# MULTI-HARMONIC MODELS FOR BUBBLE EVOLUTION IN THE RAYLEIGH-TAYLOR INSTABILITY

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Abstract. We consider the multi-harmonic model for the bubble evolution in the Rayleigh-Taylor instability in two and three dimensions. We extend the multi-harmonic model in two dimensions to a high-order and present a new class of steady-state solutions of the bubble motion. The growth rate of the bubble is expressed by a continuous family of two free parameters. The critical point in the family of solutions is identified as a saddle point and is chosen as the physically significant solution. We also present the multi-harmonic model in the cylindrical geometry and find the steady-state solution of the axisymmetric bubble. Validity and limitation of the model are also discussed.

## 1. Introduction

Unstable fluid mixing occurs frequently in basic science and engineering applications. When a heavy fluid is supported by a lighter fluid in a gravitational field, the interface between the fluids is unstable under small disturbances. This phenomenon is known as the Rayleigh-Taylor (RT) instability [17] and plays important roles in many fields ranging from astrophysics to inertial confinement fusion. To investigate dynamics of this instability, extensive researches have been carried out in last decades. For reviews, see Sharp [19] and more recently, Abarzhi [2].

Small perturbations at the unstable interface in the RT instability grow into nonlinear structures in the form of bubbles and spikes (See Fig. 1). A bubble (spike) is a portion of the light (heavy) fluid penetrating into the heavy (light) fluid. At later times, a bubble attains a constant growth rate. Eventually, a turbulent mixing caused by vortex structures around spikes breaks the ordered fluid motion [9, 12].
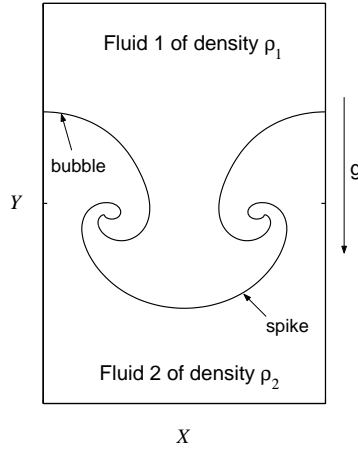
FIGURE 1. Flow description of RT instability. $g$ represents the gravitational acceleration, and $\rho_1 > \rho_2$.

Layzer [13] presented a potential-flow model, based on the approximate description of the flow near the bubble tip, and successfully described the nonlinear evolution of the single-mode RT bubble of the infinite density ratio. Since Layzer's work, it has been studied by many people [11, 25]; for example, the interface of finite density ratios [10, 14], effects of viscosity and surface tension [22], and of viscoelastic fluids [18]. Limitations of the Layzer model were reported by Mikaelian [15]. A source-flow model [7, 21], which describes the bubble as the potential with a source singularity, can be considered a similar approach as that from Layzer and provides more accurate prediction for the bubble motion.

Abarzhi [1] proposed a different type of potential-flow model, the so called *multi-harmonic model*, extending the single harmonic potential in the Layzer model to multiple harmonics. In this model, the bubble shape is parametrized by the principal curvature at the bubble tip. This approach gives a continuous family of asymptotic steady solutions for a bubble. It is consistent with the nonuniquess argument for the bubble evolution by Garabedian [8], Birkhoff and Carter [6], and recently Zudin [26]. The key idea of this approach is that the fastest solution in the family is chosen as the physically significant one. Since Abarzhi's work, the model has been developed for the interfaces of finite density contrast [3, 4] and in various polygonal geometries [5].

In this paper, we present two extensions of the multi-harmonic model for the bubble of infinite density ratio: a high-order model in two dimensions and a model in the cylindrical geometry. The multi-harmonic model for these two cases was not previously studied, whereas the Layzer-type model often considered them [10, 14]. The main purpose of this study is to complement the

previous works of the multi-harmonic model and to examine validity and limitation of the model, by identifying whether a similar approach can be applied to the two cases and the solution of the bubble evolution is improved by the high-order model.

In Section 2, we describe the potential-flow models for the evolution of an unstable interface. In Section 3, we present the high-order multi-harmonic model for the bubble evolution in two dimensions and derive the asymptotic solution from the model. In Section 4, the model for the bubble in the cylindrical flow is presented. Section 5 gives conclusions.

