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A GENERALIZATION OF THE EXPONENTIAL INTEGRAL AND SOME ASSOCIATED INEQUALITIES

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Abstract. In this paper, a generalization of the exponential integral is given. As a consequence, several inequalities involving the generalized function are derived. Among other analytical techniques, the procedure utilizes the Hölder's and Minkowski's inequalities for integrals.

1. Introduction

The classical exponential integral, $E_n(x)$ is defined for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}_0$ by [1, p. 228]

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt}\, dt \tag{1}$$

and the a-th derivative of $E_n(x)$ is given by

$$E_n^{(a)}(x) = (-1)^a \int_1^\infty t^{a-n} e^{-xt}\, dt. \tag{2}$$

The function $E_n(x)$ is related to the incomplete Gamma function, $\Gamma(r, x)$ by [5]

$$E_n(x) = x^{n-1}\Gamma(1-n, x).$$

The exponential integral belongs to the class of special functions which have been vigorously studied in recent years. For some new trends in this class of functions, one could refer to [2] and the references therein.

In [7], the author established some inequalities involving the exponential integral and its derivatives. Motivated by the results of [7] and the k-Gamma function defined in [3], the aim of this paper is to give a

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generalization of the exponential integral and to further derive some inequalities for the generalized function. The results are presented in the following sections.

2. Definition and Some Properties

Definition 2.1. *Let $k > 0$, $s \geq 1$, $n \in \mathbb{N}_0$, $a \in \mathbb{N}$ such that $a > n$. Then the (k,s)-generalization or (k,s)-analogue of the exponential integral is defined as*

$$E_{k,s,n}(x) = \int_s^\infty t^{-n} e^{-\frac{xt^k}{k}}\, dt \tag{3}$$

and the a-th derivative of $E_{k,s,n}(x)$ is given by

$$E^{(a)}_{k,s,n}(x) = \left(\frac{-1}{k}\right)^a \int_s^\infty t^{ak-n} e^{-\frac{xt^k}{k}}\, dt. \tag{4}$$

In particular, it follows easily that $E_{1,1,n}(x) = E_n(x)$, $E^{(a)}_{1,1,n}(x) = E^{(a)}_n(x)$ and $E^{(0)}_{k,s,n}(x) \equiv E_{k,s,n}(x)$.

Lemma 2.2. *The following statements are valid for $x > 0$.*

(a) $E_{k,s,n}(x)$ is decreasing.

(b) $E^{(a)}_{k,s,n}(x)$ is positive and decreasing if a is even.

(c) $E^{(a)}_{k,s,n}(x)$ is negative and increasing if a is odd.

(d) $\left|E^{(a)}_{k,s,n}(x)\right|$ is decreasing for all $a \in \mathbb{N}$.

Proof. The proofs of (a), (b) and (c) follow easily from (3) and (4). The proof of (d) is as follows. Let $x \leq y$. Then,

$$\left|E^{(a)}_{k,s,n}(x)\right| - \left|E^{(a)}_{k,s,n}(y)\right| = \frac{1}{k^a}\left[\int_s^\infty t^{ak-n}\left(e^{-\frac{xt^k}{k}} - e^{-\frac{yt^k}{k}}\right) dt\right] \geq 0$$

since e^{-x} is decreasing for $x > 0$. □

Lemma 2.3. *The function $E_{k,s,n}(x)$ is completely monotonic. That is, $(-1)^a E^{(a)}_{k,s,n}(x) \geq 0$ for every $x > 0$ and $a \in \mathbb{N}_0$.*

Proof. By (4), we obtain

$$\begin{aligned}(-1)^a E^{(a)}_{k,s,n}(x) &= (-1)^a \left(\frac{-1}{k}\right)^a \int_s^\infty t^{ak-n} e^{-\frac{xt^k}{k}}\, dt \\ &= \frac{1}{k^a}\int_s^\infty t^{ak-n} e^{-\frac{xt^k}{k}}\, dt \geq 0.\end{aligned}$$

which concludes the proof. □

3. Some Inequalities for the function $E_{k,s,n}(x)$

Theorem 3.1. *Let $k > 0$, $s \geq 1$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $n \in \mathbb{N}_0$. Then, the inequality*

$$E_{k,s,n}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \leq \left(E_{k,s,n}(x)\right)^{\frac{1}{\alpha}} \left(E_{k,s,n}(y)\right)^{\frac{1}{\beta}} \tag{5}$$

holds for $x > 0$ and $y > 0$.

