Honam Mathematical J. **39** (2017), No. 1, pp. 93–100
https://doi.org/10.5831/HMJ.2017.39.1.93

IDEALS IN THE UPPER TRIANGULAR OPERATOR ALGEBRA ALG$\mathcal{L}$

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Abstract. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with a fixed orthonormal base $\{e_1, e_2, \cdots\}$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $\{[e_1], [e_1, e_2], [e_1, e_2, e_3], \cdots\}$ and let Alg$\mathcal{L}$ be the algebra of bounded operators which leave invariant all projections in $\mathcal{L}$. Let p and q be natural numbers($p \leqslant q$). Let $\mathcal{B}_{p,q} = \{ T \in \text{Alg}\mathcal{L} \mid T_{(p,q)} = 0 \}$. Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\{0\} \subsetneq \mathcal{A} \subset \mathcal{B}_{p,q}$. If $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$, then $T_{(i,j)} = 0, p \leqslant i \leqslant q$ and $i \leqslant j \leqslant q$ for all T in $\mathcal{A}$.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with a fixed orthonormal base $\{e_1, e_2, \cdots\}$ and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. If $x_1, x_2, \cdots, x_k$ are vectors in $\mathcal{H}$, we denote by $[x_1, x_2, \cdots, x_k]$ the closed subspace spanned by the vectors $x_1, x_2, \cdots, x_k$. We denote by $\mathcal{L}$ the subspace lattice generated by the subspaces $\{[e_1], [e_1, e_2], \cdots, [e_1, e_2, \cdots, e_n], \cdots\}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. By Alg$\mathcal{L}$, we mean the algebra of bounded operators which leave invariant all subspaces in $\mathcal{L}$. It is easy to see that all such operators have the following matrix form

Received November 8, 2016. Accepted December 16, 2017.
2010 Mathematics Subject Classification. 47L35.
Key words and phrases. Linear manifold, Lie ideal, Ideal, The upper triangular operator algebra.
This work was supported by Daegu University Grant(2016).
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$$\begin{pmatrix} * & * & * & * & * & & & \\ & * & * & * & \ddots & \ddots & & \\ & & * & * & * & \ddots & \ddots & \\ & & & * & * & * & \ddots & \ddots \\ & & & & * & * & \ddots & \ddots \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

where all non-starred entries are zero. We call the algebra Alg$\mathcal{L}$ by the upper triangular operator algebra.

2. Examples of ideals in Alg$\mathcal{L}$

Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$. We say that $\mathcal{A}$ is *a left ideal* in Alg$\mathcal{L}$ if $AT \in \mathcal{A}$ for all A in Alg$\mathcal{L}$ and T in $\mathcal{A}$. $\mathcal{A}$ is called *a right ideal* in Alg$\mathcal{L}$ if $TA \in \mathcal{A}$ for all A in Alg$\mathcal{L}$ and T in $\mathcal{A}$. $\mathcal{A}$ is said to be *an ideal* in Alg$\mathcal{L}$ if $\mathcal{A}$ is a left ideal in Alg$\mathcal{L}$ and a right ideal in Alg$\mathcal{L}$. $\mathcal{A}$ is called *a prime ideal* in Alg$\mathcal{L}$ if and only if $AB \in \mathcal{A}$ for A in Alg$\mathcal{L}$ and B in Alg$\mathcal{L}$, then $A \in \mathcal{A}$ or $B \in \mathcal{A}$. $\mathcal{A}$ is called *a maximal ideal* in Alg$\mathcal{L}$ if and only if $A \neq$ Alg$\mathcal{L}$ and if there does not exist an ideal $\mathcal{M}$ in Alg$\mathcal{L}$ such that A in $\mathcal{A} \subsetneq \mathcal{M} \subset$ Alg$\mathcal{L}$, then $\mathcal{M} =$ Alg$\mathcal{L}$. Let I be the identity operator on $\mathcal{H}$ in this paper. Let $\mathbb{C}$ be the set of all complex numbers and let $\mathbb{N} = \{1, 2, \cdots\}$.

If we know the following facts, then we can easily prove the following examples.

