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EVALUATION OF SOME NEW LAPLACE TRANSFORMS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION ${}_pF_p$

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Abstract. In this paper we aim to demonstrate how one can obtain so far unknown Laplace transforms for the generalized hypergeometric functions ${}_pF_p$ for $p = 2, 3, 4,$ and 5 by employing known summation theorems available in the literature.

1. Introduction and Preliminaries

For nonnegative integers p and q, the generalized hypergeometric function in a variable(argument) z with p numerator parametres $\alpha_1, \cdots, \alpha_p$ and q denominators $\beta_1, \cdots, \beta_q$ is, as usual, defined by means of the hypergeometric series (see, *e.g.*, [13, 14, 15])

$$(1)\quad \begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p \\ \beta_1 \cdots, \beta_q\end{matrix}\,;\ z\right] &= {}_pF_q\left[\alpha_1, \cdots, \alpha_p; \beta_1 \cdots, \beta_q\ ;\ z\right] \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n\,!} \end{aligned}$$

whenever this series converges and elsewhere by analytic continuation. here Γ is the familiar Gamma function and $(.)_m$ stands for the Pochhammer(or shifted factorial) symbol defined for any complex number α and nonnegative integers m by $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. The series defining ${}_pF_q$ converges for all values of z when $p \le q$. If $p = q + 1$, then the series (1) converges when $|z| < 1$, it is absolutely convergent on the unit circle if $\Re(\beta_1 + \cdots + \beta_q - \alpha_1 - \cdots - \alpha_p) > 0$

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and it is convergent on the circle $|z| = 1$ except at $z = 1$ if $-1 < \Re(\beta_1 + \cdots + \beta_q - \alpha_1 - \cdots - \alpha_p) \leq 0$.

We begin with

$$g(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt$$

and so we define the (direct) Laplace transform of a function $f(t)$ of a real variable t as the integral $g(s)$ over a range of the complex parameter s, whenver this integral exists in the Lebesque sense. For more details, for instance, [1] or [2].
By utilizing (1) with $p \leq q$, it is an easy matter to show that the Laplace transform of a generalized hypergeometric function ${}_pF_q$ is

$$\tag{2} \begin{aligned} &\int_0^\infty e^{-st}\, t^{\nu-1}\, {}_pF_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p\\ \beta_1, \cdots, \beta_q\end{matrix}; \omega t\right] dt \\ &\qquad = \Gamma(\nu) s^{-\nu}\, {}_{p+1}F_q\left[\begin{matrix}\nu, \alpha_1, \cdots, \alpha_p\\ \beta_1, \cdots, \beta_q\end{matrix}; \frac{\omega}{s}\right] \end{aligned}$$

provided that when $p < q$, $\Re(\nu) > 0$, $\Re(s) > 0$ and ω is arbitary, or when $p = q > 0$, $\Re(\nu) > 0$, $\Re(s) > \Re(\omega)$. Note that the interchange of order summation and integration when integrating the left-hand side of (2) with respect to t is justified by the uniform convergence of the series (1).
In paticular, when $p = q = 1$, for kummer's(confluent hypergeometric) function (also referred to as the confluent hypergeometric function of the first kind) ${}_1F_1$ (see, for instance, [13], Chapter 7), we conclude that its Laplace transform

$$\tag{3} \int_0^\infty e^{-st}\, t^{b-1}\, {}_1F_1\left[\begin{matrix}a\\ c\end{matrix}; \omega t\right] dt = \Gamma(b) s^{-b} {}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix}; \frac{\omega}{s}\right],$$

is given in terms of the Gauss hypergeometric function ${}_2F_1$, where $\Re(b) > 0$ and $\Re(s) >$ max $(\Re(\omega), 0)$.

In 2012, Kim, *et al.* [6], have obtained explicit expression of

$$\tag{4} \int_0^\infty e^{-st}\, t^{b-1}\, {}_1F_1\left[\begin{matrix}a\\ \frac{1}{2}(a+b+i+1)\end{matrix}; \frac{1}{2}ts\right] dt,$$

$$\int_0^{\infty} e^{-st}\, t^{-a+i}\, {}_1F_1\left[\begin{matrix} a \\ c \end{matrix}; \frac{1}{2}ts\right] dt \tag{5}$$

and

$$\int_0^{\infty} e^{-st}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ 1+a-b+i \end{matrix}; -ts\right] dt \tag{6}$$

for $i = 0, \pm 1, \ldots, \pm 5$.

