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A REMARK ON WEAKLY HYPONORMAL WEIGHTED SHIFTS

An Hyun Kim* and Eun Young Kwon

Abstract. In this note we consider weakly hyponormal weighted shift. In particular, we focus on the weak 4-hyponormality of the weighted shift with the Bergman tail. This is related to the open question of finding a polynomially hyponormal non-subnormal weighted shift.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional complex Hilbert spaces. Let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We also write briefly $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$ and is said to be hyponormal if $T^*T \geq TT^*$. Also an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be subnormal if T has a normal extension, in other words, $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. We can easily check that if T is subnormal then T is hyponormal. We recall that if $\alpha : \alpha_0, \alpha_1, \cdots$ is a bounded sequence of positive numbers (this is called *weights* or *weighted sequence*), then the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for ℓ^2 (cf. [12]). We can easily check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if the weighted sequence $\{\alpha_n\}$ is monotonically increasing. In general it is so hard to check the subnormality of general operators because we

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*Corresponding author

should find an extension of being normal. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j}\langle T^i x_j, T^j x_i\rangle \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2], [3, II.1.9]). It is well known that this is equivalent to the following condition:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{1}$$

We note that the positivity condition (1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([8]).

We recall ([1], [8], [3], [4]) that $T \in \mathcal{B}(\mathcal{H})$ is called *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e., ([8])

$$\left\langle M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix} \right\rangle \geq 0 \quad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}. \tag{3}$$

If $k = 2$, then T is said to be *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{B}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality (cf. [5], [6], [7], [8] and etc)).

In spite of many successful works for weighted sfits, no concrete example of a weighted shift which is polynomially hyponormal, but not subnormal has yet been found (the existence of such weighted shifts was shown in [9] and [10]). In fact, until now, we were unable to get a weighted shift which is weakly 4-hyponormal, but not subnormal. In this note we examine this question.

2. The main result

If $\alpha : \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \cdots, \sqrt{\frac{n}{n+1}}, \cdots$ is a weighted sequence, then W_α is called the Bergman shift. It is well known that the Bergman shift is subnormal. On the other hand, if

$$\alpha : \frac{3}{4}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \cdots \tag{4}$$

is a weight sequence with the Bergman tail then it is known that

(i) W_α is 2-hyponormal (cf. [5]);
(ii) W_α is cubically hyponormal (cf. [11]).

Now we would like to suggset the following:

Conjecture 2.1. *If $\alpha : \frac{3}{4}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \cdots$ is a weight sequence with the Bergman tail then W_α is polynomially hyponormal.*

In this note we examine the weak 4-hyponormality of the above shift (4).

Let W_α be a hyponormal weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$. Let P_n be the orthogonal projection onto the subspace generated by $\{e_0, \cdots, e_n\}$. For $a_1, \cdots, a_{k-1} \in \mathbb{C}$, we write

$$D(a_1, \cdots, a_{k-1}) := [M^*, M]$$

(where $M := W_\alpha + a_1 W_\alpha^2 + \cdots + a_{k-1} W_\alpha^k$) and we let

$$D_n(a_1, \cdots, a_{k-1}) := P_n D(a_1, \cdots, a_{k-1}) P_n.$$

For $i = 1, 2, \cdots; j = 1, 2, \cdots$, define

$$\beta_n(i,j) = \begin{cases} \prod_{p=0}^{j-i-1} \alpha_{n+p}^2 \left(\prod_{p=j-i}^{j-1} \alpha_{n+p}^2 - \prod_{p=1}^{i} \alpha_{n-p}^2 \right)^2 & (j \geq i+1) \\ \prod_{p=n}^{n+i} \alpha_p^2 - \prod_{p=1}^{i} \alpha_{n-p}^2 & (j = i) \\ \beta_n(j,i) & (j < i) \end{cases},$$

and, for notational convenience, $\alpha_{-j} = 0$ for $j \in \mathbb{N}$.

