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Isotomic and Isogonal Conjugates Tangent Lines of Lines at Vertices of Triangle

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Abstract

In this paper we consider the two tangent lines of isogonal and isotomic conjugates of the line at both vertices of a given triangle. We find the necessary and sufficient condition for the two tangent lines of isogonal or isotomic conjugates of the line at both vertices and the median line to be concurrent. We also prove that every line whose isogonal conjugate tangent lines at both vertices are concurrent with the median line intersects at a unique point. Moreover, we show that the three intersection points correspond to the vertices of triangle are collinear.

Keywords: Barycentric Coordinates, Isotomic Conjugate, Isogonal Conjugate, Tangent Line, Ceva's Theorem, Concurrency of Three Lines

1. Introduction

The barycentric coordinates with respect to a triangle are widely used in CAGD(Computer Aided Geometric Design) as well as in Euclidean Geometry. In particular, the use of barycentric coordinates has played an important rule in a lot of methods for the conic representation and conic approximation[1-6]. In Euclidean Plane Geometry, so many works have been done based on the use of them[1,7-11].

Recently, Akopyan[12] presented the properties of the tangency of isotomically and isogonally conjugate lines of some special lines with respect to a triangle. Yoon and Ahn[13] showed that the isogonal and isotomic conjugates of conic tangent to two side lines at vertices are again conic tangent to the two side lines at vertices and classified them into ellipses, parabolas and hyperbolas using the barycentric coordinates.

In this paper we study the two tangent lines of isogonal and isotomic conjugates of the line L at both vertices B, C of the reference triangle $\triangle ABC$. We find the necessary and sufficient condition for the two tangent lines of isogonal or isotomic conjugates of the line at both vertices and a median line AM_A to be concurrent, where M_A is the midpoint of side line BC. We also prove that all lines whose isogonal conjugate tangent lines at both vertices are concurrent with the median line AM_A intersect at a unique point X_A. Moreover we show that the three intersection points X_A, X_B, X_C are collinear. All of our results are based on the barycentric coordinates.

I suggest that The contents of our paper are organized as follows. In Section 2, the basic facts in elementary plane geometry are provided, and in Section 3, our main results are presented. The results in the first half of Section 3 improves of the MS Thesis of the first author of this paper[14].

2. Preliminaries for Elementary Plane Geometry

In this section we remind the definitions of barycentric coordinates, homogeneous barycentric coordinates, isotomic conjugate, and isogonal conjugate[4,10,13,15,16].

Every point P in a reference triangle $\triangle ABC$ satisfies

$$\overrightarrow{OP} = \frac{1}{\triangle ABC}(\triangle BCP \cdot \overrightarrow{OA} + \triangle CAP \cdot \overrightarrow{OB} + \triangle ABP \cdot \overrightarrow{OC}) \tag{2.1}$$

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where O is the origin on the plane containing the triangle and $\triangle XYZ$ is the area of triangle $\triangle XYZ$[10,15]. In Eq. (2.1), the ordered triple

$$\left(\frac{\triangle BCP}{\triangle ABC}, \frac{\triangle CAP}{\triangle ABC}, \frac{\triangle ABP}{\triangle ABC}\right)$$

is called by the barycentric coordinates of P with resect to $\triangle ABC$. If (τ_0, τ_1, τ_2) is the barycentric coordinates of P, its positive scalar multiplication

$$(k\tau_0 : k\tau_1 : k\tau_2)$$

is called by the homogeneous barycentric coordinates of P with resect to $\triangle ABC$. The definition of barycentric coordinates can be extended to all points on the plane from $\triangle ABC$ as follows[4]. For every point P on the plane, $\overrightarrow{OP}$ is uniquely expressed by

$$\overrightarrow{OP} = \tau_0 \overrightarrow{OA} + \tau_1 \overrightarrow{OB} + \tau_2 \overrightarrow{OC}$$
$$\tau_0 + \tau_1 + \tau_2 = 1.$$

At this time, (τ_0, τ_1, τ_2) is called by the barycentric coordinates of P. The relationship between the signatures of barycentric coordinates and the position of P outside of $\triangle ABC$ is well-known[13,17].