Further, we take $i = 0$ in (4), (5) and (6), we respectively get

$$\int_0^{\infty} e^{-st}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+1) \end{matrix}; \frac{1}{2}ts\right] dt = s^{-b}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}, \tag{7}$$

$$\int_0^{\infty} e^{-st}\, t^{-a}\, {}_1F_1\left[\begin{matrix} a \\ c \end{matrix}; \frac{1}{2}ts\right] dt = s^{a-1}\frac{\Gamma(1-a)\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}, \tag{8}$$

and

$$\int_0^{\infty} e^{-st}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ 1+a-b \end{matrix}; -ts\right] dt = s^{-b}\frac{2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma(1+a-b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-b\right)} \tag{9}$$

where $\Re(b) > 0$, $\Re(s) > 0$ and $\Re(1-a) > 0$. Results (7) and (8) are recorded in [14], while (9) appeared in [6].
In particular, where $p = q = 2$, for generalized hypergeometric function ${}_2F_2$, we conclude that its Laplace transform

$$\int_0^{\infty} e^{-st}\, t^{\nu-1}\, {}_2F_2\left[\begin{matrix} a,\, b \\ c,\, d \end{matrix}; \omega t\right] dt = \Gamma(\nu) s^{-\nu}\, {}_3F_2\left[\begin{matrix} \nu,\, a,\, b \\ c,\, d \end{matrix}; \frac{\omega}{s}\right], \tag{10}$$

where $\Re(b) > 0$ and $\Re(s) > max(\Re(\omega), 0)$.

Very recently, Kim *et al.* [5] have obtailed explicit expression of

$$\int_0^{\infty} e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1), & 2c+j \end{matrix}; st\right] dt \tag{11}$$

for $i, j = 0, \pm 1, \pm 2$,

$$\int_0^{\infty} e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a, & b \\ 1+a-b+i, & 1+a-c+i+j \end{matrix}; st\right] dt \tag{12}$$

for with $i = 0, \pm 1, \pm 2, \pm 3$ and $j = 0, 1, 2, 3$ and

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ d,\ e \end{matrix}\ ;\ st\right]\, dt \tag{13}$$

with $a + b = 1 + i + j$, $d + e = 2c + 1 + i$ for $i, j = 0, \pm 1, \pm 2, \pm 3$.

Moreover, in (11), (12) and (13) if we take $i = j = 0$, we respectively get

$$\begin{aligned} &\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}\ ;\ st\right]\, dt \\ &= \frac{s^{-c}\, \Gamma\left(\frac{1}{2}\right)\, \Gamma(c)\, \Gamma\left(c+\frac{1}{2}\right)\, \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\, \Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\, \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\, \Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\, \Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \end{aligned} \tag{14}$$

provided $\Re e(2c - a - b) > -1$,

$$\begin{aligned} &\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a, & b \\ 1+a-b, & 1+a-c \end{matrix}\ ;\ st\right]\, dt \\ &= \frac{s^{-c}\, \Gamma(c)\, \Gamma\left(1+\frac{1}{2}a\right)\, \Gamma(1+a-b)\, \Gamma(1+a-c)\, \Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\, \Gamma\left(1+\frac{1}{2}a-b\right)\, \Gamma\left(1+\frac{1}{2}a-c\right)\, \Gamma(1+a-b-c)} \end{aligned} \tag{15}$$

provided $\Re e(a - 2b - 2c) > -2$ and

$$\begin{aligned} &\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ d,\ e \end{matrix}\ ;\ st\right]\, dt \\ &= \frac{s^{-c}\, \pi\, \Gamma(c)\, \Gamma(d)\, \Gamma(e)}{2^{2c-1}\, \Gamma\left(\frac{1}{2}a+\frac{1}{2}d\right)\, \Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\, \Gamma\left(\frac{1}{2}b+\frac{1}{2}d\right)\, \Gamma\left(\frac{1}{2}b+\frac{1}{2}e\right)} \end{aligned} \tag{16}$$

provided $\Re e(c) > 0$ and $\Re e(d + e - a - b - c) > 0$ with $a + b = 1$ and $d + e = 2c + 1$.

Motivated by these work, in our a present investigation, we aim to obtain new laplace transforms of generalized hypergeometric functions ${}_2F_2(x)$, ${}_3F_3(x)$, ${}_4F_4(x)$ and ${}_5F_5(x)$. For this, we shall need the following summation theorems (see [12] and also [10], lemma 1 and 4, [14], p. 243-244, III.10, III.12) given respectively by

$$ {}_3F_2\left[\begin{matrix} f, & a, & c+1 \\ b, & c & \end{matrix}\ ;\ 1\right] = \frac{(c-a)(\alpha-f)}{c}\, \frac{\Gamma(b)\Gamma(b-a-f-1)}{\Gamma(b-a)\Gamma(b-f)} \tag{17}$$

where $\Re(b - a - f) > 1$ and $\alpha = \frac{c(1+a-b)}{a-c}$.