Let $A = (a_{ij})$ and $T = (t_{ij})$ be operators in Alg$\mathcal{L}$. Then

(1) the (p, p)-entry of AT is $a_{pp}t_{pp}$ for all $p = 1, 2, \cdots$

(2) the (p, p)-entry of TA is $t_{pp}a_{pp}$ for all $p = 1, 2, \cdots$

(3) the (p, q)-entry of AT is

$a_{pp}t_{pq} + a_{p\ p+1}t_{p+1\ q} + \cdots + a_{p\ q-1}t_{q-1\ q} + a_{pq}t_{qq} (p < q)$

(4) the (p, q)-entry of TA is

$t_{pp}a_{pq} + t_{p\ p+1}a_{p+1\ q} + \cdots + t_{p\ q-1}a_{q-1\ q} + t_{pq}a_{qq} (p < q)$

We denote $T_{(i,j)}$ by the (i, j)-component of T for an operator T.

Example 1. Let $\mathcal{A}_0 = \{\ T \in \text{Alg}\mathcal{L} \mid T_{(i,i)} = 0,\ i \in \mathbb{N}\ \}$. Then $\mathcal{A}_0$ is an ideal in Alg$\mathcal{L}$.

Example 2. Let Λ be a nonempty subset of $\mathbb{N}$ and let $\mathcal{A}_\Lambda = \{ T \in \text{Alg}\mathcal{L} \mid T_{(i,i)} = 0, i \in \Lambda \}$. Then $\mathcal{A}_\Lambda$ is an ideal of Alg$\mathcal{L}$.

Example 3. Let I be the identity operator on $\mathcal{H}$ and let $\mathcal{A}_1 = \{ \alpha I + T \mid T \in \mathcal{A}_0, \alpha \in \mathbb{C} \}$. Then $\mathcal{A}_I$ is not an ideal in Alg$\mathcal{L}$.

Example 4. Let p and q be natural numbers such that $p \leqslant q$.
Let $\mathcal{B}_{p,q} = \{ T \in \text{Alg}\mathcal{L} \mid T_{(p,q)} = 0 \}$. If $p = q$, then $\mathcal{B}_{p,q}$ is an ideal of Alg$\mathcal{L}$. If $p < q$, then $\mathcal{B}_{p,q}$ is not an ideal of Alg$\mathcal{L}$.

Example 5. Let p and q be natural numbers($p < q$).
i) Let $\mathcal{B}^{(1)}{}_{p,q} = \{ T \in \text{Alg}\mathcal{L} \mid T_{(p+k,q)} = 0, k = 0, 1, 2, \cdots, q-p \}$. Then $\mathcal{B}^{(1)}{}_{p,q}$ is not an ideal in Alg$\mathcal{L}$.
ii) Let $\mathcal{B}^{(2)}{}_{p,q} = \{ T \in \text{Alg}\mathcal{L} \mid T_{(p,p+k)} = 0, k = 0, 1, 2, \cdots, q-p \}$. Then $\mathcal{B}^{(2)}{}_{p,q}$ is not an ideal in Alg$\mathcal{L}$.
iii) Let $\mathcal{B}^{(3)}{}_{p,q} = \{ T \in \text{Alg}\mathcal{L} \mid T_{(p+k,q)} = 0, T_{(p,p+k)} = 0, k = 0, 1, 2, \cdots, q-p \}$. Then $\mathcal{B}^{(3)}{}_{p,q}$ is not an ideal in Alg$\mathcal{L}$.
iv) Let $\mathcal{B}^{(4)}{}_{p,q} = \{ T \in \text{Alg}\mathcal{L} \mid T_{(p+k,q)} = 0, T_{(p,p+k)} = 0, T_{(p+k,p+k)} = 0, k = 0, 1, 2, \cdots, q-p \}$. Then $\mathcal{B}^{(4)}{}_{p,q}$ is not an ideal in Alg$\mathcal{L}$.

Example 6. Let p and q be natural numbers($p < q$). Let $\mathcal{A}_{p,q} = \{ T \in \text{Alg}\mathcal{L} \mid T_{(i,j)} = 0, p \leqslant i \leqslant q \text{ and } i \leqslant j \leqslant q \}$. Then $\mathcal{A}_{p,q}$ is an ideal in Alg$\mathcal{L}$.

3. Properties of ideals of Alg$\mathcal{L}$

Theorem 1. *Let k be a natural number. Then*
(1) $\mathcal{A}_{\{k\}}$ is prime and (2) $\mathcal{A}_{\{k\}}$ is maximal.

