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James-Stein Type Estimators Shrinking towards Projection Vector When the Norm is Restricted to an Interval

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Abstract

Consider the problem of estimating a $p\times 1$ mean vector θ ($p-q\geq 3$), $q=\mathrm{rank}(P_V)$ with a projection matrix P_v under the quadratic loss, based on a sample $X_1, X_2, \cdots, X_n$. We find a James-Stein type decision rule which shrinks towards projection vector when the underlying distribution is that of a variance mixture of normals and when the norm $\|\theta - P_V\theta\|$ is restricted to a known interval, where P_V is an idempotent and projection matrix and $\mathrm{rank}(P_V)=q$. In this case, we characterize a minimal complete class within the class of James-Stein type decision rules. We also characterize the subclass of James-Stein type decision rules that dominate the sample mean.

Keywords: James-Stein Type Decision Rule, Mean Vector, Quadratic Loss, Underlying Distribution

1. Introduction

The The problem considered is that of estimating with quadratic loss function the mean vector of a compound multinormal distribution when the norm $\| \theta - P_V\theta \|$ is restricted known interval. The class of estimation rules considered will consist of Lindley type estimators only. Such a class was introduced by James-Stein[1] and Lindley[2] in order to prove that some of its members dominate the sample mean in the multinormal case. Strawderman[3] also derived a similar result for the more general case considered in this paper of a compound multinormal distribution. The problem of estimation of a mean under constraint has an old origin and recently focussed again in the context of curved model in the works of Amari[4], Kariya[5], Perron and Giri[6], Merchand and Giri[7], and Baek[8] among others. A study of compound multinormal distributions and the estimation of their location vectors was carried out by Berger[9].

In section 2, we present the general setting of our problem and develop necessary notations. In section 3, we examine the estimation problem based on a Lindley type decision rule when the norm $\| \theta - P_V\theta \|$ is restricted to a known interval. In this case, we give to the subclass of Lindley type estimators which

dominate the sample mean when the norm is restricted to a known interval.

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2. Notation and Preliminaries

Let $x=(x_1, \cdots, x_p)'$, $p-q\geq 3$, be an observation from a compound multinormal distribution with unknown location parameter $\theta\,(p\times 1)$ and mixture parameter $H(\,\bullet\,)$, where $H(\,\bullet\,)$ represents a known c.d.f defined on the interval $(0, \infty)$. In other words, we assume that the random variable X generating our observation x admits the representation,

$$L(X \mid Z=z) = N_p(\theta, zI_p)\ \forall\, z>0 \tag{2.1}$$

Z being the positive random variable with c.d.f. $H(\,\bullet\,)$. Our problem concerns the estimation of the location parameter θ with loss function.

$L(\theta, \delta(x)) = (\delta(x)-\theta)'(\delta(x)-\theta)$, with, $\theta\in\Theta_{\lambda_2}^{\lambda_1} = \theta\in R^p \mid \|\theta - P_V\theta\| \in [\lambda_1, \lambda_2], 0\leq\lambda_1\leq\lambda_2\leq\infty$ where P_V is an idempotent and projection matrix with $\mathrm{rank}(P_V)=q$ and the decision rule δ, $\delta(\,\cdot\,)$: $R^p \to R^p$, is of the form

$$\delta(\boldsymbol{x}) = P_V x + \left(1 - \frac{c}{(\boldsymbol{x} - P_V x)'(\boldsymbol{x} - P_V x)}\right)(\boldsymbol{x} - P_V x),$$

$c \in \boldsymbol{R}$ Restated in terms of the family of probability density functions of $\boldsymbol{X}$, the distributional assumption give by expression (2.1) and the restriction on the location parameter θ indicate that the p.d.f. of $\boldsymbol{X}$ is

$$P_\theta(\boldsymbol{x}) = \int_{(0,\infty)} (2\pi z)^{-p/2} \exp\left(\frac{\| \boldsymbol{x} - \boldsymbol{\theta} \|^2}{2z}\right) dH(z) \quad (2.2)$$