Proof. By (3) we obtain

$$\begin{aligned} E_{k,s,n}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) &= \int_s^\infty t^{-n} e^{-\frac{t^k}{k}\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)}\, dt \\ &= \int_s^\infty t^{-n\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} e^{-\frac{t^k}{k}\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)}\, dt \\ &= \int_s^\infty t^{-\frac{n}{\alpha}} e^{-\frac{xt^k}{\alpha k}} t^{-\frac{n}{\beta}} e^{-\frac{yt^k}{\beta k}}\, dt \\ &\leq \left(\int_s^\infty t^{-n} e^{-\frac{xt^k}{k}}\, dt\right)^{\frac{1}{\alpha}} \left(\int_s^\infty t^{-n} e^{-\frac{yt^k}{k}}\, dt\right)^{\frac{1}{\beta}} \\ &= \left(E_{k,s,n}(x)\right)^{\frac{1}{\alpha}} \left(E_{k,s,n}(y)\right)^{\frac{1}{\beta}}. \end{aligned}$$

□

Theorem 3.2. *Let $k > 0$, $s \geq 1$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $m, n \in \mathbb{N}_0$ such that $\alpha m, \beta n \in \mathbb{N}_0$. Then, the inequality*

$$E_{k,s,m+n}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \leq \left(E_{k,s,\alpha m}(x)\right)^{\frac{1}{\alpha}} \left(E_{k,s,\beta n}(y)\right)^{\frac{1}{\beta}} \tag{6}$$

holds for $x > 0$ and $y > 0$.

Proof. By (3) we obtain

$$\begin{aligned} E_{k,s,m+n}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) &= \int_s^\infty t^{-(m+n)} e^{-\frac{t^k}{k}\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)}\, dt \\ &= \int_s^\infty t^{-m} e^{-\frac{xt^k}{\alpha k}} t^{-n} e^{-\frac{yt^k}{\beta k}}\, dt \\ &\leq \left(\int_s^\infty t^{-\alpha m} e^{-\frac{xt^k}{k}}\, dt\right)^{\frac{1}{\alpha}} \left(\int_s^\infty t^{-\beta n} e^{-\frac{yt^k}{k}}\, dt\right)^{\frac{1}{\beta}} \\ &= \left(E_{k,s,\alpha m}(x)\right)^{\frac{1}{\alpha}} \left(E_{k,s,\beta n}(y)\right)^{\frac{1}{\beta}}. \end{aligned}$$

□

Corollary 3.3. *Let $k > 0$, $s \geq 1$ and $m, n \in \mathbb{N}_0$. Then, the inequality*

$$(E_{k,s,m+n}(x))^2 \leq E_{k,s,2m}(x).E_{k,s,2n}(x)$$

holds for $x > 0$.

Proof. This follows from Theorem 3.2 by setting $x = y$ and $\alpha = \beta = 2$. □

Remark 3.4. *Theorem 3.2 generalizes the result of Theorem 2.2 of* [7].

Theorem 3.5. *Let $k > 0$, $s \geq 1$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $m, n \in \mathbb{N}_0$ such that $\frac{m}{\alpha} + \frac{n}{\beta} \in \mathbb{N}_0$. Then, the inequality*

$$E_{k,s,\frac{m}{\alpha}+\frac{n}{\beta}}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \leq (E_{k,s,m}(x))^{\frac{1}{\alpha}} (E_{k,s,n}(y))^{\frac{1}{\beta}} \tag{7}$$

holds for $x > 0$ and $y > 0$.