Now set $f = -n$, where n is a nonnegative integer. Hence we give

$$(18) \qquad {}_3F_2\left[\begin{matrix} -n, & a, & c+1 \\ b, & c & \end{matrix}; 1\right] = \frac{(\alpha+n)}{\alpha}\frac{(b-a-1)_n}{(b)_n},$$

$$(19) \qquad {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c & \end{matrix}; -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)},$$

$(\Re(a-2b-2c) > -2)$

$$(20) \qquad {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c & \end{matrix}; 1\right]$$
$$= \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b\right)\left(\frac{1}{2}+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)}.$$

In (20), putting $c = -n$, we obtain

$$(21) \qquad {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & -n \\ \frac{1}{2}a, & 1+a-b, & 1+a+n & \end{matrix}; 1\right] = \frac{(1+a)_n(\frac{1}{2}+\frac{1}{2}a-b)_n}{(1+a-b)_n(\frac{1}{2}+\frac{1}{2}a)_n},$$

$$(22) \qquad {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & -n \\ \frac{1}{2}a, & 1+a-b, & 1+2b-n & \end{matrix}; 1\right] = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n},$$

$$(23) \qquad {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & -n \\ \frac{1}{2}a, & 1+a-b, & 1+2b-n & \end{matrix}; 1\right]$$
$$= \frac{(a-2b-1)_n(\frac{1}{2}a+\frac{1}{2}-b)_n(-b-1)_n}{(1+a-b)_n\,(\frac{1}{2}a-\frac{1}{2}-b)_n(-2b-1)_n},$$

$$(24) \qquad {}_4F_3\left[\begin{matrix} a, & \frac{1}{2}+\frac{1}{2}a, & b+n, & -n \\ \frac{1}{2}b, & \frac{1}{2}b+\frac{1}{2}, & 1+a & \end{matrix}; 1\right] = \frac{(b-a)_n}{(b)_n},$$

$$
(25)\quad {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d & \end{matrix} ; 1\right]
$$
$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} ,
$$

$$
(26)\quad {}_6F_5\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix} ; -1\right]
$$
$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(a)\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(a+c+d)}
$$
$$
\times \frac{\Gamma(1+\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}e)(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}d-\frac{1}{2}c)}{\Gamma(1+\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}d)(\Gamma(1+\frac{1}{2}a-\frac{1}{2}c-\frac{1}{2}e)}
$$

where $1 = b+c = d+e$.

The main objective of this paper is to demonstrate how one can obtain so far unknown Laplace transforms for the generalized hypergeometric function ${}_pF_p$ for $p = 2, 3, 4$ and 5 by employing known summation theorems available in the literature.

2. New class of Laplce transforms of the generalized hypergeometric functions ${}_pF_p$ $(p = 2, 3, 4, 5)$

In (10), if we set respectively $\omega = s$, $\nu = a$, $a = f$, $b = c+1$, and $d = b$, then

(27)
$$
\mathcal{L}\left\{t^{a-1}\,{}_2F_2\left[\begin{matrix} f, c+1 \\ b, c \end{matrix} ; st\right] ; s\right\} = \int_0^\infty e^{-st}\,t^{a-1}\,{}_2F_2\left[\begin{matrix} f, c+1 \\ b, c \end{matrix} ; st\right] dt
$$
$$
= \Gamma(a)s^{-a}{}_3F_2\left[\begin{matrix} f, & a, & c+1 \\ b, & c & \end{matrix} ; 1\right] .
$$

Applying result (17) to (27), we get a new Laplace transform

$$\begin{aligned}\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} a, & c+1 \\ b, & c \end{matrix}; st\right] dt \\ = \frac{(c-a)(\alpha-f)}{c}\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(b-a-f-1)}{s^a\Gamma(b-a)\,\Gamma(b-f)},\end{aligned} \tag{28}$$

where $\Re(b-a-f)>1$ and $\alpha=\frac{c(1+a-b)}{(a-c)}$. Further, letting $f=-n$ in (28), by using (18), we obtain

$$\begin{aligned}\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} -n, & c+1 \\ b, & c \end{matrix}; st\right] dt \\ = \frac{(c-a)(\alpha+n)}{cs^a}\frac{\Gamma(b)\,\Gamma(b-a-1)\,(b-a-1)_n}{\Gamma(b-a)\,(b)_n}.\end{aligned} \tag{29}$$

Indeed, if we set in (2) $p=q=3$, we conclude that its Laplace transform (30)

$$\begin{aligned}\mathcal{L}\left\{t^{d-1}{}_3F_3\left[\begin{matrix} a,\, b,\, c \\ e,\, f,\, g \end{matrix}; \omega t\right]; s\right\} &= \int_0^\infty e^{-st}\, t^{d-1}\, {}_3F_3\left[\begin{matrix} a,\, b,\, c \\ e,\, f,\, g \end{matrix}; \omega t\right] dt \\ &= \Gamma(d)s^{-d}{}_4F_3\left[\begin{matrix} a, & b, & c, & d \\ e, & f, & g & \end{matrix}; \frac{\omega}{s}\right],\end{aligned}$$

where $\Re(d)>0$ and $\Re(s)>\Re(\omega)>0$.