Proof. (1) Let $A = (a_{ij})$ and $T = (t_{ij})$ be elements of Alg$\mathcal{L}$. If $(AT)_{(k,k)} = a_{kk}t_{kk} = 0$, then $a_{kk} = 0$ or $t_{kk} = 0$. So $A \in \mathcal{A}_{\{k\}}$ or $T \in \mathcal{A}_{\{k\}}$.
(2) Let $\mathcal{M}$ be an ideal in Alg$\mathcal{L}$ such that $\mathcal{A}_{\{k\}} \subset \mathcal{M} \subset \text{Alg}\mathcal{L}$. Let $\mathcal{A}_{\{k\}} \neq \mathcal{M}$. Then there exists an operator $T = (t_{ij})$ in $\mathcal{M}$ and $T \notin \mathcal{A}_{\{k\}}$, i.e. $T_{(k,k)} \neq 0$. Let $A = (a_{ij}) \in \text{Alg}\mathcal{L}$. If $a_{kk} = 0$, then $A \in \mathcal{A}_{\{k\}}$. Since $\mathcal{A}_{\{k\}} \subset \mathcal{M}$, $A \in \mathcal{M}$. Let $a_{kk} \neq 0$. Let A_1 be an operator defined by

$$\begin{cases} A_{1(k,k)} = 0 \\ A_{1(i,j)} = a_{ij} \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}_{\{k\}}$. Since $\mathcal{A}_{\{k\}} \subset \mathcal{M}$, $A_1 \in \mathcal{M}$. Let T_1 be an operator defined by

$$\begin{cases} T_{1(k,k)} = 0 \\ T_{1(i,j)} = -T_{(i,j)} \text{ otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}_{\{k\}}$. Since $\mathcal{A}_{\{k\}} \subset \mathcal{M}$, $T_1 \in \mathcal{M}$. Put $T_2 = T + T_1$. Then $T_2 \in \mathcal{M}$, $T_{2(k,k)} = T_{(k,k)}$ and $T_{2(i,j)} = 0$ otherwise. Let $\alpha = \frac{a_{kk}}{T_{(k,k)}}$. Then $\alpha T_2 + A_1 = A$ and $A \in \mathcal{M}$. Hence $\mathcal{M} = \text{Alg}\mathcal{L}$. □

Theorem 2. *Let p and q be natural numbers($p < q$). Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\{0\} \subsetneq \mathcal{A} \subset \mathcal{B}_{p,q}$. If $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$, then $T_{(i,j)} = 0, p \leqslant i \leqslant q$ and $i \leqslant j \leqslant q$ for all T in $\mathcal{A}$ $\cdots\cdots\cdots (*)$ i.e. $\mathcal{A} \subset \mathcal{A}_{p,q}$.*

Proof. Let $\mathcal{A}$ be an ideal in Alg$\mathcal{L}$. Let $T \in \mathcal{A}$ and let A be in Alg$\mathcal{L}$. Then $AT \in \mathcal{A}$ and $TA \in \mathcal{A}$.

Since $(AT)_{(p,q)} = A_{(p,p)}T_{(p,q)} + A_{(p,p+1)}T_{(p+1,q)} + \cdots + A_{(p,q-1)}T_{(q-1,q)} + A_{(p,q)}T_{(q,q)} = 0$ for all A in Alg$\mathcal{L}$, $T_{(p,q)} = 0, T_{(p+1,q)} = 0, \cdots, T_{(q-1,q)} = 0, T_{(q,q)} = 0$.

Since $(TA)_{(p,q)} = T_{(p,p)}A_{(p,q)} + T_{(p,p+1)}A_{(p+1,q)} + \cdots + T_{(p,q-1)}A_{(q-1,q)} + T_{(p,q)}A_{(q,q)} = 0$ for all A in Alg$\mathcal{L}$, $T_{(p,p)} = 0, T_{(p,p+1)} = 0, \cdots, T_{(p,q-1)} = 0, T_{(p,q)} = 0$.

Since $(TA)_{(p+1,q)} = T_{(p+1,p+1)}A_{(p+1,q)} + T_{(p+1,p+2)}A_{(p+2,q)} + \cdots + T_{(p+1,q-1)}A_{(q-1,q)} + T_{(p+1,q)}A_{(q,q)} = 0$ for all A in Alg$\mathcal{L}$, $T_{(p+1,p+1)} = 0$, $T_{(p+1,p+2)} = 0, \cdots, T_{(p+1,q-1)} = 0, T_{(p+1,q)} = 0$.