## 2. Potential-flow models

We consider an interface in a vertical channel between fluids with infinite density ratio in two dimensions. The upper fluid is heavier than the lower fluid (vacuum). We assume that the fluid is incompressible, inviscid, and irrotational. Let $(x, y)$ be the Cartesian coordinates. The channel width $L$, $[L] = $ m, and the gravitational acceleration $g$, $[g] = $ m/s$^2$, are the basic scales of the instability. The velocity potential satisfies

$$\Delta \phi = 0, \tag{1}$$

with the zero velocity condition at the far field

$$\frac{\partial \phi}{\partial y} \to 0 \qquad \text{for } y \to \infty. \tag{2}$$

The bubble moves in the positive $y$-direction with the tip velocity $U$. It is convenient to choose a frame of reference comoving with the tip of the bubble. The interface near the bubble tip is approximated as

$$\eta(x, y, t) = y + \sum_{j=1}^{\infty} \zeta_j(t) x^{2j} = 0, \tag{3}$$

where $\zeta_1(t)$ is the principal curvature at the bubble tip. The evolution of the interface is determined by the kinematic condition and the Bernoulli equation

$$\frac{d\eta(x, y, t)}{dt} = v + \sum_{j=1}^{\infty} \frac{d\zeta_j}{dt} x^{2j} + 2 \sum_{j=1}^{\infty} j \, \zeta_j \, u \, x^{2j-1} = 0, \tag{4}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \left( g + \frac{dU}{dt} \right) y = 0, \tag{5}$$

where $u$ and $v$ are $x$- and $y$-components of the interface velocity taken from the fluid.

The velocity potential can be taken as

$$\phi(x, y, t) = \sum_{m=1}^{\infty} a_m(t) \left( \frac{1}{mk} e^{-mky} \cos(mkx) + y \right), \tag{6}$$

where $k = 2\pi/L$ is the wave number. Layzer [13] expanded the kinematic condition (4) and the Bernoulli equation (5) by the second order in $x$, using the first harmonic in the potential (6), and obtained the asymptotic solution for the bubble growth

$$(7) \qquad \zeta_1 = \frac{k}{6}, \qquad U = \sqrt{\frac{g}{3k}}.$$

Abarzhi [1] presented the second-order solution for the bubble motion, retaining the first and second harmonics in (6), i.e., $m = 1$ and 2. The velocity is expressed as a function of the curvature

$$(8) \qquad U = \frac{3}{2}\sqrt{\frac{g}{k}}\sqrt{\frac{2\zeta_1/k[1 - 4(\zeta_1/k)^2]^2}{1 + 4\zeta_1/k + 4(\zeta_1/k)^2}}.$$

Abarzhi chose the fastest solution in the family (8). Imposing the condition,

$$(9) \qquad \frac{\partial U}{\partial \zeta_1}(\zeta_1 = \zeta_1^*) = 0, \qquad \frac{\partial^2 U}{\partial \zeta_1^2}(\zeta_1 = \zeta_1^*) < 0,$$

she obtained the solution

$$(10) \qquad \zeta_1^* = \frac{k}{6}, \qquad U = \sqrt{\frac{g}{3k}},$$

which is the same as the solution (7) by Layzer. A stability analysis for steady solutions is also given in [1]. It was shown that the region of stable solutions is narrow and includes the fastest solution. The dependence of the velocity on the parameter in this region is weak.

## 3. High-order model

We present a high-order extension of the multi-harmonic model in two dimensions. We take the four harmonics in (6), $1 \leq m \leq 4$, and approximate the interface in the fourth-order in $x$,

$$(11) \qquad \eta(x, y, t) = y + \zeta_1(t)x^2 + \zeta_2(t)x^4 = 0.$$

The quantities in (4) and (5) are approximated as

$$\frac{\partial \phi}{\partial t} = \sum_{m=1}^{4} \frac{1}{mk}\frac{da_m}{dt} - \frac{1}{2}\frac{dM_1}{dt}x^2 + \left(\frac{1}{2}\frac{dM_1}{dt}\zeta_1^2 - \frac{1}{2}\frac{dM_1}{dt}\zeta_1 + \frac{1}{24}\frac{dM_3}{dt}\right)x^4,$$

$$\frac{\partial \phi}{\partial x} = -M_1 x + \left(-M_2\zeta_1 + \frac{1}{6}M_3\right)x^3,$$

$$\frac{\partial \phi}{\partial y} = \left(-M_1\zeta_1 + \frac{1}{2}M_2\right)x^2 + \left(-M_1\zeta_2 - \frac{1}{2}M_2\zeta_1^2 + \frac{1}{2}M_3\zeta_1 - \frac{1}{24}M_4\right)x^4,$$