$\boldsymbol{x} \in \boldsymbol{R}^p$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\lambda_2}^{\lambda_1}$. It will be also assumed that $E(Z) < \infty$ which will guarantee the existence of the covariance matrix $\Sigma = Cov(\boldsymbol{X}) = E(Z)\boldsymbol{I}_p$ and the mean vector $E(\boldsymbol{X}) = \boldsymbol{\theta}$. The performance of the estimator δ will be measured by its risk function $R(\boldsymbol{\theta}, \delta) = E_{\boldsymbol{\theta}}[L(\boldsymbol{\theta}, \delta(\boldsymbol{X}))] = E_{\boldsymbol{\theta}}[(\delta(\boldsymbol{X}) - \boldsymbol{\theta})'(\delta(\boldsymbol{X}) - \boldsymbol{\theta})]$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\lambda_2}^{\lambda_1}$ Define

$$D_{Lind} = \left\{ \begin{array}{l} \delta : \boldsymbol{R}^p \to \boldsymbol{R}^p \mid \delta^c(\boldsymbol{X}) \\ = P_V X + \left(1 - \frac{c}{(\boldsymbol{X} - P_V \boldsymbol{X})'(\boldsymbol{X} - P_V \boldsymbol{X})}(\boldsymbol{X} - P_V \boldsymbol{X})\right), \\ c \in \boldsymbol{R} \end{array} \right\}$$

where the parameter space is of the form $\boldsymbol{\Theta}_{\lambda_2}^{\lambda_1} = \boldsymbol{\Theta}_\lambda = \{\boldsymbol{\theta} \in \boldsymbol{R}^p \mid \| \boldsymbol{\theta} - P_V \boldsymbol{\theta} \| = \lambda\}$, $\lambda \geq 0$. Then under the assumptions $\boldsymbol{\theta} \in \boldsymbol{\Theta}_\lambda$, $p - q \geq 3$ and $E[Z] < \infty$, we can show that

$$\begin{aligned} R(\boldsymbol{\theta}, \delta^c) &= E_\theta[(\delta^c(\boldsymbol{X}) - \delta^c)'(\delta^c(\boldsymbol{X}) - \delta^c)] \\ &= pE(Z) + \left\{\int_{(0,\infty)} \left[\frac{c}{z} - 2c(p-q-2)\right] f_p(\lambda, z) dH(z)\right\} \end{aligned} \quad (2.3)$$

using the method by Baek[8]. By expression (2.3), the unique best estimator within the class D_{Lind} is given by $\delta^{c^*(\lambda)}$ where

$$c^*(\lambda) = (p-q-2)\frac{\int_{(0,\infty)} f_p(\lambda, z) dH(z)}{\int_{(0,\infty)} f_p(\lambda, z)\frac{dH(z)}{z}} \quad (2.4)$$

and its risk is

$$R(\boldsymbol{\theta}, \delta^{c^*(\lambda)}) = pE(Z) - (p-q-2)^2 \frac{\left[\int_{(0,\infty)} f_p(\lambda, z) dH(z)\right]^2}{\int_{(0,\infty)} f_p(\lambda, z)\frac{dH(z)}{z}},$$

$\boldsymbol{\theta} \in \boldsymbol{\Theta}_\lambda$.