$\cdots\cdots\cdots$

Since $(TA)_{(q-1,q)} = T_{(q-1,q-1)}A_{(q-1,q)} + T_{(q-1,q)}A_{(q,q)} = 0$ for all A in Alg$\mathcal{L}$, $T_{(q-1,q-1)} = 0, T_{(q-1,q)} = 0$. Thus $(*)$ holds. Since $(TA)_{(q,q)} = T_{(q,q)}A_{(q,q)} = 0$ for all A in Alg$\mathcal{L}$, $T_{(q,q)} = 0$.

If $(*)$ holds for all T in $\mathcal{A}$, then $\mathcal{A} \subset \mathcal{A}_{p,q}$. □

Corollary 3. *Let p and q be natural numbers such that $p < q$ and let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{B}_{p,q}$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,q}$.*

Corollary 4. *Let p and q be natural numbers such that $p < q$ and let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{B}^{(1)}{}_{p,q}$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,q}$.*

Corollary 5. *Let p and q be natural numbers such that $p < q$ and let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{B}^{(2)}{}_{p,q}$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,q}$.*

Corollary 6. *Let p and q be natural numbers such that $p < q$ and let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{B}^{(3)}{}_{p,q}$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,q}$.*

Corollary 7. *Let p and q be natural numbers such that $p < q$ and let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{B}^{(4)}{}_{p,q}$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,q}$.*

If we repeat the proof of Theorem 2, then we can prove Theorem 8.

Theorem 8. *Let p a natural number and let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{B}_{p,p+1} = \{T \in Alg\mathcal{L} \mid T_{(p,p+1)} = 0\} \subset \mathcal{A} \subset Alg\mathcal{L}$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = Alg\mathcal{L}$.*

Proof. Let $\mathcal{A}$ be an ideal in Alg$\mathcal{L}$. Since $\mathcal{B}_{p,p+1}$ is not an ideal in Alg$\mathcal{L}$, there exists an operator T in $\mathcal{A}$ but $T \notin \mathcal{B}_{p,p+1}$, i.e. $T_{(p,p+1)} \neq 0$. Let $A \in$ Alg$\mathcal{L}$. If $A_{(p,p+1)} = 0$, then $A \in \mathcal{B}_{p,p+1}$. Since $\mathcal{B}_{p,p+1} \subset \mathcal{A}$, $A \in \mathcal{A}$. Let $A_{(p,p+1)} \neq 0$. Let G be an operator defined by

$$\begin{cases} G_{(p,p+1)} = 0 \\ G_{(i,j)} = -T_{ij} \text{ otherwise.} \end{cases}$$

Then $G \in \mathcal{A}$. Let $T_1 = T + G$. Then $T_1 \in \mathcal{A}$. Let T_2 be an operator defined by

$$\begin{cases} T_{2(p,p+1)} = 0 \\ T_{2(i,j)} = A_{(i,j)} \text{ otherwise.} \end{cases}$$

Then $T_2 \in \mathcal{A}$. Let $x = \frac{A_{(p,p+1)}}{T_{(p,p+1)}}$. Then $A = xT_1 + T_2$ and $A \in \mathcal{A}$. Hence $\mathcal{A} =$ Alg$\mathcal{L}$. □

Theorem 9. *Let p be a natural number and let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,p+1} \subset \mathcal{A} \subset \mathcal{D}_1 = \{T \in Alg\mathcal{L} \mid T_{(p,p+1)} = 0$ and $T_{(p,p)} = 0\}$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,p+1}$.*

Proof. Let $\mathcal{A}$ be an ideal in Alg$\mathcal{L}$. Suppose that $\mathcal{A} \neq \mathcal{A}_{p,p+1}$. Then there exists an operator T in $\mathcal{A}$ and $T \notin \mathcal{A}_{p,p+1}$, i.e. $T_{(p+1,p+1)} \neq 0$. Let $A \in \mathcal{D}_1$. If $A_{(p+1,p+1)} = 0$, then $A \in \mathcal{A}_{p,p+1}$. Since $\mathcal{A}_{p,p+1} \subset \mathcal{A}$, $A \in \mathcal{A}$. Let $A_{(p+1,p+1)} \neq 0$. Let A_1 be an operator defined by

$$\begin{cases} A_{1(p+1,p+1)} = 0 \\ A_{1(i,j)} = -T_{(i,j)} \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}_{p,p+1}$. Since $\mathcal{A}_{p,p+1} \subset \mathcal{A}$, $A_1 \in \mathcal{A}$. Put $T_1 = T + A_1$. Then $T_1 \in \mathcal{A}$. Let T_2 be an operator defined by

$$\begin{cases} T_{2(p+1,p+1)} = 0 \\ T_{2(i,j)} = A_{(i,j)} \text{ otherwise.} \end{cases}$$