Proof. We proceed as follows.

$$\begin{aligned} E_{k,s,\frac{m}{\alpha}+\frac{n}{\beta}}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) &= \int_s^\infty t^{-(\frac{m}{\alpha}+\frac{n}{\beta})} e^{-\frac{t^k}{k}\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)} \, dt \\ &= \int_s^\infty t^{-\frac{m}{\alpha}} e^{-\frac{xt^k}{\alpha k}} t^{-\frac{n}{\beta}} e^{-\frac{yt^k}{\beta k}} \, dt \\ &\leq \left(\int_s^\infty t^{-m} e^{-\frac{xt^k}{k}} \, dt\right)^{\frac{1}{\alpha}} \left(\int_s^\infty t^{-n} e^{-\frac{yt^k}{k}} \, dt\right)^{\frac{1}{\beta}} \\ &= (E_{k,s,m}(x))^{\frac{1}{\alpha}} (E_{k,s,n}(y))^{\frac{1}{\beta}} . \end{aligned}$$

□

Remark 3.6. *By letting $k = s = 1$ and $x = y$ in Theorem 3.5, we obtain the result of Theorem 4.1 of* [6]. *If in addition, $\alpha = \beta = 2$, then we obtain the result*

$$E_{\frac{m+n}{2}}(x) \leq E_m(x)E_n(x)$$

which was mentioned in [4].

Theorem 3.7. *Let $k > 0$, $s \geq 1$, $n \in \mathbb{N}_0$ and $x, y, \alpha > 1$ such that $\frac{1}{x} + \frac{1}{y} \leq 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then, the inequality*

$$E_{k,s,n}(xy) \leq (E_{k,s,n}(\alpha x))^{\frac{1}{\alpha}} (E_{k,s,n}(\beta y))^{\frac{1}{\beta}} \tag{8}$$

is valid.

Proof. From the hypothesis, we have $x+y \leq xy$. Then, since $E_{k,s,n}(x)$ is decreasing for $x>0$, we obtain

$$\begin{aligned} E_{k,s,n}(xy) \leq E_{k,s,n}(x+y) &= \int_s^\infty t^{-n\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} e^{-\frac{t^k}{k}(x+y)}\,dt \\ &= \int_s^\infty t^{-\frac{n}{\alpha}} e^{-\frac{xt^k}{k}} t^{-\frac{n}{\beta}} e^{-\frac{yt^k}{k}}\,dt \\ &\leq \left(\int_s^\infty t^{-n} e^{-\frac{\alpha x t^k}{k}}\,dt\right)^{\frac{1}{\alpha}} \left(\int_s^\infty t^{-n} e^{-\frac{\beta y t^k}{k}}\,dt\right)^{\frac{1}{\beta}} \\ &= \left(E_{k,s,n}(\alpha x)\right)^{\frac{1}{\alpha}} \left(E_{k,s,n}(\beta y)\right)^{\frac{1}{\beta}}. \end{aligned}$$

□

Remark 3.8. *Theorem 3.7 generalizes the result of Theorem 2.4 of* [7].

Theorem 3.9. *Let $k>0$, $s\geq 1$, $m,n\in\mathbb{N}_0$ and $u\in\mathbb{Z}^+$. Then, the inequality*

$$[E_{k,s,m}(x)+E_{k,s,n}(y)]^{\frac{1}{u}} \leq [E_{k,s,m}(x)]^{\frac{1}{u}} + [E_{k,s,n}(y)]^{\frac{1}{u}} \tag{9}$$

is valid for $x>0$ and $y>0$.