The midpoints of the side lines BC, CA, AB are denoted by M_A, M_B, M_C, respectively. For the point P inside the triangle $\triangle ABC$, let the points P_A, P_B, P_C be the intersection points of the lines AP, BP, CP and side lines BC, CA, AB, respectively, and let P_A°, P_B°, P_C° be the symmetric point of P_A, P_B, P_C with respect to M_A, M_B, M_C, respectively. Then the three lines AP_A°, BP_B°, CP_C° are concurrent at a point, which is called by the isotomic conjugate of P and denoted by P°. It is also well-known[10,15,16] that P° satisfies

$$\triangle BCP : \triangle CAP : \triangle ABP = \frac{1}{\triangle BCP^\circ} : \frac{1}{\triangle CAP^\circ} : \frac{1}{\triangle ABP^\circ}$$

and its homogeneous barycentric coordinates is $\left(\frac{1}{\tau_0} : \frac{1}{\tau_1} : \frac{1}{\tau_2}\right)$.

The angle bisectors at the vertices A, B, C of $\triangle ABC$ are denoted by L_A, L_B, L_C, respectively, which are concurrent at the incenter of $\triangle ABC$. For the point P inside the triangle $\triangle ABC$, let the lines L_A^*, L_B^*, L_C^* be the symmetric lines of the lines AP, BP, CP with respect to L_A, L_B, L_C, respectively. The three lines L_A^*, L_B^*, L_C^* are concurrent at a point, which is called by the isogonal conjugate of P and denoted by P^*. It is also well-known[10,15,16] that

$$\triangle BCP : \triangle CAP : \triangle ABP = \frac{a^2}{\triangle BCP^\circ} : \frac{b^2}{\triangle CAP^\circ} : \frac{c^2}{\triangle ABP^\circ}$$

and P^* has the homogeneous barycentric coordinates $\left(\frac{a^2}{\tau_0} : \frac{b^2}{\tau_1} : \frac{c^2}{\tau_2}\right)$.

Ceva's theorem[14] will be use to prove our main theorems.

Theorem 2.1 (Ceva's Theorem)

Let the points X, Y, Z be on the side lines BC, CA, AB of a triangle $\triangle ABC$, respectively. The lines AX, BY, CZ are concurrent if and only if

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1.$$

3. Tangent Lines of Isotomic and Isogonal Conjugates of Line at Vertices of Triangle

In this section, we consider a line L which intersects the side lines AB and AC of a reference triangle $\triangle ABC$ at two points D, E, respectively.

Theorem 3.1

The two tangent lines of the isotomic conjugate curve of L at the vertices B, C, and the median line AM_A are concurrent if and only if the line L is parallel to the side line BC.

Proof.

Let $p_0 = \overrightarrow{OA}$, $p_1 = \overrightarrow{OB}$, $p_2 = \overrightarrow{OC}$. There are real numbers $\delta_1, \delta_2 \in (0,1)$ such that

$$\overrightarrow{OD} = (1-\delta_1)\overrightarrow{OA} + \delta_1 \overrightarrow{OB}$$
$$\overrightarrow{OE} = (1-\delta_2)\overrightarrow{OA} + \delta_2 \overrightarrow{OC}.$$

The line L has the parametric equation

$$r(t) = \big(1-\delta_1 + t(\delta_1 - \delta_2)\big)p_0 + (1-t)\delta_1 p_1 + t\delta_2 p_2 \tag{3.1}$$

and the homogeneous barycentric coordinates

$$((1-\delta_1+t(\delta_1-\delta_2)):(1-t)\delta_1:t\delta_2) \quad (3.2)$$

So, the isotomic conjugate curve $r^\circ(t)$ has the homogeneous barycentric coordinates

$$\left(\frac{1}{(1-\delta_1+t(\delta_1-\delta_2))}:\frac{1}{(1-t)\delta_1}:\frac{1}{t\delta_2}\right)$$

and the parametric equation of $r^\circ(t)$ is

$$r^\circ(t)=[(1-t)\delta_1 t\delta_2 p_0+(1-\delta_1+t(\delta_1-\delta_2))t\delta_2 p_1 +(1-\delta_1+t(\delta_1-\delta_2))(1-t)\delta_1 p_2]/w(t)$$

where

$$w(t)=(1-t)\delta_1 t\delta_2+(1-\delta_1+t(\delta_1-\delta_2))t\delta_2 +(1-\delta_1+t(\delta_1-\delta_2))(1-t)\delta_1.$$