When $\| \boldsymbol{\theta} - P_V \boldsymbol{\theta} \| = \lambda$, the use of other estimators of the Lindley class other estimators of the Lindley class other that will incur risk which is a strictly increasing function of distance $| c - c^*(\lambda) |$. To see this, we can define $t(\lambda)$ such that $c = t(\lambda)c^*(\lambda)$ and, using expression (2.3), express $R(\boldsymbol{\theta}, \delta^c)$ as

$$pE(Z) + (p-q-2)^2[t^2(\lambda) - 2t(\lambda)]\frac{\left[\int_{(0,\infty)} f_p(\lambda, z) dH(z)\right]^2}{\int_{(0,\infty)} f_p(\lambda, z)\frac{dH(z)}{z}}$$

From this we can write

$$R(\boldsymbol{\theta}, \delta^c) - R(\boldsymbol{\theta}, \delta^{c^*(\lambda)}) = | c - c^*(\lambda) |^2 \int_{(0,\infty)} f_p(\lambda, z)\frac{dH(z)}{z} \quad (2.6)$$

The natural estimator $\delta^0(\boldsymbol{X}) = \boldsymbol{X}$ is a member of the Lindley class and has a constant risk function equal to $pE(Z)$. Using the expression (2.5), we can verify that the Lindley type estimator δ^c dominates the natural estimator δ^0 if and only if $0 < c < 2 < c^*(\lambda)$ for the Lindley type estimator δ^c dominates the natural estimator δ^0 if and only if $0 < c < 2 < c^*(\lambda)$ for $\boldsymbol{\theta} \in \boldsymbol{\Theta}_\lambda$.

3. Estimation when the Norm is Restricted to an Interval

In this section, we study the case where the mean $\boldsymbol{\theta}$ is restricted to a known interval $[\lambda_1, \lambda_2]$ case, no optimal Lindley type decision rule will exist whenever $\lambda_1 \leq \lambda_2$ (but see the discussion following Corollary 3.7 for asymptotic considerations). We can also characterize the subclass of Lindley type decision rules that dominate the natural estimator $\delta^0 = \boldsymbol{X}$ when $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\lambda_2}^{\lambda_1}$. In the following, we will denote $\underline{c}^*[\lambda_1, \lambda_2] = inf_{\lambda \in [\lambda_1, \lambda_2]} c^*(\lambda)$ and $\bar{c}^*[\lambda_1, \lambda_2] = sup_{\lambda \in [\lambda_1, \lambda_2]} c^*(\lambda)$

Theorem 3.1 Let x be a single observation from a p-dimensional location parameter with p.d.f. of the form given by expression (2.1). Under the assumptions $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\lambda_2}^{\lambda_1}$, $0 \leq \lambda_1 \leq \lambda_2 \leq \infty$; $p - q \geq 3$ and

$E(Z) < \infty$, (a) the subclass $\left\{\delta^c \in D_{Lind} \mid \underline{c}^*[\lambda_1, \lambda_2] \le c \le \overline{c}^*[\lambda_1, \lambda_2]\right\}$ is a minimal complete class within the class D_{Lind} and (b) the decision rule δ^c will be dominate the natural estimator δ^0 if $0 < c < \underline{c}^*[\lambda_1, \lambda_2]$.

Proof. (a) Let c_0 be a real number such that $c_0 \notin \left[\underline{c}^*[\lambda_1, \lambda_2], \overline{c}^*[\lambda_1, \lambda_2]\right]$. Then, using expression (2.6), if $c_0 < \underline{c}^*[\lambda_1, \lambda_2]$, we may write the difference in risks

$$\begin{aligned} &R(\theta, \delta^{c_0}) - R(\theta, \delta^{\underline{c}^*[\lambda_1, \lambda_2]}) \\ &= \left[R(\theta, \delta^{c_0}) - R(\theta, \delta^{c^*(\|\theta - P_V\theta\|)})\right] - \\ &\left[R(\theta, \delta^{\underline{c}^*[\lambda_1, \lambda_2]}) - R(\theta, \delta^{c^*(\|\theta - P_V\theta\|)})\right] \\ &= \int_{(0,\infty)} f_p(\lambda, z)\frac{dH(z)}{z} \\ &\left\{ |c_0 - c^*(\|\theta - P_V\theta\|)|^2 - |\underline{c}^*[\lambda_1, \lambda_2] - c^*(\|\theta - P_V\theta\|)|^2 \right\} \end{aligned}$$