Proof. Here we employ the Minkowski's inequality for integrals, and the fact that $a^u+b^u \leq (a+b)^u$, for $a,b\geq 0$ and $u\in\mathbb{Z}^+$. We proceed as follows.

$$\begin{aligned} [E_{k,s,m}(x)+E_{k,s,n}(y)]^{\frac{1}{u}} &= \left[\int_s^\infty t^{-m} e^{-\frac{xt^k}{k}}\,dt + \int_s^\infty t^{-n} e^{-\frac{yt^k}{k}}\,dt\right]^{\frac{1}{u}} \\ &= \left[\int_s^\infty \left(\left(t^{-\frac{m}{u}} e^{-\frac{xt^k}{ku}}\right)^u + \left(t^{-\frac{n}{u}} e^{-\frac{yt^k}{ku}}\right)^u\right) dt\right]^{\frac{1}{u}} \\ &\leq \left[\int_s^\infty \left(\left(t^{-\frac{m}{u}} e^{-\frac{xt^k}{ku}}\right) + \left(t^{-\frac{n}{u}} e^{-\frac{yt^k}{ku}}\right)\right)^u dt\right]^{\frac{1}{u}} \\ &\leq \left[\int_s^\infty t^{-m} e^{-\frac{xt^k}{k}}\,dt\right]^{\frac{1}{u}} + \left[\int_s^\infty t^{-n} e^{-\frac{yt^k}{k}}\,dt\right]^{\frac{1}{u}} \\ &= [E_{k,s,m}(x)]^{\frac{1}{u}} + [E_{k,s,n}(y)]^{\frac{1}{u}}. \end{aligned}$$

□

4. Some Inequalities for the function $E^{(a)}_{k,s,n}(x)$

Theorem 4.1. *Let $k > 0$, $s \geq 1$, $n \in \mathbb{N}_0$ and $a \in \mathbb{N}$. Then for $x > 0$ and $y > 0$, the following inequalities are valid.*

$$E^{(a)}_{k,s,n}(x+y) \leq E^{(a)}_{k,s,n}(x) + E^{(a)}_{k,s,n}(y) \tag{10}$$

if a is even, and

$$E^{(a)}_{k,s,n}(x+y) \geq E^{(a)}_{k,s,n}(x) + E^{(a)}_{k,s,n}(y) \tag{11}$$

if a is odd.

Proof. Let a be even and $\phi(x) = E^{(a)}_{k,s,n}(x+y) - E^{(a)}_{k,s,n}(x) - E^{(a)}_{k,s,n}(y)$. Then for a fixed y, we obtain

$$\begin{aligned}
\phi'(x) &= E^{(a+1)}_{k,s,n}(x+y) - E^{(a+1)}_{k,s,n}(x) \\
&= -\frac{1}{k^{a+1}} \int_s^\infty t^{(a+1)k-n} \left(e^{-\frac{(x+y)t^k}{k}} - e^{-\frac{xt^k}{k}} \right) dt \\
&\geq 0.
\end{aligned}$$

Thus, $\phi(x)$ is increasing. Further,

$$\begin{aligned}
\lim_{x\to\infty} \phi(x) &= \lim_{x\to\infty} \left[E^{(a)}_{k,s,n}(x+y) - E^{(a)}_{k,s,n}(x) - E^{(a)}_{k,s,n}(y) \right] \\
&= \left(\frac{-1}{k} \right)^a \lim_{x\to\infty} \left[\int_s^\infty t^{ak-n} \left(e^{-\frac{(x+y)t^k}{k}} - e^{-\frac{xt^k}{k}} - e^{-\frac{yt^k}{k}} \right) dt \right] \\
&= -\left(\frac{-1}{k} \right)^a \int_s^\infty t^{ak-n} e^{-\frac{yt^k}{k}}\, dt \\
&= -\frac{1}{k^a} \int_s^\infty t^{ak-n} e^{-\frac{yt^k}{k}}\, dt \\
&\leq 0.
\end{aligned}$$

Therefore, $\phi(x) \leq 0$ yielding the result (10). Next, suppose a is odd. Then by the same technique, we obtain $\phi'(x) \leq 0$ and $\lim_{x\to\infty} \phi(x) \geq 0$ implying that $\phi(x) \geq 0$ which yields the result (11). □

Remark 4.2. *Theorem 4.1 generalizes and extends the result of Theorem 2.3 of* [7].