Note that $r^\circ(t)$ is passing through the points C,B when $t=0,1$, respectively. For $i=0,1$, let T_i be the tangent line of $r^\circ(t)$ at $t=i$, and let F,G be the intersection points of T_0 and AB, T_1 and AC, respectively. Since

$$r^{\circ\prime}(0)=\frac{\delta_2}{(1-\delta_1)\delta_1}(\delta_1 p_0+(1-\delta_1)p_1-p_2)$$
$$r^{\circ\prime}(1)=\frac{-\delta_1}{(1-\delta_2)\delta_2}(\delta_2 p_0-p_1+(1-\delta_2)p_2),$$

we have

$$AF:FB=1-\delta_1:\delta_1$$
$$AG:GC=1-\delta_2:\delta_2.$$

By Ceva's Theorem, the three lines, AM_A, T_0, and T_1 are concurrent if and only if

$$\frac{AF}{FB}\cdot\frac{BM_A}{M_AC}\cdot\frac{CG}{GA}=1.$$

Since

$$\frac{AF}{FB}\cdot\frac{BM_A}{M_AC}\cdot\frac{CG}{GA}=\frac{1-\delta_1}{\delta_1}\cdot\frac{\delta_2}{1-\delta_2}=1$$

is equivalent to $\delta_1=\delta_2$, the two tangent lines T_0, T_1, and

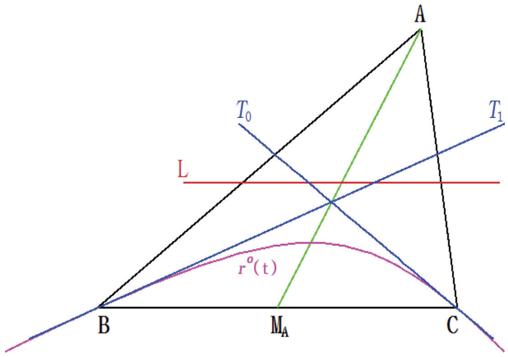

Fig. 1. Isotomic conjugate for $\delta_1=\delta_2=0.55$

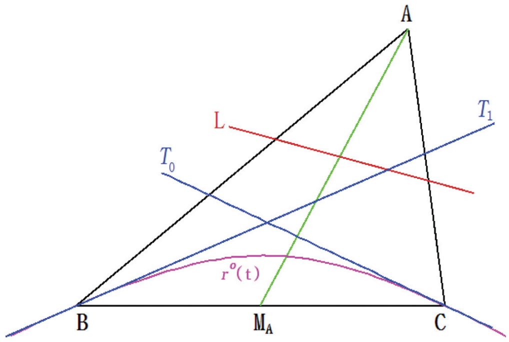

Fig. 2. Isotomic conjugate for $\delta_1=0.4$, $\delta_2=0.55$

the median line AM_A are concurrent if and only if the line L is parallel to the side line BC. □

Figs. 1-2 illustrate Theorem 3.1. In Figs. 1-5, $a=10$, $b=\sqrt{50}$ and $c=\sqrt{130}$. In the case that the line L (orange color) and the side line BC are parallel, the two tangent lines T_0, T_1 (blue line) of the isotomic conjugate curve $r^\circ(t)$ (magenta color) at vertices C,B, and the median line AM_A (green color) are concurrent, as shown in Fig. 1. If the line L and the side line BC are not parallel, then T_0, T_1, and AM_A are not concurrent, as shown in Fig. 2.

Theorem 3.2

The two tangent lines of the isogonal conjugate curve of the line DE at the vertices B,C of triangle $\triangle ABC$, and the median line AM_A are concurrent if and only if the line DE satisfies

$$\frac{b^2 DB}{AD} = \frac{c^2 EC}{AE} \tag{3.3}$$

Proof.

By Eqs. (3.1)-(3.2), the isogonal conjugate curve $r^*(t)$ of the line L has the homogeneous barycentric coordinates

$$\left(\frac{a^2}{1-\delta_1+t(\delta_1-\delta_2)} : \frac{b^2}{(1-t)\delta_1} : \frac{c^2}{t\delta_2}\right)$$

and the parametric equation of $r^*(t)$ is

$$r^*(t) = [a^2(1-t)\delta_1 t\delta_2 p_0 + b^2(1-\delta_1+t(\delta_1-\delta_2))t\delta_2 p_1 + c^2(1-\delta_1+t(\delta_1-\delta_2))(1-t)\delta_1 p_2]/w^*(t)$$

where

$$w^*(t) = a^2(1-t)\delta_1 t\delta_2 + b^2(1-\delta_1+t(\delta_1-\delta_2))t\delta_2 + c^2(1-\delta_1+t(\delta_1-\delta_2))(1-t)\delta_1.$$