Then we have:

Theorem 2.2. *Let W_α be a hyponormal weighted shift. Then the following are equivalent.*

(i) *W_α is weakly k-hyponormal;*
(ii) *For any $a_1, \cdots, a_{k-1} \in \mathbb{C}$ and $\mathbf{x} = (x_0, x_1, \cdots) \in l^2$,*

$$F(\mathbf{x}, a_1, \cdots, a_{k-1}) \equiv \sum_{i=1}^{k-1} \left\langle \Theta_i \begin{pmatrix} \overline{a_{k-i}} x_0 \\ \overline{a_{k-i+1}} x_1 \\ \vdots \\ \overline{a_{k-1}} x_{i-1} \end{pmatrix}, \begin{pmatrix} \overline{a_{k-i}} x_0 \\ \overline{a_{k-i+1}} x_1 \\ \vdots \\ \overline{a_{k-1}} x_{i-1} \end{pmatrix} \right\rangle$$
$$+ \sum_{i=0}^{\infty} \left\langle \Delta_i \begin{pmatrix} x_i \\ \overline{a_1} x_{i+1} \\ \vdots \\ \overline{a_{k-1}} x_{i+k-1} \end{pmatrix}, \begin{pmatrix} x_i \\ \overline{a_1} x_{i+1} \\ \vdots \\ \overline{a_{k-1}} x_{i+k-1} \end{pmatrix} \right\rangle \geq 0,$$

where Θ_i is an $(i \times i)$ hermitian matrix whose (m,n)- entry, $\Theta_i(m,n)$, is given by

$$\Theta_i(m,n) := \begin{cases} \sqrt{\beta_{m-1}(k-i+m,\ k-i+n)} & (m < n) \\ \beta_{m-1}(k-i+m,\ k-i+m) & (m = n) \end{cases}$$

and Δ_i is a $(k \times k)$ hermitian matrix whose (m,n)-entry, $\Delta_i(m,n)$, is given by

$$\Delta_i(m,n) := \begin{cases} \sqrt{\beta_{i+m-1}(m,n)} & (m < n) \\ \beta_{i+m-1}(m,m) & (m = n) \end{cases}.$$

Proof. Observe

$$\begin{aligned} D_n(a_1,\cdots,a_{k-1}) &= P_n D(a_1,\cdots,a_{k-1})P_n \\ &= P_n \left(\sum_{1\le i,j\le k} \overline{a_{i-1}} a_{j-1} [W_\alpha^{*i},\ W_\alpha^j] \right) P_n \quad (a_0 := 1) \end{aligned}$$

and

$$\begin{aligned} \big\langle [W_\alpha^{*i},\ W_\alpha^j] e_n\ e_n \big\rangle &= \begin{cases} \displaystyle\prod_{p=0}^{j-i-1} \alpha_{n+p} \left(\prod_{p=j-i}^{j-1} \alpha_{n+p}^2 - \prod_{p=1}^{i} \alpha_{n-p}^2 \right) & (j \ge i+1) \\ \displaystyle\prod_{p=n}^{n+j} \alpha_p^2 - \prod_{p=1}^{j} \alpha_{n-p}^2 & (j = i) \end{cases} \\ &= \begin{cases} \sqrt{\beta_n(i,j)} & (j \ge i+1) \\ \beta_n(i,i) & (j=i) \end{cases}. \end{aligned}$$

Thus if $\mathbf{x}_n := (x_0, x_1, \cdots, x_n) \in \mathbb{C}^{n+1}$ then

$$\big\langle P_n [W_\alpha^{*i},\ W_\alpha^j]\, P_n \mathbf{x}_n,\ \mathbf{x}_n \big\rangle = \begin{cases} \displaystyle\sum_{s=0}^{n-j+i} \sqrt{\beta_s(i,j)}\ x_s\, \overline{x_{s+j-i}} & (j \ge i+1) \\ \displaystyle\sum_{s=0}^{n} \beta_s(i,i)\, |x_s|^2 & (j=i). \end{cases}$$