where $M_j$ are defined by the moments

$$M_j(t) = \sum_{m=1}^{4} a_m(t)(km)^j$$

for $0 \leq j \leq 4$. Then, the second- and fourth-order equations in $x$ of the kinematic condition (4) are given by

$$\frac{d\zeta_1}{dt} - 3\zeta_1 M_1 + \frac{M_2}{2} = 0, \tag{12}$$

$$\frac{d\zeta_2}{dt} - 5M_1\zeta_2 - \frac{5}{2}M_2\zeta_1^2 + \frac{5}{6}M_3\zeta_1 - \frac{1}{24}M_4 = 0. \tag{13}$$

The second- and fourth-order equations of the Bernoulli equation (5) are

$$\frac{1}{2}\frac{dM_1}{dt} - \zeta_1\frac{dM_0}{dt} - \frac{M_1^2}{2} + \zeta_1 g = 0, \tag{14}$$

$$\frac{1}{2}\frac{dM_1}{dt}\zeta_1^2 - \frac{1}{2}\frac{dM_2}{dt}\zeta_1 + \frac{1}{24}\frac{dM_3}{dt} + \frac{dM_0}{dt}\zeta_2 + \frac{1}{2}M_1M_2\zeta_1$$

$$+\frac{1}{2}M_1^2\zeta_1^2 - \frac{1}{6}M_1M_3 + \frac{1}{8}M_2^2 - g\zeta_2 = 0. \tag{15}$$

Equations (12)-(15) determine the dynamics of the bubble.

We find the asymptotic (or steady state) solution of the model. All the time derivatives in (12)-(15) are taken as zero. The definition of the moment gives

$$a_1 = \frac{24M_1k^3 - 26M_2k^2 + 9M_3k - M_4}{6k^4},$$

$$a_2 = \frac{-12M_1k^3 + 19M_2k^2 - 8M_3k + M_4}{4k^4},$$

$$a_3 = \frac{8M_1k^3 - 14M_2k^2 + 7M_3k - M_4}{6k^4},$$

$$a_4 = \frac{-6M_1k^3 + 11M_2k^2 - 6M_3k + M_4}{24k^4}.$$

From the far field boundary condition, in the comoving frame, the bubble velocity is written as

$$U = -M_0 = -(a_1 + a_2 + a_3 + a_4)$$

$$= \frac{-50M_1k^3 + 35M_2k^2 - 10M_3k + M_4}{24k^4}. \tag{16}$$

From (12) and (13), $\zeta_1$ and $\zeta_2$ at a late time are

$$\zeta_1 = \frac{M_2}{6M_1}, \tag{17}$$

(18)
$$\zeta_2 = -\frac{1}{72}\frac{M_2^3}{M_1^3} + \frac{1}{36}\frac{M_2 M_3}{M_1^2} - \frac{1}{120}\frac{M_4}{M_1}.$$

At a late time, (14) gives

$$M_1 = -\sqrt{2g\zeta_1}.$$

Here, the negative sign is taken from the quadratic equation. Substituting this into (17), we have

$$M_2 = -6\sqrt{2g}\,\zeta_1^{3/2}.$$

Then, (15) and (18) give the expressions for $M_3$ and $M_4$,

$$M_3 = 3\sqrt{2g}\left(\frac{-16\zeta_1^3 + \zeta_2}{\sqrt{\zeta_1}}\right),$$

$$M_4 = 60\sqrt{2g}\left(-10\zeta_1^{7/2} + 3\sqrt{\zeta_1}\zeta_2\right).$$

The bubble velocity is expressed as a function of $\zeta_1$ and $\zeta_2$,

(19)
$$U = \frac{5\sqrt{2}}{4}\sqrt{\frac{g}{k}}\left[\frac{5}{3}\sqrt{\frac{\zeta_1}{k}} - 7\left(\frac{\zeta_1}{k}\right)^{\frac{3}{2}} + 16\left(\frac{\zeta_1}{k}\right)^{\frac{5}{2}} - 20\left(\frac{\zeta_1}{k}\right)^{\frac{7}{2}}\right.$$
$$\left. + \left(6 - \frac{1}{\frac{\zeta_1}{k}}\right)\sqrt{\frac{\zeta_1}{k}}\frac{\zeta_2}{k^3}\right].$$

The bubble velocity is a function of *two* parameters in the high-order model, while it is a function of one parameter in the low-order model.