Theorem 4.3. *Let $k > 0$, $s \geq 1$, $n \in \mathbb{N}_0$ and $x, y, \alpha > 1$ such that $\frac{1}{x} + \frac{1}{y} \leq 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then, the inequality*

$$\left| E^{(a)}_{k,s,n}(xy) \right| \leq \left| E^{(a)}_{k,s,n}(\alpha x) \right|^{\frac{1}{\alpha}} \left| E^{(a)}_{k,s,n}(\beta y) \right|^{\frac{1}{\beta}} \tag{12}$$

is holds for $a \in \mathbb{N}$.

Proof. Note that $x+y \le xy$ from the hypothesis. Then since $\left|E_{k,s,n}^{(a)}(x)\right|$ is decreasing for $x > 0$, we obtain

$$\begin{aligned}
\left|E_{k,s,n}^{(a)}(xy)\right| \le \left|E_{k,s,n}^{(a)}(x+y)\right| &= \frac{1}{k^a}\int_s^\infty t^{ak-n} e^{-\frac{t^k}{k}(x+y)}\, dt \\
&= \left(\frac{1}{k^a}\right)^{\frac{1}{\alpha}+\frac{1}{\beta}} \int_s^\infty t^{\frac{ak-n}{\alpha}} e^{-\frac{xt^k}{k}} t^{\frac{ak-n}{\beta}} e^{-\frac{yt^k}{k}}\, dt \\
&\le \left(\frac{1}{k^a}\int_s^\infty t^{ak-n} e^{-\frac{\alpha x t^k}{k}}\, dt\right)^{\frac{1}{\alpha}} \left(\frac{1}{k^a}\int_s^\infty t^{ak-n} e^{-\frac{\beta y t^k}{k}}\, dt\right)^{\frac{1}{\beta}} \\
&= \left|E_{k,s,n}^{(a)}(\alpha x)\right|^{\frac{1}{\alpha}} \left|E_{k,s,n}^{(a)}(\beta y)\right|^{\frac{1}{\beta}}.
\end{aligned}$$

□

Theorem 4.4. *Let* $k > 0$, $s \ge 1$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta} = 1$, $a \in \mathbb{N}$ *and* $m, n \in \mathbb{N}_0$ *such that* $\alpha m, \beta n \in \mathbb{N}_0$. *Then, the inequality*

$$\left|E_{k,s,m+n}^{(a)}\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)\right| \le \left|E_{k,s,\alpha m}^{(a)}(x)\right|^{\frac{1}{\alpha}} \left|E_{k,s,\beta n}^{(a)}(y)\right|^{\frac{1}{\beta}} \tag{13}$$

is holds for $x, y > 0$.

Proof. We proceed as follows.

$$\begin{aligned}
\left|E_{k,s,m+n}^{(a)}\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)\right| &= \frac{1}{k^a}\int_s^\infty t^{ak-(m+n)} e^{-\frac{t^k}{k}\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)}\, dt \\
&= \left(\frac{1}{k^a}\right)^{\frac{1}{\alpha}+\frac{1}{\beta}} \int_s^\infty t^{\frac{ak}{\alpha}-m} e^{-\frac{xt^k}{\alpha k}} t^{\frac{ak}{\beta}-n} e^{-\frac{yt^k}{\beta k}}\, dt \\
&\le \left(\frac{1}{k^a}\int_s^\infty t^{ak-\alpha m} e^{-\frac{xt^k}{k}}\, dt\right)^{\frac{1}{\alpha}} \\
&\quad \times \left(\frac{1}{k^a}\int_s^\infty t^{ak-\beta n} e^{-\frac{yt^k}{k}}\, dt\right)^{\frac{1}{\beta}} \\
&= \left|E_{k,s,\alpha m}^{(a)}(x)\right|^{\frac{1}{\alpha}} \left|E_{k,s,\beta n}^{(a)}(y)\right|^{\frac{1}{\beta}}.
\end{aligned}$$

□

Corollary 4.5. *Let* $k > 0$, $s \ge 1$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta} = 1$, $a \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$. *Then, the inequality*

$$\left|E_{k,s,n}^{(a)}\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)\right| \le \left|E_{k,s,n}^{(a)}(x)\right|^{\frac{1}{\alpha}} \left|E_{k,s,n}^{(a)}(y)\right|^{\frac{1}{\beta}} \tag{14}$$

is holds for $x, y > 0$.

