose that (X, d) is a metric space and f, $g : X \to X$. Then f and g are said to satisfy the (CLRg) property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$$

for some $x \in X$ (see [19]). The property (CLRg) is seen to be stronger than the property (E.A)

For example, Let $X = \mathbb{C}_2$ and d be a bicomplex-valued metric on X. Define f, $g : X \to X$ by $fu = 2u + i_2$ and $gu = 3u - 1$, for all $u \in X$. Consider a sequence $\{u_n\} = \left\{i_2 + 1 + \frac{1}{n}\right\}$ in X. Then

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} \left(3i_2 + 2 + \frac{2}{n}\right) = 3i_2 + 2$$

and

$$\lim_{n\to\infty} gu_n = \lim_{n\to\infty} \left(3i_2 + 2 + \frac{3}{n}\right) = 3i_2 + 2 = g(i_2 + 1).$$

Thus f and g satisfy the (CLRg) property. Here this pair also satisfies the (CLRf) property.

The max function for the partial order $\precsim_{i_2}$ on $\mathbb{C}_2$ is defined as follows:

(i) $\max\{u, v\} = v \Leftrightarrow u \precsim_{i_2} v$;
(ii) $u \precsim_{i_2} \max\{u, v\} \Rightarrow u \precsim_{i_2} v$;
(iii) $u \precsim_{i_2} \max\{v, w\} \Rightarrow u \precsim_{i_2} v$ or $u \precsim_{i_2} w$.

For any $0 \precsim_{i_2} u$, $0 \precsim_{i_2} v$, we can easily prove that $\|\max\{u, v\}\| = \max\{\|u\|, \|v\|\}$.

3. Lemmas

Here we recall two assertions which will be required in the sequel (see [7]).

Lemma 3.1. *Let (X, d) be a bicomplex-valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.*

Lemma 3.2. *Let (X, d) be a bicomplex-valued metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then, for any $a \in X$,*

$$\lim_{n\to\infty} \|d(x_n, a)\| = \|d(x, a)\| .$$

4. Main Results

Here we present some fixed point theorems on bicomplex-valued metric spaces by modifying some known results.

Theorem 4.1. *Let (X, d) be a bicomplex-valued metric space and S, $T : X \to X$ be weakly compatible mappings such that*

(i) *S and T satisfy* (CLR$_\text{S}$) *property and*

(ii) *$d(Tx, Ty) \precsim_{i_2} p\, d(Sx, Sy) + q\, d(Tx, Sy)$ $(x,\, y \in X)$, where $p,\, q \in \mathbb{R}_0^+$ with $p + q < 1$.*

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy (CLR$_\text{S}$) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sa \tag{5}$$

for some $a \in X$.

Applying Lemma 3.1 in (5), we find that, as $n \to \infty$,

$$\|d(Sx_n, Sa)\| \to 0 \quad \text{and} \quad \|d(Tx_n, Sa)\| \to 0. \tag{6}$$

Replacing x, y by x_n, a in (ii), respectively, we obtain

$$d(Tx_n, Ta) \precsim_{i_2} p\, d(Sx_n, Sa) + qd\,(Tx_n, Sa).$$

which, in view of (4), gives

$$\|d(Tx_n, Ta)\| \leq p\, \|d(Sx_n, Sa)\| + q\, \|d(Tx_n, Sa)\| . \tag{7}$$

Taking the limit of both sides of (7) and considering Lemma 3.1 and (6), we obtain

$$\lim_{n\to\infty} \|d(Tx_n, Ta)\| = 0 \Longleftrightarrow \lim_{n\to\infty} Tx_n = Ta. \tag{8}$$

We find from (5) and (8) that

$$Sa = Ta. \tag{9}$$

Since S and T are weakly compatible, we have

$$TTa = TSa = STa = SSa. \tag{10}$$

Replacing x, y by x_n, Ta in (ii), respectively, we obtain

$$d(Tx_n, TTa) \precsim_{i_2} p\, d(Sx_n, STa) + q\, d(Tx_n, STa). \tag{11}$$

Setting the norm in (3) of both sides of (11) and taking the limit of both sides of the resulting inequality together with (5) and (9), we find

$$\begin{aligned} \|d(Ta, TTa)\| &\le p\,\|d(Sa, TTa)\| + q\,\|d(Ta, TTa)\| \\ &= (p+q)\,\|d(Ta, TTa)\|\,. \end{aligned}$$