We now find the critical point in the family (19), applying the similar idea of the low-order model. This approach imposes the solution to satisfy the conditions

(20)
$$\frac{\partial U}{\partial \zeta_1}(\zeta_1 = \zeta_1^*,\ \zeta_2 = \zeta_2^*) = 0,$$

(21)
$$\frac{\partial U}{\partial \zeta_2}(\zeta_1 = \zeta_1^*,\ \zeta_2 = \zeta_2^*) = 0.$$

The second condition is not coupled in the two parameters, because (19) is a linear function in $\zeta_2$. We thus obtain the asymptotic bubble curvature,

(22)
$$\zeta_1^* = \frac{k}{6}.$$

The condition (20) gives

(23)
$$\zeta_2^* = \frac{7}{324}k^3.$$

Equation (19) has only one critical point $(\zeta_1^*, \zeta_2^*)$, which is identified as a saddle point, because $(U_{\zeta_1\zeta_2})^2 - U_{\zeta_1\zeta_1}U_{\zeta_2\zeta_2} > 0$. The contour plot of (19) near the critical point is shown in Fig. 2. The bubble velocity is constant on the line $\zeta_1/k = 1/6$. The geometric structure in Fig. 2 shows that the solution is chosen
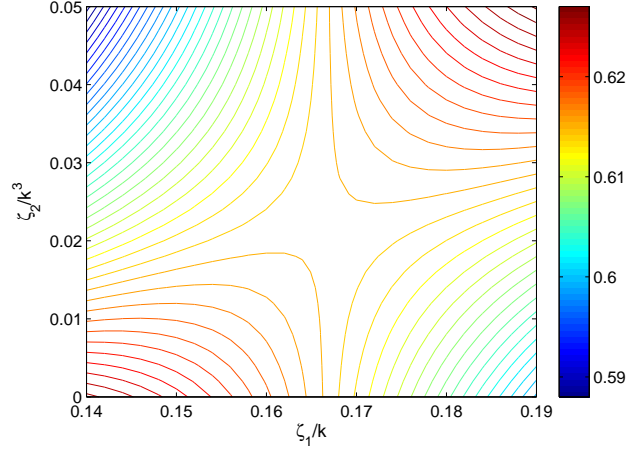
FIGURE 2. Contour plot of the bubble velocity in $\zeta_1/k$ and $\zeta_2/k^3$. It shows that the critical point $(\frac{1}{6}, \frac{7}{324}) = (0.1667, 0.0217)$ is a saddle point.

as the *fastest* in $\zeta_1$, but is *constant* in $\zeta_2$. The asymptotic bubble velocity is therefore obtained by

$$(24) \qquad U = \frac{115}{108}\sqrt{\frac{g}{3k}}.$$

The bubble velocity has the correction factor $115/108 = 1.065$, but the bubble curvature is the same as the second-order solution (10).

## 4. Model in the cylindrical flow

In this section, we consider an axisymmetric interface in a cylindrical tube of the radius $R$ (See Fig. 1 in [13]). The interface near the tip of the bubble can be written as

$$(25) \qquad z = \eta(r,t) = \sum_{j=0}^{\infty} \zeta_j(t)\, r^{2j},$$

where $r$ and $z$ represent the axisymmetric coordinates. The velocity potential of the fluid is taken as

$$(26) \qquad \phi(x,y,t) = \sum_{m=1}^{\infty} a_m(t)\left(\frac{1}{k_m} J_0(k_m r)\, e^{-k_m z} + z\right),$$

where $k_m = \beta_m/R$ and $\beta_m$ is the $m$-th zero of the Bessel function $J_1(r)$. Note that Goncharov [10] proposed $J_0(mkr)$ instead of $J_0(k_m r)$ in the potential of a high order model. The potential should be expressed by (26), because $\{J_0(k_m r)\}$ forms a complete orthogonal basis for a function in $0 \le r \le R$.