Proof. This follows from Theorem 4.4 by replacing m and n by $\frac{n}{\alpha}$ and $\frac{n}{\beta}$ respectively. □

Theorem 4.6. *Let $k > 0$, $s \geq 1$, $m, n \in \mathbb{N}_0$, $a, u \in \mathbb{N}$ such that $a > m, n$. Then, the inequality*

$$\left(\left|E^{(a)}_{k,s,m}(x)\right| + \left|E^{(a)}_{k,s,n}(y)\right|\right)^{\frac{1}{u}} \leq \left|E^{(a)}_{k,s,m}(x)\right|^{\frac{1}{u}} + \left|E^{(a)}_{k,s,n}(y)\right|^{\frac{1}{u}} \tag{15}$$

holds for $x > 0$ and $y > 0$.

Proof. We proceed as follows.

$$\begin{aligned}
&\left(\left|E^{(a)}_{k,s,m}(x)\right| + \left|E^{(a)}_{k,s,n}(y)\right|\right)^{\frac{1}{u}} \\
&= \left(\frac{1}{k^a}\int_s^\infty t^{ak-m}e^{-\frac{xt^k}{k}}\,dt + \frac{1}{k^a}\int_s^\infty t^{ak-n}e^{-\frac{yt^k}{k}}\,dt\right)^{\frac{1}{u}} \\
&= \left(\frac{1}{k^a}\right)^{\frac{1}{u}}\left(\int_s^\infty \left[\left(t^{\frac{ak-m}{u}}e^{-\frac{xt^k}{ku}}\right)^u + \left(t^{\frac{ak-n}{u}}e^{-\frac{yt^k}{ku}}\right)^u\right]dt\right)^{\frac{1}{u}} \\
&\leq \left(\frac{1}{k^a}\right)^{\frac{1}{u}}\left(\int_s^\infty \left[\left(t^{\frac{ak-m}{u}}e^{-\frac{xt^k}{ku}}\right) + \left(t^{\frac{ak-n}{u}}e^{-\frac{yt^k}{ku}}\right)\right]^u dt\right)^{\frac{1}{u}} \\
&\leq \left(\frac{1}{k^a}\int_s^\infty t^{ak-m}e^{-\frac{xt^k}{k}}\,dt\right)^{\frac{1}{u}} + \left(\frac{1}{k^a}\int_s^\infty t^{ak-n}e^{-\frac{yt^k}{k}}\,dt\right)^{\frac{1}{u}} \\
&= \left|E^{(a)}_{k,s,m}(x)\right|^{\frac{1}{u}} + \left|E^{(a)}_{k,s,n}(y)\right|^{\frac{1}{u}}.
\end{aligned}$$

□

Theorem 4.7. *Let $k > 0$, $s \geq 1$, $n \in \mathbb{N}_0$, $a \in \mathbb{N}$ and $\beta \geq 1$. Then, the inequalities*

$$\left(\exp E^{(a)}_{k,s,n}(x)\right)^{\beta} \geq \exp E^{(a+1)}_{k,s,n}(y). \exp E^{(a-1)}_{k,s,n}(y), \text{ if } a \text{ is even} \tag{16}$$

$$\left(\exp E^{(a)}_{k,s,n}(x)\right)^{\beta} \leq \exp E^{(a+1)}_{k,s,n}(y). \exp E^{(a-1)}_{k,s,n}(y), \text{ if } a \text{ is odd} \tag{17}$$

are satisfied for $x > 0$.