For $i=0,1$, let T_i^* be the tangent line of $r^*(t)$ at $t=0,1$, and let F, G be the intersection points of T_0^* and $\overline{AB}$, T_1^* and $\overline{AC}$, respectively. Since

$$r^{*\prime}(0) = \frac{\delta_2}{c^2(1-\delta_1)\delta_1}\left(a^2\delta_1 p_0 + b^2(1-\delta_1)p_1 - (a^2\delta_1 + b^2(1-\delta_1))p_2\right)$$

$$r^{*\prime}(1) = \frac{-\delta_1}{b^2(1-\delta_2)\delta_2}\left(a^2\delta_2 p_0 - (a^2\delta_2 + c^2(1-\delta_2))p_1 + c^2(1-\delta_2)p_2\right)$$

we have

$$AF : FB = b^2(1-\delta_1) : a^2\delta_1$$
$$AG : GC = c^2(1-\delta_2) : a^2\delta_2.$$

Since

$$\frac{AF}{FB}\cdot\frac{BM_A}{M_A C}\cdot\frac{CG}{GA} = \frac{b^2(1-\delta_1)}{\delta_1}\cdot\frac{\delta_2}{c^2(1-\delta_2)},$$

the two tangent lines T_0^*, T_1^*, and the median line AM_A are concurrent if and only if the line DE satisfies

$$\frac{b^2 DB}{AD} = \frac{c^2 EC}{AE}. \qquad \square$$

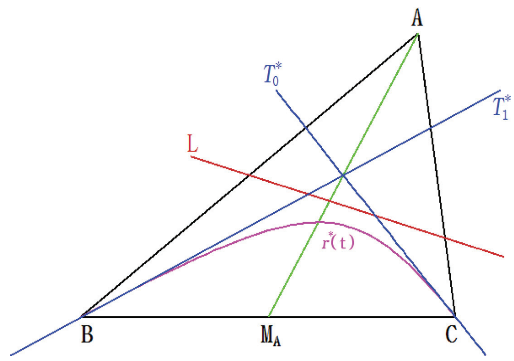

Fig. 3. Isogonal conjugate for $\delta_1 = \frac{1}{2}$, $\delta_1 = \frac{13}{18}$.

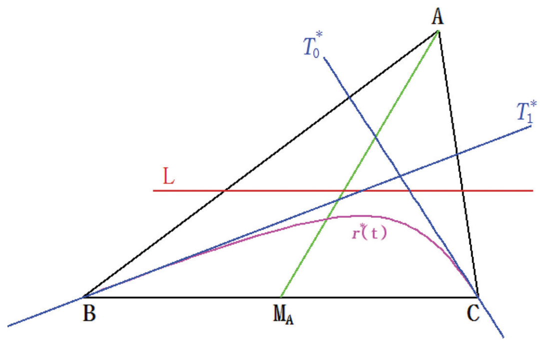

Fig. 4. Isogonal conjugate for $\delta_1 = \delta_2 = 0.6$.

Figs. 3-4 illustrate Theorem 3.2. In Fig. 3, $\delta_1 = \frac{1}{2}$, $\delta_2 = \frac{13}{18}$ and it shows that if the line L (orange color) passes through D, E satisfying Eq. (3.3), then the two tangent lines T_0^*, T_1^* (blue line) of the isogonal conjugate curve $r^*(t)$ (magenta color) of the line L at vertices B, C, and the median line AM_A are concurrent. Fig. 4 shows that if the line L does not satisfy Eq. (3.3), then T_0^*, T_1^*, and AM_A are not concurrent.

Definition 3.3

The tangent lines of isogonal conjugate curve of the line L at B, C are called by the isogonal conjugate tangent line of L at B, C, respectively.

Theorem 3.4

All lines whose isogonal conjugate tangent lines at B, C are concurrent with the median line AM_A intersect at a unique point X_A which is the externally dividing point of B, C in the ratio of $c^2 : b^2$, i.e.,

$$X_A = \begin{cases} \dfrac{b^2 \cdot B - c^2 \cdot C}{b^2 - c^2} & (b \neq c) \\ \infty & (b = c). \end{cases} \qquad (3.4)$$

proof.

