J. Chosun Natural Sci.
Vol. 10, No. 1 (2017) pp. 40 – 42
https://doi.org/10.13160/ricns.2017.10.1.40

On Some Polynomials with Weighted Sums

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Abstract

Abstract. In this note, we study a generalization of a certain polynomial $z^n - \sum_{k=0}^{n-1} a_k z^k$, where $\sum_{k=0}^{n-1} a_k = 1$, $a_k \geq 0$ for each k, whose all zeros except for $z=1$ lie on the circle of radius $1/2$ with center at the origin.

Keywords: Polynomial, Weighted Sum

1. Introduction

Throughout this paper, n is an integer ≥ 3, $p > 1$, and we denote $C(r)$ by the circle of radius r with center at the origin. All polynomials in this paper will be assumed to have real coefficients. It follows from Eneström-Kakeya theorem for the statement and its proof[1] to

$$\frac{z^n - \sum_{k=0}^{n-1} a_k z^k}{z-1} \tag{1}$$

where $\sum_{k=0}^{n-1} a_k = 1$, $a_k \geq 0$ for each k that all zeros of (1) do not lie outside $C(1)$. Kim[2] studied polynomials of type (1),

$$z^n - \sum_{k=0}^{n-1} a_k z^k,$$

whose all zeros except for $z=1$ lie on $C(1/p)$, where $p > 1$. For convenience, we call these polynomials $C(1/p)$-polynomials, and $\sum_{k=0}^{n-1} a_k z^k$ their weighted sums, respectively. Kim[2] showed that, given $p > 1$, there exist $C(1/p)$-polynomials whose degree of weighted sum is $n-1$. However, by estimating some coefficients of lacunary polynomials, he obtained sufficient conditions for nonexistence of certain lacunary $C(1/p)$-polynomials. Perhaps the most basic example of $C(1/2)$-polynomials is

$$z^{2n+1} - \frac{1}{2^{2n}}\left(1 + \sum_{k=1}^{2n} 2^{k-1} z^k\right) \tag{2}$$

For this, see Proposition 1 of [2]. In this paper, we study a generalization

$$p(z) = z^{2n+1} - \frac{1}{2^{2n}}\left(1 + \sum_{k=1}^{2n} 2^{k-1} z^k + tz^n - tz^{n+1}\right)$$

of the polynomial (2).

2. Results and Questions

The polynomial

$$p(z) = z^{2n+1} - \frac{1}{2^{2n}}\left(1 + \sum_{k=1}^{2n} 2^{k-1} z^k + tz^n - tz^{n+1}\right)$$

can be computed by

$$p(z) = \frac{1}{2^{2n}(2z-1)}(z-1)(2^{2n+1} z^{2n+1} + 2tz^{n+1} - tz^n - 1). \tag{3}$$

For rare choices of t, the polynomial $p(z)$ are nicely factored. For example,

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(Received: January 18, 2017, Accepted: March 25, 2017)

$$p(z)=\begin{cases}\frac{1}{2^{2n}(2z-1)}(z-1)((2z)^n+1)((2z)^{n+1}-1), & t=2^n\\ \frac{1}{2^{2n}(2z-1)}(z-1)((2z)^n-1)((2z)^{n+1}+1), & t=-2^n\end{cases}$$

Less nice examples are when $t=-2^{2n+1}+1$,

$$p(x)=\frac{1}{2^{2n}(2z-1)}(z-1)^2\big(2^{2n+1}(z^{2n}+z^{2n-1}+\cdots+z^{n+1})+(2-2^{2n+1})z^n+z^{n-1}+\cdots+z+1\big)$$

and when $t=-2^n(2n+1)$,

$$p(z)=\frac{1}{2^{2n+1}}\left(z-\frac{1}{2}\right)^2 u(z),$$

where

$$\begin{aligned}u(z)&=2^{2n+1}z^{2n-2}+\left(\sum_{k=1}^{2}k\right)2^{2n}z^{2n-3}+\left(\sum_{k=1}^{3}k\right)2^{2n-1}z^{2n-4}\\&+\left(\sum_{k=1}^{4}k\right)2^{2n-2}z^{2n-5}+\cdots+\left(\sum_{k=1}^{n}k\right)2^{n+2}z^{n-1}\\&+\left(\sum_{k=1}^{n-1}k\right)2^{n+1}z^{n-2}+\left(\sum_{k=1}^{n-2}k\right)2^{n}z^{n-3}+\cdots\\&+\left(\sum_{k=1}^{3}k\right)2^5z^2+\left(\sum_{k=1}^{2}k\right)2^4z+2^3.\end{aligned}$$