We thus have

$$(1-p-q)\,\|d(Ta, TTa)\| \le 0.$$

Since $0 \le p+q < 1$, we have $\|d(Ta, TTa)\| = 0$, which implies $TTa = Ta$. We therefore find from (10) that

$$STa = TTa = Ta.$$

Hence Ta is a common fixed point of S and T.

For uniqueness of the fixed point, let $b \in X$ be such that $Sb = Tb = b$. Replacing x, y by a, b, respectively, in (ii), we get

$$d(Ta, b) = d(Ta, Tb) \precsim_{i_2} pd(Sa, Sb) + qd(Ta, Sb),$$

which, upon taking norm and using (9), yields

$$\|d(Ta, b)\| \le p\,\|d(Ta, b)\| + q\,\|d(Ta, b)\|\,.$$

Similarly as before,

$$d(Ta, b) = 0 \Longleftrightarrow Ta = b.$$

Hence Ta is the unique common fixed point of S and T. □

Corollary 4.2. *Let (X, d) be a bicomplex valued metric space and S, $T : X \to X$ be weakly compatible mappings such that*

(i) *S and T satisfy* $(\mathrm{CLR_T})$ *property,*
(ii) *$TX \subset SX$ and*
(iii) *$d(Tx, Ty) \precsim_{i_2} p\, d(Sx, Sy) + q\, d(Tx, Sy)$ $(x,\, y \in X)$, where p, $q \in \mathbb{R}_0^+$ with $p + q < 1$.*

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy ($\mathrm{CLR_T}$) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{x \longrightarrow \infty} Sx_n = \lim_{x \longrightarrow \infty} Tx_n = Tb$$

for some $b \in X$. Since $TX \subset SX$, then $Tb = Sa$ for some $a \in X$. Thus

$$\lim_{x \longrightarrow \infty} Sx_n = \lim_{x \longrightarrow \infty} Tx_n = Sa$$

for some $a \in X$. Therefore S and T satisfy ($\mathrm{CLR_S}$) property. Hence, by Theorem 4.1, S and T have a unique common fixed point. □

Corollary 4.3. *Let* (X, d) *be a bicomplex valued metric space and* $S, T : X \to X$ *be weakly compatible mappings such that*

(i) S *and* T *satisfy* (E.A) *property,*
(ii) SX *is a complete subspace of* X *and*
(iii) $d(Tx, Ty) \precsim_{i_2} p\, d(Sx, Sy) + q\, d(Tx, Sy)$ $(x, y \in X)$,
where $p, q \in \mathbb{R}_0^+$ *with* $p + q < 1$.

Then S *and* T *have a unique common fixed point.*

Proof. Since S and T satisfy (E.A) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{x \longrightarrow \infty} Sx_n = \lim_{x \longrightarrow \infty} Tx_n = t$$

for some $t \in X$. Since SX is a complete subspace of X, $t = Sa$ for some $a \in X$. Thus

$$\lim_{x \longrightarrow \infty} Sx_n = \lim_{x \longrightarrow \infty} Tx_n = Sa$$

for some $a \in X$. Therefore S and T satisfy ($\mathrm{CLR_S}$) property. Hence, by Theorem 4.1, S and T have a unique common fixed point. □

We give an example, which supports Theorem 4.1.

Example 4.4. Let $X = [0, \infty)$ with the bicomplex valued metric

$$d(x, y) = (1 + 2i_2)\,|x - y| \quad (x, y \in X).$$

Define $S, T : X \longrightarrow X$ by

$$Sx = x^2 + \frac{1}{4} \quad \text{and} \quad Tx = \frac{1}{2}$$

for all $x \in X$.

Then it is easy to check that S and T satisfy all the three conditions in Theorem 4.1, and S and T have the unique common fixed point $\frac{1}{2}$.