Then a straightforward calculation shows that

$$\langle D_n(a_1,\cdots,a_{k-1})\mathbf{x}_n,\ \mathbf{x}_n \rangle = \left\langle \Omega_n \begin{pmatrix} 1 \\ \overline{a_1} \\ \vdots \\ \overline{a_{k-1}} \end{pmatrix},\ \begin{pmatrix} 1 \\ \overline{a_1} \\ \vdots \\ \overline{a_{k-1}} \end{pmatrix} \right\rangle,$$

where Ω_n is a $(k \times k)$ hermitian matrix whose (i,j)-entry, $\Omega_n(i,j)$, is given by

$$\Omega_n(i,j) = \begin{cases} \displaystyle\sum_{s=0}^{n-j+i} \sqrt{\beta_s(i,j)}\ x_s \overline{x_{s+j-i}} & (j \ge i+1) \\ \displaystyle\sum_{s=0}^{n} \beta_s(i,i)\, |x_s|^2 & (j=i) \end{cases}.$$

Therefore for any $a_1, \cdots, a_{k-1} \in \mathbb{C}$ and $\mathbf{x} \in l^2$,

$$\langle D(a_1, \cdots, a_{k-1})\mathbf{x},\ \mathbf{x}\rangle = \left\langle \Omega \begin{pmatrix} 1 \\ \overline{a_1} \\ \vdots \\ \overline{a_{k-1}} \end{pmatrix}, \begin{pmatrix} 1 \\ \overline{a_1} \\ \vdots \\ \overline{a_{k-1}} \end{pmatrix} \right\rangle,$$

where Ω is a $(k \times k)$ hermitian matrix whose (i,j)-entry $\Omega(i,j)$ is given by

$$\Omega(i,j) = \begin{cases} \sum_{s=0}^{\infty} \sqrt{\beta_s(i,j)}\ x_s \overline{x_{s+j-i}} & (j \geq i+1) \\ \sum_{s=0}^{\infty} \beta_s(i,i)\, |x_s|^2 & (j = i) \end{cases}.$$

Again a direct computation shows that

$$\langle D(a_1, \cdots, a_{k-1})\mathbf{x},\ \mathbf{x}\rangle = F(\mathbf{x}, a_1, \cdots, a_{k-1}).$$

□

If $k = 4$ in Theorem 2.2, we have:

Corollary 2.3. *Let W_α is a hyponormal weighted shift. Then the following are equivalent:*

(i) *W_α is weakly 4-hyponormal;*
(ii) *For any $a_1, a_2, a_3 \in \mathbb{C}$, $\mathbf{x} = (x_0, x_1, \cdots) \in l^2$,*

(5)

$$F(\mathbf{x}, a_1, a_2, a_3) := |a_3|^2\,(\beta_0(4,4))\,|x_0|^2 + \left\langle \Theta_2 \begin{pmatrix} \overline{a_2}x_0 \\ \overline{a_3}x_1 \end{pmatrix}, \begin{pmatrix} \overline{a_2}x_0 \\ \overline{a_3}x_1 \end{pmatrix} \right\rangle$$

$$+ \left\langle \Theta_3 \begin{pmatrix} \overline{a_1}x_0 \\ \overline{a_2}x_1 \\ \overline{a_3}x_2 \end{pmatrix}, \begin{pmatrix} \overline{a_1}x_0 \\ \overline{a_2}x_1 \\ \overline{a_3}x_2 \end{pmatrix} \right\rangle + \sum_{i=0}^{\infty} \left\langle \Delta_i \begin{pmatrix} x_i \\ \overline{a_1}x_{i+1} \\ \overline{a_2}x_{i+2} \\ \overline{a_3}x_{i+3} \end{pmatrix}, \begin{pmatrix} x_i \\ \overline{a_1}x_{i+1} \\ \overline{a_2}x_{i+2} \\ \overline{a_3}x_{i+3} \end{pmatrix} \right\rangle \geq 0$$