Then $T_2 \in \mathcal{A}_{p,p+1}$. Since $\mathcal{A}_{p,p+1} \subset \mathcal{A}$, $T_2 \in \mathcal{A}$. Put $x = \frac{A_{(p+1,p+1)}}{T_{(p+1,p+1)}}$. Then $xT_1 \in \mathcal{A}$, $A = xT_1 + T_2$ and $A \in \mathcal{A}$. So $\mathcal{A} = \mathcal{D}_1$. It is a contradiction. Hence $\mathcal{A} = \mathcal{A}_{p,p+1}$. □

We can prove the following theorem by the similar proof of Theorem 9.

Theorem 10. *Let p be a natural number and let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,p+1} \subset \mathcal{A} \subset \mathcal{D}_2 = \{T \in Alg\mathcal{L} \mid T_{(p,p+1)} = 0$ and $T_{(p+1,p+1)} = 0\}$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,p+1}$.*

We denote $\mathcal{A}_{p,q} \cap \mathcal{A}_0$ by $\mathcal{A}_{p,q}{}^{(0)}$.

Theorem 11. *Let p and q be natural numbers($p < q$). Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,p+1}{}^{(0)} \subset \mathcal{A} \subset \mathcal{A}_0$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,p+1}{}^{(0)}$ or $\mathcal{A} = \mathcal{A}_0$.*

Proof. Let $\mathcal{A}$ be an ideal in Alg$\mathcal{L}$. Assume that $\mathcal{A} \neq \mathcal{A}^{(0)}_{p,p+1}$. Then there exists an operator $T \in \mathcal{A}$ and $T \notin \mathcal{A}^{(0)}{}_{p,p+1}$, i.e. $T_{(p,p+1)} \neq 0$ and $T_{(i,i)} = 0$ for all $i \in \mathbb{N}$. Let $A \in \mathcal{A}_0$. If $A_{(p,p+1)} = 0$, then $A \in \mathcal{A}_{p,p+1}{}^{(0)}$ and so $A \in \mathcal{A}$. Let $A_{(p,p+1)} \neq 0$. Let A_1 be an operator defined by

$$\begin{cases} A_{1(p,p+1)} = 0 \\ A_{1(i,j)} = A_{(i,j)} \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}^{(0)}{}_{p,p+1}$. Since $\mathcal{A}^{(0)}{}_{p,p+1} \subset \mathcal{A}$, $A_1 \in \mathcal{A}$. Define an operator T_1 by

$$\begin{cases} T_{1(p,p+1)} = 0 \\ T_{1(i,j)} = -T_{(i,j)} \text{ otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}^{(0)}{}_{p,p+1}$. Since $\mathcal{A}^{(0)}{}_{p,p+1} \subset \mathcal{A}$, $T_1 \in \mathcal{A}$. Let $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$ and $T_{2(p,p+1)} = T_{(p,p+1)} + T_{1(p,p+1)} = T_{(p,p+1)}$ and $T_{2(i,j)} = 0$ otherwise. Put $x = \frac{A_{(p,p+1)}}{T_{(p,p+1)}}$. Then $xT_2 + A_1 = A$ and so $A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_0$. □

Theorem 12. *Let p and q be natural numbers($p < q$). Then*
i) $\mathcal{A}_{p,p+1} \supset \mathcal{A}_{p,p+2} \supset \mathcal{A}_{p,p+3} \supset \cdots$.

ii) $\mathcal{A}_{p,p+1}{}^{(0)} \supset \mathcal{A}_{p,p+2}{}^{(0)} \supset \mathcal{A}_{p,p+3}{}^{(0)} \supset \cdots$.
iii) $\mathcal{A}_{p,q} \subset \mathcal{A}_{p+1,q} \subset \mathcal{A}_{p+2,q} \subset \cdots \subset \mathcal{A}_{q-1,q}$.
iv) $\mathcal{A}_{p,q}{}^{(0)} \subset \mathcal{A}_{p+1,q}{}^{(0)} \subset \mathcal{A}_{p+2,q}{}^{(0)} \subset \cdots \subset \mathcal{A}^{(0)}{}_{q-1,q}$.
v) $\mathcal{A}_{p,q} \supset \mathcal{A}_{p,q+1} \supset \mathcal{A}_{p,q+2} \supset \cdots$.
vi) $\mathcal{A}_{p,q}{}^{(0)} \supset \mathcal{A}_{p,q+1}{}^{(0)} \supset \mathcal{A}_{p,q+2}{}^{(0)} \supset \cdots$.