A polynomial $P(z)$ of degree n is said to be self-reciprocal if it satisfies $P(z)=z^nP(1/z)$. The zeros of a self-reciprocal polynomial either lie on $C(1)$ or occur in pairs conjugate to $C(1)$. Cohn obtained a sufficient condition for a self-reciprocal polynomial $P(z)$ to have all its zeros on $C(1)$; if all zeros of $P'(z)$ lie in $|z|\le 1$, then all zeros of $P(z)$ lie on $C(1)$. For this[3], see p. 230 of [3]. Using this and Eneström-Kakeya theorem, we can prove the following.

Proposition 1 If $-2^n/n\le t\le 2^n/n$, then $p(z)$ has all its zeros except for 1 lying on $C(1/2)$.

Proof Let

$$f(z)=2^{2n+1}z^{2n+1}+2tz^{n+1}-tz^n-1$$

that is the last factor of $p(z)$ in (3). Observe that $f(1/2)=0$, and for $z\ne 1/2,1$, the zeros of the polynomial $p(z)$ satisfy $f(z)=0$. Assume that $z\ne 1/2,1$. Then

$$\begin{aligned}p(z)=0&\Leftrightarrow 2^{2n+1}z^{2n+1}=2+\sum_{k=1}^{2n}2^kz^k+2tz^n-2tz^{n+1}\\&\Leftrightarrow 2^{2n+1}z^{2n+1}-2tz^n+2tz^{n+1}-1=1+\sum_{k=1}^{2n}2^kz^k\\&\Leftrightarrow -tz^n=1+\sum_{k=1}^{2n}2^kz^k\Leftrightarrow 1+\sum_{k=1}^{2n}2^kz^k+tz^n=0\end{aligned}$$

Let $g(z)=1+\sum_{k=1}^{2n}2^kz^k+tz^n$. Changing variable $y=2z$, we have

$$\begin{aligned}g(y)&=1+y+y^2+\cdots+y^{n-1}\\&+(1+t/2^n)y^n+y^{n+1}+\cdots+y^{2n}\end{aligned}$$

that is self-reciprocal. Then

$$\begin{aligned}g'(y)&=2ny^{2n-1}+(2n-1)y^{2n-2}+\cdots\\&+(n+1)y^n+n(1+t/2^n)y^{n-1}\\&+(n-1)y^{n-2}+\cdots+2y+1.\end{aligned}$$

So if $n+1\ge n(1+t/2^n)\ge n-1$, i.e. $-2^n/n\le t\le 2^n/n$, then by Eneström-Kakeya theorem and Cohn's theorem, $g(y)$ has all its zeros on $C(1)$, which implies the result. □

Remark 2. By Proposition 1, the zeros of $p(z)$ lie on $C(1/2)$ for a wide range of values of t. But it follows from numerical computations that for t sufficiently large, the zeros start to leave $C(1/2)$. But it seems that $p(z)$ with large t mostly form pairs of zeros α, β such that $\sqrt{|\alpha\beta|}=1/2$. Thus they “remember” the circle $C(1/2)$.

The polynomial $p(z)$ seems to have the discriminant with three nice factors. Perhaps the discriminant has only real zeros. More specifically, we conjecture the following.

Conjecture 3 The discriminant of the polynomial $p(z)$ is

$$\Delta_z(p(z))=(-1)^{\frac{n(n+1)}{2}}\frac{1}{2^{4n\left(n+\left[\frac{n+1}{2}\right]\right)}}(t+(-1)^n2^n)(t+2^n(2n+1))(t+(2^{2n+1}-1))^2a(t)^2,$$

where $a(t)$ is a polynomial of degree $n-1$ with integer coefficients whose all zeros are real.

References

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