where

$$\Theta_2 := \begin{pmatrix} \beta_0(3,3) & \sqrt{\beta_0(3,4)} \\ \sqrt{\beta_0(3,4)} & \beta_1(4,4) \end{pmatrix},$$

$$\Theta_3 := \begin{pmatrix} \beta_0(2,2) & \sqrt{\beta_0(2,3)} & \sqrt{\beta_0(2,4)} \\ \sqrt{\beta_0(2,3)} & \beta_1(3,3) & \sqrt{\beta_1(3,4)} \\ \sqrt{\beta_0(2,4)} & \sqrt{\beta_1(3,4)} & \beta_2(4,4) \end{pmatrix}$$

$$\Delta_i := \begin{pmatrix} \beta_i(1,1) & \sqrt{\beta_i(1,2)} & \sqrt{\beta_i(1,3)} & \sqrt{\beta_i(1,4)} \\ \sqrt{\beta_i(1,2)} & \beta_{i+1}(2,2) & \sqrt{\beta_{i+1}(2,3)} & \sqrt{\beta_{i+1}(2,4)} \\ \sqrt{\beta_i(1,3)} & \sqrt{\beta_{i+1}(2,3)} & \beta_{i+2}(3,3) & \sqrt{\beta_{i+2}(3,4)} \\ \sqrt{\beta_i(1,4)} & \sqrt{\beta_{i+1}(2,4)} & \sqrt{\beta_{i+2}(3,4)} & \beta_{i+3}(4,4) \end{pmatrix}$$
$$(i = 0, 1, \cdots).$$

We also have:

Corollary 2.4. *If* $\alpha : \frac{3}{4}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \cdots$ *is a weight sequence with the Bergman tail then every matrix* Θ_2*,* Θ_3 *and* Δ_i *(*$i = 0, 1, \cdots$*) in the sum (5) is positive semi-definite except for* Δ_1*.*

Proof. Since α has a Bergman tail from the second weight α_1, it follows that $\Delta_2 \geq 0$ for all $i \geq 2$ because Δ_i ($i \geq 2$) is independent of α_0. Now a direct calculation for the remaining matrices gives the result. □

Remark 2.5. *For the weak 4-hyponormality of the weighted shift (4) with the Bergman tail, in view of Corollary 2.4, it will suffice to show that*

(6)

$$B(4) := |a_3|^2 \left(\beta_0(4,4)\right) |x_0|^2 + \left\langle \Theta_2 \begin{pmatrix} \overline{a_2}x_0 \\ \overline{a_3}x_1 \end{pmatrix}, \begin{pmatrix} \overline{a_2}x_0 \\ \overline{a_3}x_1 \end{pmatrix} \right\rangle$$
$$+ \left\langle \Theta_3 \begin{pmatrix} \overline{a_1}x_0 \\ \overline{a_2}x_1 \\ \overline{a_3}x_2 \end{pmatrix}, \begin{pmatrix} \overline{a_1}x_0 \\ \overline{a_2}x_1 \\ \overline{a_3}x_2 \end{pmatrix} \right\rangle + \sum_{i=0}^{4} \left\langle \Delta_i \begin{pmatrix} x_i \\ \overline{a_1}x_{i+1} \\ \overline{a_2}x_{i+2} \\ \overline{a_3}x_{i+3} \end{pmatrix}, \begin{pmatrix} x_i \\ \overline{a_1}x_{i+1} \\ \overline{a_2}x_{i+2} \\ \overline{a_3}x_{i+3} \end{pmatrix} \right\rangle$$