this last expression being positive for all $\theta \in \Theta_{\lambda_2}^{\lambda_1}$ given that $c_0 < \underline{c}^*[\lambda_1, \lambda_2]$. In the same manner, the decision rule δ^c with $c = \overline{c}^*[\lambda_1, \lambda_2]$ will dominate the decision rule δ^{c_0} if $c_0 > \overline{c}^*[\lambda_1, \lambda_2]$, the intermediate value theorem ($c^*(\lambda)$ is easily shown to be continuous) assures us that $R(\theta, \delta^c) - R(\theta, \delta^{c_0}) > 0$, $\forall c \ne c_0$, when $c^*(\|\theta - P_V\theta\|) = c_0$. These last results guarantee that all the rules δ^c with $c \notin \left[\underline{c}^*[\lambda_1, \lambda_2], \overline{c}^*[\lambda_1, \lambda_2]\right]$ are inadmissible within the class D_{Lind} and the rules δ^c with c belonging to the interval $\left[\underline{c}^*[\lambda_1, \lambda_2], \overline{c}^*[\lambda_1, \lambda_2]\right]$ cannot be improved upon by another rule of the class D_{Lind}. Thus, the result of part (a) follows.

(b) Similar to last part in Section 2, the decision rule δ^c will dominate the decision rule δ^0 if

$$\begin{aligned} &R(\theta, \delta^c) < R(\theta, \delta^0), \ \forall \ \theta \in \Theta_{\lambda_2}^{\lambda_1} \\ &\Leftrightarrow 0 < c < 2c^*(\|\theta - P_V\theta\|), \\ &\quad \forall \ \|\theta - P_V\theta\| \in [\lambda_1, \lambda_2] \\ &\Leftrightarrow 0 < c < 2\underline{c}^*[\lambda_1, \lambda_2] \end{aligned}$$

It may also be remarked that the rule δ^c with $c = 2\underline{c}^*[\lambda_1, \lambda_2]$ will also dominate δ^0 under the conditions of the theorem when $\lambda_1 < \lambda_2$ and that all the decisions rules δ^c with $c > 2c^*[\lambda_1, \lambda_2]$ do not dominate δ^0 under the conditions of the theorem. The results above would be more explicit if the function $\underline{c}^*[\lambda_1, \lambda_2] = c^*(\lambda_1)$ and $\overline{c}^*[\lambda_1, \lambda_2] = c^*(\lambda_2)$.

The case with no restrictions on the norm $\|\theta - P_V\theta\|$ (i. e. , $\lambda_1 = 0$, and $\lambda_2 = \infty$) can be expanded using by Strawderman's result[3] and it can be showed that the decision rules δ^c with $0 \le c \le 2(p-q-2)E^{-1}(Z^{-1})$ are minimax rules by showing that their risk functions are uniformly less than or equal to the risk function ($= pE(Z)$) of the minimax decision rule δ^c. This result is derived below as a particular case of Theorem 3.1. To do so, we need to determine the quantity $\underline{c}^*[0, \infty]$. The following three Lemmas will prove useful in determining $\underline{c}^*[0, \infty]$ and, also, $\underline{c}^*[\lambda_1, \lambda_2]$.

Lemma 3.2. Let X be an arbitrary random variable and let f and g be two real nondecreasing functions on the support of X. Then, if the quantities $E[f(x)]$ and $E[g(x)]$ exist, $Cov(f(x), g(x)) \ge 0$ with the inequality being strict if f and g are strictly increasing and X is nondegenerate.

Proof. A neat proof of Lemma 3.2. is given by Chow and Wang[10].