If $b = c$, then the assertion is clearly true. If $b \neq c$, then, by Eq. (3.4),

$$\overrightarrow{DX_A} = -\frac{DB}{AB}\overrightarrow{OA} + \left(\frac{b^2}{b^2-c^2} - \frac{AD}{AB}\right)\overrightarrow{OB} - \frac{c^2}{b^2-c^2}\overrightarrow{OC}$$

$$\overrightarrow{EX_A} = -\frac{EC}{AC}\overrightarrow{OA} + \frac{b^2}{b^2-c^2}\overrightarrow{OB} + \left(-\frac{c^2}{b^2-c^2} - \frac{AE}{AC}\right)\overrightarrow{OC}$$

and by Eq. (3.3),

$$\overrightarrow{DX_A} = -\frac{DB}{AB}\overrightarrow{OA} + \frac{c^2 \cdot AC \cdot DA}{(b^2-c^2) \cdot AB \cdot EA}\overrightarrow{OB} - \frac{c^2}{b^2-c^2}\overrightarrow{OC}$$

$$\overrightarrow{EX_A} = -\frac{EC}{AC}\overrightarrow{OA} + \frac{b^2}{b^2-c^2}\overrightarrow{OB} - \frac{b^2 \cdot AE \cdot AB}{(b^2-c^2) \cdot AC \cdot AD}\overrightarrow{OC}$$

Thus we have $\overrightarrow{DX_A} = \frac{c^2 \cdot AC \cdot AD}{b^2 \cdot AB \cdot AE}\overrightarrow{EX_A}$, and so, all lines passing through D, E intersect at the point X_A. Since the slopes of all lines are different mutually, the intersection point X_A is unique. □

Similarly as X_A denotes the unique intersection point in Theorem 3.3, we define the point X_B (or X_C) by the unique intersection point of all lines whose isogonal conjugate tangent lines at C, A (or A, B) concurrent with the median line BM_B (or CM_C). Then X_B and X_C are the externally dividing point of C, A in the ratio of $a^2 : c^2$, and of A, B in the ratio of $b^2 : a^2$, respectively.

Theorem 3.5

The three points X_A, X_B, X_C are collinear.

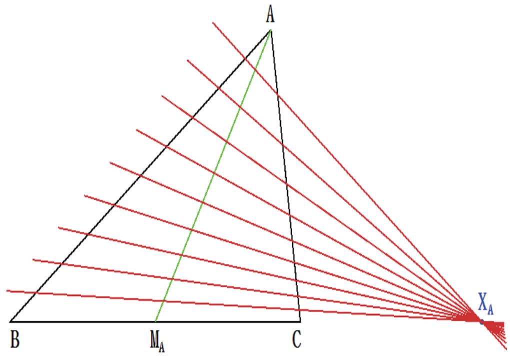

Fig. 5. Lines satisfying Eq. (3.3).

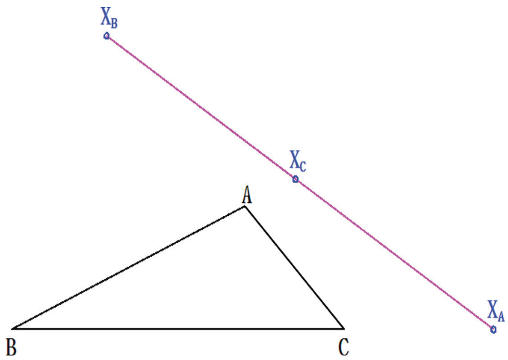

Fig. 6. The collinearity of three points.

proof.

If $\triangle ABC$ is a isosceles triangle, then at least one of three points X_A, X_B, X_C is the infinite point. Thus the three points are trivially collinear. Otherwise,

$$X_A = \frac{b^2 \cdot B - c^2 \cdot C}{b^2 - c^2}$$

$$X_B = \frac{a^2 \cdot A - c^2 \cdot C}{a^2 - c^2}$$

$$X_C = \frac{a^2 \cdot A - b^2 \cdot B}{a^2 - b^2}$$

yeld that

$$\overrightarrow{X_AX_B} = \frac{a^2}{a^2-c^2}A - \frac{b^2}{b^2-c^2}B + \frac{c^2(a^2-b^2)}{(b^2-c^2)(a^2-c^2)}C$$

$$\overrightarrow{X_BX_C} = \frac{a^2(b^2-c^2)}{(a^2-b^2)(a^2-c^2)}A - \frac{b^2}{a^2-b^2}B - \frac{c^2}{a^2-c^2}C$$

Thus we have $\overrightarrow{X_A X_B} = \frac{a^2 - b^2}{b^2 - c^2} \overrightarrow{X_B X_C}$, and so, the three points X_A, X_B, X_C are collinear. □

Fig. 6 illustrates Theorem 3.5. It shows that the three points X_A, X_B, X_C are collinear for the triangle ABC with $a = 10$, $b = \sqrt{18}$, $c = \sqrt{58}$.

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