Now, let $\omega=-s$, $d=1+\frac{1}{2}a$, $e=\frac{1}{2}a$, $f=1+a-b$ and $g=1+a-c$ in (30), then, we find

$$\begin{aligned}&\mathcal{L}\left\{t^{\frac{1}{2}a}{}_3F_3\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; -st\right]; s\right\} \\ &= \int_0^\infty e^{-st}\, t^{\frac{1}{2}a}\, {}_3F_3\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; -st\right] dt \\ &= \Gamma(1+\frac{1}{2}a)s^{-(1+\frac{1}{2}a)}{}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b,\, c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; -1\right].\end{aligned} \tag{31}$$

Upon applying the result (19), it is quite easy to see that the following relations can be obtained from the equation (31) :

$$\begin{aligned}
&\mathcal{L}\left\{ t^{\frac{1}{2}a} {}_3F_3\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; -st\right]; s\right\} \\
(32)\qquad &= \int_0^\infty e^{-st}\, t^{\frac{1}{2}a}\, {}_3F_3\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; -st\right] dt \\
&= \sqrt{\pi}\, s^{-(1+\frac{1}{2}a)} \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{2^a\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(1+a-b-c)}.
\end{aligned}$$

where $\Re(a) > 0$ and $\Re(s) > \Re(\omega) > 0$ and $\Re(a-2b-2c) > -1$.

Similarly, we also find

$$\begin{aligned}
&\mathcal{L}\left\{ t^{\frac{1}{2}a} {}_3F_3\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; st\right]; s\right\} \\
(33)\qquad &= \int_0^\infty e^{-st}\, t^{\frac{1}{2}a}\, {}_3F_3\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; st\right] dt \\
&= \Gamma(1+\frac{1}{2}a) s^{-(1+\frac{1}{2}a)} {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b,\ c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; 1\right].
\end{aligned}$$

Using (20) and Legendre's duplication formula (see [13], p.24, Eq. (2)), we obtain the following another Laplace transform

(34)

$$\begin{aligned}
&\mathcal{L}\left\{ t^{\frac{1}{2}a} {}_3F_3\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}; st\right]; s\right\} \\
&= \sqrt{\pi}\, s^{-(1+\frac{1}{2}a)} \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}a-b-c)}{2^a\Gamma(\frac{1}{2}+\frac{1}{2}a-b)\Gamma(\frac{1}{2}+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.
\end{aligned}$$

In (33), if we take $c = -n$, then by using (21), we have

$$\mathcal{L}\left\{ t^{\frac{1}{2}a}{}_3F_3\left[\begin{matrix} a, & b, & -n \\ \frac{1}{2}a, & 1+a-b, & 1+a+n \end{matrix}; st\right]; s\right\} \tag{35}$$

$$= \Gamma(1+\frac{1}{2}a)s^{-(1+\frac{1}{2}a)}\frac{(1+a)_n(\frac{1}{2}+\frac{1}{2}a-b)_n}{(\frac{1}{2}+\frac{1}{2}a)_n(1+a-b)_n}.$$

In (2), setting $p = q = 4$ and applying (24), we obtain its Laplace transform :

(36)

$$\mathcal{L}\left\{ t^{b-1}{}_4F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}; st\right]; s\right\}$$

$$= \Gamma(b)s^{b-1}\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}.$$

Similarly, if we take $p = q = 5$ and applying the summation formula(26), then we get

(37)

$$\mathcal{L}\left\{ t^{b-1}{}_5F_5\left[\begin{matrix} a, \ 1+\frac{1}{2}a, \ c, \ d, \ e \\ \frac{1}{2}a, \ 1+a-b, \ 1+a-c, \ 1+a-d, \ 1+a-e \end{matrix}; -st\right]; s\right\}$$

$$= \Gamma(b)s^{b-1}\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$

$$\times \frac{\Gamma(1+\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}e)}{\Gamma(1+\frac{1}{2}a-\frac{1}{2}c-\frac{1}{2}e)}.$$

where $1 = b + c = d + e$.

3. Concluding remarks.

In this research paper, we have obtained unknown so far Laplace transforms for the generalized hypergeometric functions ${}_pF_q$ for $p = 2, 3, 4$ and 5 by employing known results listed in [10, 14]. These results may be useful in theoretical physics, engineering and applied mathematics.

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