Lemma 3.3. Let L be a Poisson random variable with mean $\gamma(>0)$ and $f_p^*(\gamma) = E^L[(p-q+2L-2)^{-1}]$, $p \ge 4$ then

(i) $f_{p-q}^*(\gamma) = e^{-\gamma}\int_{[0,1]} t^{p-q-3}e^{\gamma t^2}dt$ and

(ii) $f_{p-q+2}(\gamma) = (2\gamma)^{-1}[1-(p-q-2)f_{p-q}^*(\gamma)]$ (3.1)

Proof. We can prove this lemma using the method by Egerton and Laycock[11].

Lemma 3.4. Let $f_p^*(\cdot)$, $p \ge 4$ be a function defined on $[0, \infty]$ and equal to $f_p^*(\gamma) = E^L[(p-q+2L-2)^{-1}]$, $\gamma \ge 0$, where L is a Poisson random variable with mean γ. Then,

(i) $f_{p-q}^*(\cdot)$ is a strictly decreasing function,

(ii) $\lim_{\gamma\to 0^+} f^*_{p-q}(\gamma) = (p-q-2)^{-1}$, $\lim_{\gamma\to\infty} f^*_{p-q}(\gamma) = 0$

(iii) if $p \geq 5$, $\gamma f^*_{p-q}(\gamma)$ is strictly increasing function for $\gamma \geq 0$.

Proof. (i) Using part (i) of Lemma 3.3, we have for $\gamma_2 > \gamma_1 > 0$, $f^*_{p-q}(\gamma_2) - f^*_{p-q}(\gamma_1)$

$$= \int_{[0,1]} t^{p-q-3}\left(e^{\gamma_2(t^2-1)} - e^{\gamma_1(t^2-1)}\right) dt < 0$$

(ii) By the dominated convergence theorem,

$$\lim_{\gamma\to 0^+} f^*_{p-q}(\gamma) = \lim_{\gamma\to 0^+} \int_{[0,1]} t^{p-q-3}\left(e^{\gamma(t^2-1)}\right) dt$$
$$= \int_{[0,1]} t^{p-q-3}\left(\lim_{\gamma\to 0^+}\left(e^{\gamma(t^2-1)}\right)\right) dt$$
$$= \int_{[0,1]} t^{p-q-3} dt = (p-q-2)^{-1}$$

and

$$\lim_{\gamma\to\infty} f^*_{p-q}(\gamma) = \lim_{\gamma\to\infty} \int_{[0,1]} t^{p-q-3} e^{\gamma(t^2-1)} dt$$
$$= \int_{[0,1]} t^{p-q-3}\left(\lim_{\gamma\to\infty} e^{\gamma(t^2-1)}\right) dt = 0$$

(iii) Using Lemma 3.3, we have $\gamma f^*_5(\gamma) = \frac{1}{2}(1-e^{-\gamma})$, which is easily seen to be strictly increasing. For $p \geq 6$ we obtain by the recurrence formula given by expression (3.1),

$$\gamma f^*_{p-q}(\gamma) = \frac{1}{2}\left(1-(p-q-4) f^*_{p-q-2}(\gamma)\right), \gamma > 0.$$

which must be strictly increasing given that function $f_{p-q-2}(\cdot)$ is strictly decreasing by part (i).

In the following, we will set $E^{-1}[Z^{-1}]$ equal to zero if the expectation $E[Z^{-1}] = \infty$.

Theorem 3.5. The function $c^*(\cdot)$ defined by expression (2.4) satisfies the following properties :

(a) $\infty\ c^*(\lambda) = (p-q-2) E[Z^{-1}]\ \lambda \geq 0$

(b) $c^*(\gamma) = k \Rightarrow Z$ is constant with probability one and,

(c) for $p \geq 5$,

Proof. (a) Expression (2.4) can be rewritten as

$$c^*(\lambda) = (p-q-2)\frac{E^Z[f_{p-q}(\lambda, Z)]}{E^Z[Z^{-1} f_{p-q}(\lambda, Z)]}, \lambda \geq 0.$$