Layzer [13] used the first harmonic and presented the asymptotic solution for the bubble,

$$(27) \qquad \zeta_1 = \frac{k_1}{8}, \qquad U = \sqrt{\frac{g}{k_1}}.$$

We take the first and second harmonics in (26), $m = 1$ and 2, and approximate the interface in the second-order in $r$,

$$(28) \qquad \eta(r, t) = z + \zeta_1(t) r^2 = 0.$$

Satisfying the kinematic condition and the Bernoulli equation in the second-order in $r$, we have the equations for the bubble growth,

$$(29) \qquad \frac{d\zeta_1}{dt} - 2\zeta_1 M_1 + \frac{M_2}{4} = 0,$$

$$(30) \qquad \frac{1}{4}\frac{dM_1}{dt} - \zeta_1 \frac{dM_0}{dt} - \frac{M_1^2}{8} + \zeta_1\, g = 0.$$

Here, the moments are defined as $M_j(t) = \sum_{m=1}^{2} a_m(t)(k_m)^j$. The bubble velocity is then expressed as a function of the curvature,

$$(31) \qquad U = \frac{2\sqrt{2g}}{k_1 k_2} \left[ \zeta_1^{\frac{1}{2}}(k_1 + k_2) - 8\zeta_1^{\frac{3}{2}} \right].$$

We choose the fastest solution in this family, i.e., $U'(\zeta_1^*) = 0$, and find the asymptotic bubble curvature,

$$(32) \qquad \zeta_1^* = \frac{1}{24}(k_1 + k_2).$$

From (31), the asymptotic bubble velocity is obtained as

$$(33) \qquad U = \frac{2}{3\sqrt{3}} \frac{\sqrt{(k_1 + k_2)^3 g}}{k_1 k_2}.$$

We emphasize that the solution (32) and (33) depends on both wavenumbers $k_1$ and $k_2$ and has a different functional form from the solution of the Layzer model (27). Recall that in two dimensions, the second-order solution of the multi-harmonic model is the same as Layzer's. Nevertheless, the quantitative difference between the solutions (32) and (33), and (27) is small; the solution (32) and (33) can be rewritten as

$$(34) \qquad \zeta_1^* = \frac{k_1}{8.48}, \qquad U = 1.0013\sqrt{\frac{g}{k_1}}$$

from the fact $\beta_2/\beta_1 = 7.0156/3.8317 = 1.8309$. The difference of the bubble curvature of the two solutions is only 6%, and the bubble velocities are nearly the same.

## 5. Discussion and conclusions

We have presented the multi-harmonic models for the RT instability in two and three dimensions and have found new types of solutions of the bubble motion. In the high-order model in two dimensions, the asymptotic solutions are expressed in the family of the two free parameters, and the critical point is chosen as the physically significant solution. Interestingly, the asymptotic bubble velocity is, locally, the fastest in the parameter of the curvature, but is constant in the parameter $\zeta_2$. Therefore, the bubble curvature still acts as the key parameter in the approach of the high-order model, similarly as the low-order model. As a result of the high-order model, we obtain the correction factor of 7% in the asymptotic bubble velocity.

Although we did not attempt the stability analysis for the asymptotic solution, we expect that the stability region is very narrow, including the critical point. Abarzhi [1] found that as the order of approximation of the model is higher, the stability region becomes narrowed and bifurcations points are brought together. The stability analysis for the high-order model requires much more works of numerical calculations and we leave it for future study.

A different high-order model was proposed by Goncharov [10], taking only odd harmonics in the potential (6). This approach requires no free parameters in the equations. Using the odd harmonics up to $m \leq 7$, he showed the asymptotic solution

$$(35) \qquad \zeta_1 = \frac{k}{4.88}, \qquad U = 1.025\sqrt{\frac{g}{3k}}.$$

Note that the predictions for the bubble velocity are little different among the models, but the solutions of the bubble curvature are fairly different. Numerical results showed the convergence of the curvature to $\zeta_1 \approx k/4$, at late times [20].

We have also presented the multi-harmonic model in the cylindrical geometry and have obtained a new type of asymptotic solution of the axisymmetric bubble. The functional form of the solution of the present model is different from that of Layzer model, while the solutions of the two models are the same in two dimensions. In fact, the different type of solution comes from the use of the orthogonal functions for the potential in the present model.

In summary, we have successfully extended the multi-harmonic model to a high-order in two dimensions and to the cylindrical flow. We conclude that the approach of the multi-harmonic model is appropriate for the interface of infinite density ratio, although it has a limitation for quantitative prediction for the bubble curvature.

For the case of a finite density contrast, the validity of the approach of the multi-harmonic model is under question [23]. The solution of the multi-harmonic model differs significantly from numerical and experimental results, for low density ratios [16, 24]. So far there is no theoretical model which gives

quantitatively correct prediction for the RT bubble and simultaneously satisfies the far field boundary conditions. This subject is open to researchers.

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