Theorem 4.5. *Let (X,d) be a bicomplex valued metric space and S, $T : X \to X$ be weakly compatible mappings such that*

(i) *S and T satisfy* (CLR$_\mathrm{S}$) *property and*

(ii) $d(Tx,Ty) \precsim_{i_2} k \max\{d(Sx,Sy),\, d(Tx,Sy)\}$
for all x, $y \in X$ and for some $k \in \mathbb{R}$ with $0 < k < 1$.

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy (CLR$_\mathrm{S}$) property, there exists a $\{x_n\}$ sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sa \tag{12}$$

for some $a \in X$. Replacing x, y by x_n, a, respectively, in (ii), we get

$$d(Tx_n, Ta) \precsim_{i_2} k \cdot \max\{d(Sx_n, Sa), d(Tx_n, Sa)\},$$

which, upon taking norm and the limit on both sides of the resulting inequality as $n \to \infty$ together with (12), yields

$$\|d(Sa,Ta)\| \le k \cdot \max\{\|d(Sa,Sa)\|, \|d(Sa,Sa)\|\}.$$

We thus have

$$\|d(Sa,Ta)\| \le 0 \Leftrightarrow \|d(Sa,Ta)\| = 0 \Leftrightarrow Sa = Ta.$$

Since S and T are weakly compatible, we have

$$TTa = TSa = STa = SSa\ . \tag{13}$$

Replacing x, y by Ta, x_n, respectively, in (ii), we obtain

$$d(TTa, Tx_n) \precsim k \cdot \max\{d(STa, Sx_n), d(TTa, Sx_n)\}.$$

Similarly as before,

$$\|d(TTa,Ta)\| \le k \cdot \max\{\|d(TTa,Ta)\|, \|d(TTa,Ta)\|\}, \tag{14}$$

or, equivalently,

$$\|d(TTa,Ta)\| \le k \cdot \|d(TTa,Ta)\|.$$

Since $0 < k < 1$, we have $d(TTa,Ta) = 0 \Leftrightarrow TTa = Ta$. Therefore, in view of (13), $STa = TTa = Ta$. Hence Ta is a common fixed point of S and T.

We can prove uniqueness of the fixed point in a similar way as in the proof of Theorem 4.1. We omit the details. □

Corollary 4.6. *Let (X,d) be a bicomplex valued metric space and S, $T : X \to X$ be weakly compatible mappings such that*

(i) *S and T satisfy* (CLR$_\mathrm{T}$) *property,*

(ii) *$TX \subset SX$ and*
(iii) $d(Tx, Ty) \precsim_{i_2} k \max\{d(Sx, Sy), d(Tx, Sy)\}$
for all x, $y \in X$ and for some $k \in \mathbb{R}$ with $0 < k < 1$.

Then S and T have a unique common fixed point.

Proof. We can prove this result by a similar argument as in the proof of Corollary 4.2. We omit the details. □

Corollary 4.7. *Let (X, d) be a bicomplex valued metric space and S, $T : X \to X$ be weakly compatible mappings such that*

(i) *S and T satisfy* (E.A) *property,*
(ii) *SX is a complete subspace of X and*
(iii) $d(Tx, Ty) \precsim_{i_2} k \max\{d(Sx, Sy), d(Tx, Sy)\}$
for all x, $y \in X$ and for some $k \in \mathbb{R}$ with $0 < k < 1$.

Then S and T have a unique common fixed point.

Proof. The proof runs parallel to that of Corollary 4.3, considering the condition (iii). We omit the details. □

The following example supports Theorem 4.5.

Example 4.8. Let $X = \mathbb{C}_2$ with the bicomplex valued metric

$$d(x, y) = (1 + i_1 + i_2)\,\|x - y\| \quad (x,\, y \in X).$$

Define S, $T : X \to X$ by

$$Sx = 3x - 2i_2 \quad \text{and} \quad Tx = i_2$$

for all $x \in X$. Then it is easy to check that S and T satisfy all the three conditions in Theorem 4.5 and i_2 is the unique common fixed point of S and T.

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