By applying Lemma 3.2 to the functions $f_{p-q}(\lambda, Z)$ and Z^{-1}, the function $f_{p-q}(\lambda, Z)$ being an increasing function by part (i) of Lemma 3.4, we have for $\lambda \geq 0$,

$$Cov\left(f_{p-q}(\lambda, Z), -Z^{-1}\right) \geq 0$$
$$\Rightarrow E^Z[Z^{-1} f_{p-q}(\lambda, Z)] \geq E[Z^{-1}]\, E^Z[f_{p-q}(\lambda, Z)]$$
$$\Rightarrow c^*(\lambda) \geq (p-q-2) E^{-1}[Z^{-1}]$$
$$\Rightarrow \overset{\infty}{\lambda \geq 0}\ c^*(\lambda) \geq (p-q-2) E^{-1}[Z^{-1}]$$

The reverse inequality is obtained by observing that $c^*(0) = (p-q-2) E^{-1}[Z^{-1}]$.

(b) The constancy of $c^*(\lambda)$ implies

$c^*(\lambda) = k = c^*(0) = (p-q-2) E^{-1}[Z^{-1}]$
$\forall\ \lambda > 0$,

and $\int_{(0,\infty)}\left(p-q-1-\frac{k}{z}\right) f_{p-q}(\lambda, z) dH(z) = 0$

Since both $f_{p-q}(\lambda, Z)$ and $-kz^{-1}$ are strictly increasing function of z, we have by Lemma 3.2, for nondegenerate Z,

$$Cov\left(f_{p-q}(\lambda, Z), p-q-2-kZ^{-1}\right) \geq 0$$
$$\Rightarrow E[p-q-2-kZ^{-1}) f_{p-q}(\lambda, Z)] >$$
$$E[p-q-2-kZ^{-1})]\ E[f_{p-q}(\lambda, Z)] = 0$$

which results in a contradiction implying Z is constant with probability one.

(c) By applying Lemma 3.2 to the functions $-z^{-1} f_{p-q}(\lambda, Z)$ and z, the function $-z^{-1} f_{p-q}(\lambda, Z)$ being an increasing function by virtue of part (iii) of Lemma 3.4, we have for $p \geq 5$ and $\lambda \geq 0$,

$$Cov\left(-Z^{-1} f_{p-q}(\lambda, Z), Z\right) \geq 0$$
$$\Rightarrow E^Z[f_{p-q}(\lambda, Z)] \leq E[Z^{-1} f_{p-q}(\lambda, Z)]\, E[Z]$$
$$\Rightarrow c^*(\lambda) \leq (p-q-2) E[Z]$$
$$\Rightarrow \overset{sup}{\lambda \geq 0}\ c^*(\lambda) \leq (p-q-2) E[Z]$$

The reverse inequality is obtained by verifying that $\lim_{\lambda\to\infty} c^*(\lambda) = (p-q-2)E[Z]$ whenever $p \geq 5$. To do so, it will be useful to express the function $c^*(\cdot)$ in the following way,

$$c^*(\lambda) = (p-q-2)\frac{\int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j+1}}{j!(p-q+2y-2)}zdH(z)}{\int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j+1}}{j!(p-q+2y-2)}dH(z)}$$

$$= (p-q-2)\frac{\int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j+1}}{j!}\frac{2j}{p-q+2y-4}zdH(z)}{\int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j+1}}{j!}\frac{2j}{p-q+2y-4}dH(z)},$$

$\lambda > 0$.