is positive semi-definite. To do so we replace the (2,2)-entry of Θ_2*,* Θ_3*,* Δ_0 *and the (1,1)-entry of* $\Delta_2, \Delta_3, \Delta_4$ *by extremal values so that each determinant of those matrices is zero and the resulting matrix is denoted by* $\widetilde{\Theta_2}$*,* $\widetilde{\Theta_3}$*, and* $\widetilde{\Delta_i}$ *(*$i = 0, 2, 3, 4$*), respectively and the resulting* *remainder is denoted by* $\delta_1|a_3|^2|x_1|^2$, $\delta_2|a_2|^2|x_1|^2$, $\delta_3|a_3|^2|x_1|^2$, $\delta_4|x_2|^2$, $\delta_5|x_3|^2$ *and* $\delta_6|x_4|^2$, *respectively. Then we can write*

$$\begin{aligned} B(4) &= |a_3|^2\left(\beta_0(4,4)\right)|x_0|^2 + \left\langle \widetilde{\Theta_2}\begin{pmatrix}\overline{a_2}x_0\\ \overline{a_3}x_1\end{pmatrix}, \begin{pmatrix}\overline{a_2}x_0\\ \overline{a_3}x_1\end{pmatrix}\right\rangle \\ &+ \left\langle \widetilde{\Theta_3}\begin{pmatrix}\overline{a_1}x_0\\ \overline{a_2}x_1\\ \overline{a_3}x_2\end{pmatrix}, \begin{pmatrix}\overline{a_1}x_0\\ \overline{a_2}x_1\\ \overline{a_3}x_2\end{pmatrix}\right\rangle + \sum_{\substack{1\le i\le 4\\ i\neq 1}} \left\langle \widetilde{\Delta_i}\begin{pmatrix}x_i\\ \overline{a_1}x_{i+1}\\ \overline{a_2}x_{i+2}\\ \overline{a_3}x_{i+3}\end{pmatrix}, \begin{pmatrix}x_i\\ \overline{a_1}x_{i+1}\\ \overline{a_2}x_{i+2}\\ \overline{a_3}x_{i+3}\end{pmatrix}\right\rangle \\ &+ \left\langle \Delta_1^{\delta}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}, \begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}\right\rangle, \end{aligned}$$

where

$$\Delta_1^{\delta} := \begin{pmatrix} \beta_1(1,1)+\delta_1|a_3|^2+\delta_2|a_2|^2+\delta_3|a_1|^2 & \overline{a_1}\sqrt{\beta_1(1,2)} & \overline{a_2}\sqrt{\beta_1(1,3)} & \overline{a_3}\sqrt{\beta_1(1,4)} \\ a_1\sqrt{\beta_1(1,2)} & |a_1|^2\beta_2(2,2)+\delta_4 & a_1\overline{a_2}\sqrt{\beta_2(2,3)} & a_1\overline{a_3}\sqrt{\beta_2(2,4)} \\ a_2\sqrt{\beta_1(1,3)} & \overline{a_1}a_2\sqrt{\beta_2(2,3)} & |a_2|^2\beta_3(3,3)+\delta_5 & a_2\overline{a_3}\sqrt{\beta_3(3,4)} \\ a_3\sqrt{\beta_1(1,4)} & \overline{a_1}a_3\sqrt{\beta_2(2,4)} & \overline{a_2}a_3\sqrt{\beta_3(3,4)} & |a_3|^2\beta_4(4,4)+\delta_6 \end{pmatrix}.$$

So for the weak 4-hyponormality of the weighted shift (4) with the Bergman tail, it suffices to prove that Δ_1^{δ} *is positive semidefinite for any* $a_1, a_2, a_3 \in \mathbb{C}$.

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An Hyun Kim
Department of Mathematics, Changwon National University,
Changwon 641-773, Korea.
E-mail: ahkim@changwon.ac.kr

Eun Young Kwon
Department of Mathematics, Changwon National University,
Changwon 641-773, Korea.
E-mail: key3506@changwon.ac.kr