Let $\mathcal{A}$ be an ideal in Alg$\mathcal{L}$. Let $X = \{ (p,q) \mid T_{(p,q)} = 0$ for all $T \in \mathcal{A} \}$. Let i, j be natural numbers and let $E_{i\ j}$ be the operator whose (i,j)-component is 1 and all other entries are 0. Let $k \in \mathbb{N}$ and let $n \in \mathbb{N}$. Put $E_n{}^{(k)} = \sum_{i=1}^{n} E_{i\ i+k}$, $E^{(k)} = \sum_{i=1}^{\infty} E_{i\ i+k}$. Then $E_n{}^{(k)} \longrightarrow E^{(k)}$(strongly).

Lemma 13. *Let $\mathcal{A}$ be a strongly closed ideal in Alg$\mathcal{L}$. Assume that $X = \{ (p,q) \mid T_{(p,q)} = 0$ for all $T \in \mathcal{A} \} = \emptyset$. Then $E^{(k)} \in \mathcal{A}$ for all $k \in \mathbb{N}$.*

Proof. Let k be a natural number. Since $X = \emptyset$, there exists $T^{(k,i)} \in \mathcal{A}$ such that $T^{(k,i)}{}_{(i,i+k)} \neq 0$ for each $i \in \mathbb{N}$. Let $T_i' = E_{i\ i}T^{(k,i)}E_{i+k\ i+k} = T^{(k,i)}{}_{(i,i+k)}E_{i\ i+k}$. Then $T_i' \in \mathcal{A}$ because $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$. Since $T^{(k,i)}{}_{(i,i+k)} \neq 0$ and $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$, $E_{i\ i+k} \in \mathcal{A}$ for each $i \in \mathbb{N}$. Since $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$, $E_i{}^{(k)} = \sum_{i=1}^{n} E_{i\ i+k} \in \mathcal{A}$ for all $n \in \mathbb{N}$. Since $E_n{}^{(k)} \longrightarrow E^{(k)}$(strongly) and $\mathcal{A}$ is strongly closed, $E^{(k)} \in \mathcal{A}$ for all $k \in \mathbb{N}$. □

Theorem 14. *Let $\mathcal{A}$ be a strongly closed ideal in Alg$\mathcal{L}$. If $X = \emptyset$, then $\mathcal{A}_0 \subset \mathcal{A}$.*

Proof. Let $A = (a_{ij}) \in \mathcal{A}_0$. Let B be an operator defined by

$$\begin{cases} B_{(1,k)} = 0 (k = 1, 2, \cdots) \\ B_{(i,j)} = a_{i-1\ j} (i = 2, 3, \cdots \ \textit{and}\ j = 1, 2, \cdots). \end{cases}$$

Then $B \in$ Alg$\mathcal{L}$. By Lemma 13, $E^{(1)} = \sum_{i=1}^{\infty} E_{i\ i+1}$ is in $\mathcal{A}$. Hence $E^{(1)}B = A \in \mathcal{A}$. □

Theorem 15. *Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_0 \subset \mathcal{A} \subset \mathcal{A}_1$. Then $\mathcal{A}$ is an ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_0$.*

Proof. Let $\mathcal{A}$ be an ideal in Alg$\mathcal{L}$. Since $\mathcal{A}_1$ is not an ideal in Alg$\mathcal{L}$, there exists an operator T in $\mathcal{A}_1$ such that $T \notin \mathcal{A}$. Let $T_{(i,i)} = \alpha (i = 1, 2, \cdots)$. Let $A \in \mathcal{A}$ and let $A_{(i,i)} = \beta (i = 1, 2, \cdots)$. If $\beta = 0$, then $A \in \mathcal{A}_0$. If $\beta \neq 0$, then $A - (A - \beta I) = \beta I \in \mathcal{A}$. Since $\beta \neq 0$, $I \in \mathcal{A}$. So

$IS = S \in \mathcal{A}$ for all S in Alg$\mathcal{L}$. Hence $\mathcal{A} =$ Alg$\mathcal{L}$. It is a contradiction. So $\beta = 0$ and hence $\mathcal{A} = \mathcal{A}_0$. □

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