Moreover, we can write

$$\lim_{\lambda\to\infty} c^*(\lambda) = (p-q-2)\frac{\lim_{\lambda\to\infty}\left\{\int_{(0,\infty)}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j}}{j!}\frac{2j}{p-q+2y-4}zdH(z)\right\}}{\lim_{\lambda\to\infty}\left\{\int_{(0,\infty)}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j}}{j!}\frac{2j}{p-q+2y-4}dH(z)\right\}}$$

if both limits exist and the denominator is not equal to zero. By the dominated converge theorem, we can then write $\lim_{\lambda\to\infty} c^*(\lambda)$ as

$$(p-q-2)\frac{\int_{(0,\infty)}\lim_{\lambda\to\infty}E^{L_z}\left[\frac{2L_z}{p-q+2L_z-4}1_{(1,2,\cdots)}(L_z)\right]zdH(z)}{\int_{(0,\infty)}\lim_{\lambda\to\infty}E^{L_z}\left[\frac{2L_z}{p-q+2L_z-4}1_{(1,2,\cdots)}(L_z)\right]dH(z)}$$

where, for $z > 0$, L_z is a Poisson random variable with mean $\lambda^2/2z$. Finally by noting that,

$$\forall\, z > 0,\ \lim_{\lambda\to\infty}E^{L_z}\left[\frac{2L_z}{p-q+2L_z-4}1_{(1,2,\cdots)}(L_z)\right] = 1$$

because the integrand tends $2L_z(p-q+2L_z-4)^{-1}$ tends to one when $L_z \to \infty$ we obtain

$$\lim_{\lambda\to\infty} c^*(\lambda) = (p-q-2)\frac{\int_{(0,\infty)}zdH(z)}{\int_{(0,\infty)}dH(z)} = (p-q-2)E(Z)$$

Having evaluated the quantities $\underline{c}^*[0,\infty]$ and $\overline{c}^*[0,\infty]$, and Theorem 3.1 yields the following result.

Corollary 3.6. Let x be a single observation from a p-dimensional location parameter family with p.d.f. of the form given by expression (2.1). with $p-q \geq 3$, and under the assumption $\theta \in R^p$ and $E[Z] < \infty$,

(a) the subclass

$\{\delta^c \in D_{Lind} \mid (p-q-2)E^{-1}[Z^{-1}] \leq c \leq (p-q-2)E[Z]\}$ is a minimal complete class D_{Lind} for $p-q \geq 4$,

(b) the decision rule δ^c will dominate the decision rule δ^0 if $0 < c < 2(p-q-2)E^{-1}[Z^{-1}]$.

Proof. These results above are a direct application of Theorem 3.1 and 3.5. We pursue with some remarks.

Remark 3.1. Under the conditions of Corollary 3.6, the decision rule δ^c is a minimax rule if and only if $0 \leq c \leq 2(p-q-2)E^{-1}[Z^{-1}]$. This condition can also be obtained using part (a) of Theorem 3.5 and similar to last part in Section 2 which, under the same conditions, would specify that

$$R(\theta, \delta^c) \leq p \iff 0 \leq c \leq 2c^*(\|\theta - P_V\theta\|).$$

It is interesting to note that the natural estimator δ^0 represents the only minimax rule within the class D_{Lind} when the quantity $E[Z^{-1}]$ does not exist.

Remark 3.2. The results above of Theorem 3.1 and Corollary 3.6 can be extended to the case where the experimental information consist of a sample $X_1, \cdots, X_n$ with p.d.f. of the form in (2.1) and the class of decision rules considered consists of the decision rules of the form

$$\delta^c(X_1, \cdots, X_n),\ c \in R,$$

$$= P_V\overline{X} + \left(1 - \frac{c}{(\overline{X} - P_V\overline{X})'(\overline{X} - P_V\overline{X})}\right)(\overline{X} - P_V\overline{X})$$

where $\overline{X}$ is the sample mean and P_V is an idempotent and projection matrix. This can be seen by nothing that the probability law of sample mean $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$; $X_1, \cdots, X_n$ being n independently and identically distributed random vectors admitting the representations.