Proof. We proceed as follows.

$$\begin{aligned}&E^{(a)}_{k,s,n}(x) - E^{(a+1)}_{k,s,n}(x) - E^{(a-1)}_{k,s,n}(x)\\&= \left(\frac{-1}{k}\right)^a \int_s^\infty \left(t^{ak-n} + \frac{t^{(a+1)k-n}}{k} + kt^{(a-1)k-n}\right) e^{-\frac{xt^k}{k}}\, dt\\&\geq (\leq) 0\end{aligned}$$

respectively for even(odd) a. This implies,

$$E^{(a)}_{k,s,n}(x) \geq E^{(a+1)}_{k,s,n}(x) + E^{(a-1)}_{k,s,n}(x)$$

and

$$E^{(a)}_{k,s,n}(x) \leq E^{(a+1)}_{k,s,n}(x) + E^{(a-1)}_{k,s,n}(x)$$

respectively for even and odd a. Then for $\beta \geq 1$, we obtain

$$\beta E^{(a)}_{k,s,n}(x) \geq E^{(a)}_{k,s,n}(x) \geq E^{(a+1)}_{k,s,n}(x) + E^{(a-1)}_{k,s,n}(x)$$

and

$$\beta E^{(a)}_{k,s,n}(x) \leq E^{(a)}_{k,s,n}(x) \leq E^{(a+1)}_{k,s,n}(x) + E^{(a-1)}_{k,s,n}(x)$$

respectively. Finally by exponentiation, we obtain the inequalities (16) and (17). □

Theorem 4.8. *Let $k > 0$, $s \geq 1$, $n \in \mathbb{N}_0$, $a, \beta \in \mathbb{N}$ and $x_i > 0$ for each $i = 1, 2, \ldots, \beta$. Then the inequality*

$$\prod_{i=1}^{\beta} E^{(a)}_{k,s,n}(x_i) \geq \left[E^{(a)}_{k,s,n}\left(\sum_{i=1}^{\beta} x_i\right)\right]^{\beta} \tag{18}$$

holds if a is even.

Proof. Suppose that a is even. Then

$$\begin{aligned}E^{(a)}_{k,s,n}(x_1) - E^{(a)}_{k,s,n}\left(\sum_{i=1}^{\beta} x_i\right) &= \frac{1}{k^a} \int_s^\infty t^{ak-n} \left(e^{-\frac{t^k}{k}x_1} - e^{-\frac{t^k}{k}\sum_{i=1}^{\beta} x_i}\right) dt\\&\geq 0.\end{aligned}$$

Hence,

$$E^{(a)}_{k,s,n}(x_1) \geq E^{(a)}_{k,s,n}\left(\sum_{i=1}^{\beta} x_i\right) > 0.$$

Proceeding in this manner, we obtain the following.

$$\begin{aligned} E^{(a)}_{k,s,n}(x_2) &\geq E^{(a)}_{k,s,n}\left(\sum_{i=1}^{\beta} x_i\right) > 0, \\ E^{(a)}_{k,s,n}(x_3) &\geq E^{(a)}_{k,s,n}\left(\sum_{i=1}^{\beta} x_i\right) > 0, \\ &\vdots \qquad \vdots \\ E^{(a)}_{k,s,n}(x_\beta) &\geq E^{(a)}_{k,s,n}\left(\sum_{i=1}^{\beta} x_i\right) > 0. \end{aligned}$$

Then multiplying these inequalities yields,

$$\prod_{i=1}^{\beta} E^{(a)}_{k,s,n}(x_i) \geq \left[E^{(a)}_{k,s,n}\left(\sum_{i=1}^{\beta} x_i\right)\right]^{\beta}$$

which concludes the proof. □

Remark 4.9. *In particular, by letting $\beta = 2$, $x_1 = x$ and $x_2 = y$ in Theorem 4.8, we obtain the inequality*

$$E^{(a)}_{k,s,n}(x)E^{(a)}_{k,s,n}(y) \geq \left[E^{(a)}_{k,s,n}(x+y)\right]^2.$$

5. Conclusion

In this study, a generalization of the exponential integral has been given and some basic monotonicity properties discussed. As applications, some interesting inequalities involving the generalized function are derived.

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