$$L(X_j \mid Z_j = z_j) = N_p(\theta, z_j I_p),\ j = 1, \cdots, n.$$

for all values $z_1, \cdots, z_n$ of n independent copies $Z_1, \cdots, Z_n$ of a positive random variable Z; admits the representation

$$L(Z \mid Z_1 = z_1, \cdots, Z_n = z_n) = N_p(\theta, n^{-2}\sum_{j=1}^{n} z_j I_p), \text{ or}$$
$$L(\overline{X} \mid W = w) = N_p(\theta, w I_p), \ \forall w > 0$$

where W is a random variable such that

$$L(W) = L(n^{-2}\sum_{j=1}^{n} Z_j). \tag{3.2}$$

Thus the optimal estimator of the Lindley type is; with the conditions $\theta \in \Theta_\lambda$, $E[Z] < \infty$, $p-q \geq 3$; given by expression (2.4), and is equal to

$$\delta^{c_n^*(\lambda)} = P_V\overline{X} + \left(1 - \frac{c_n^*(\lambda)}{(\overline{X} - P_V\overline{X})'(\overline{X} - P_V\overline{X})}\right)(\overline{X} - P_V\overline{X})$$

where

$$c_n^*(\lambda) = (p-q-2)\frac{\int_{(0,\infty)} f_p(\lambda, w)\, dH_n^*(w)}{\int_{(0,\infty)} f_p(\lambda, w)\frac{dH_n^*(w)}{w}},$$

$H_n^*(\cdot)$ representing the c.d.f. of the random variable W defined by expression (3.2). Furthermore, the result specifying a minimal complete class within the class

$$D_{JS} = \left\{\delta^c : R^p \to R^p \mid \delta^c(\overline{X}) = P_V\overline{X} + \left(1 - \frac{c}{(\overline{X} - P_V\overline{X})'(\overline{X} - P_V\overline{X})}\right)(\overline{X} - P_V\overline{X})\right\}$$

as well as the result giving a subclass of Lindley type rues that dominate the sample mean $\delta^0(\overline{X}) = \overline{X}$ and be applied to the case where the experimental information consists of a sample. In particular, by rewriting Corollary 3.6, we obtain the following result. Part (b) of this corollary has been proved by Bravo and MacGibbon[12] under a more general setting.

Corollary 3.7. Let $X_1, \cdots, X_n$ be a sample generated by a common random vector X which admits the representation given by expression (2.1). Under the conditions $\theta \in R^p$, $p-q \geq 3$ and $E[Z] < \infty$

(a) for $p-q \geq 4$, the subclass

$$\left\{\delta^c \in D_{JS} \mid n^{-1}(p-q-2)E^{-1}\left[\left(\sum_{i=1}^{n} Z_i\right)^{-1}\right] \leq c \leq n^{-1}(p-q-2)E[Z]\right\}$$

is a minimal complete class with the class D_{JS}, and

(b) the decision rule δ^c will dominate the sample mean

$$\text{if } 0 < c < 2n^{-2}(p-q-2)E^{-1}\left[\left(\sum_{i=1}^{n} Z_i\right)^{-1}\right] \tag{3.3}$$

Proof. These results are a direct application of Corollary 3.6 and the discussion above expression (3.2).

However, the results concerning the minimax criteria given by Strawerman cannot be applied to the decision rules $\delta^c(\overline{x})$ since the statistic $\overline{X}$ does not represent in general a sufficient statistic(the multinormal case being a well known exception). Finally it is interesting to note that,

$$E^{-1}\left[\left(\sum_{i=1}^{n} Z_i\right)^{-1}\right] \leq E\left[\sum_{i=1}^{n} Z_i\right] = nE[Z],$$

(the above inequality can be seen us a consequence of Lemma 3.2), implying that the interval

$$\left(0, 2n^{-1}(p-q-2)E^{-1}\left[\left(\sum_{i=1}^{n} Z_i\right)^{-1}\right]\right) \to \varnothing \text{ as } n \to \infty$$

which, by expression (3.3), indicates that the subclass of Lindley type decision rules dominating the sample mean can be made arbitrarily small by increasing the sample size n.

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