J. Korean Math. Soc. **54** (2017), No. 1, pp. 281–302
https://doi.org/10.4134/JKMS.j150728
pISSN: 0304-9914 / eISSN: 2234-3008

GENERALIZED BROWDER, WEYL SPECTRA AND THE POLAROID PROPERTY UNDER COMPACT PERTURBATIONS

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To Robin Harte, a dear friend, on his 78th birthday

Abstract. For a Banach space operator $A \in B(\mathcal{X})$, let $\sigma(A)$, $\sigma_a(A)$, $\sigma_w(A)$ and $\sigma_{aw}(A)$ denote, respectively, its spectrum, approximate point spectrum, Weyl spectrum and approximate Weyl spectrum. The operator A is polaroid (resp., left polaroid), if the points iso$\sigma(A)$ (resp., iso$\sigma_a(A)$) are poles (resp., left poles) of the resolvent of A. Perturbation by compact operators preserves neither SVEP, the single-valued extension property, nor the polaroid or left polaroid properties. Given an $A \in B(\mathcal{X})$, we prove that a sufficient condition for: (i) $A+K$ to have SVEP on the complement of $\sigma_w(A)$ (resp., $\sigma_{aw}(A)$) for every compact operator $K \in B(\mathcal{X})$ is that $\sigma_w(A)$ (resp., $\sigma_{aw}(A)$) has no holes; (ii) $A+K$ to be polaroid (resp., left polaroid) for every compact operator $K \in B(\mathcal{X})$ is that iso$\sigma_w(A) = \emptyset$ (resp., iso$\sigma_{aw}(A) = \emptyset$). It is seen that these conditions are also necessary in the case in which the Banach space $\mathcal{X}$ is a Hilbert space.

1. Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Banach (resp., Hilbert) space into itself. A well known result on "the permanence of distinguished parts of the spectrum of an operator" says that if an $A \in B(\mathcal{X})$ commutes with a Riesz operator $R \in B(\mathcal{H})$, $[A, R] = AR - RA = 0$, then $\sigma_x(A+K) = \sigma_x(A)$ for a variety of choices of the spectrum σ_x, amongst them the Browder spectrum σ_b, the Weyl spectrum σ_w, the approximate Browder spectrum σ_{ab} and the approximate Weyl spectrum σ_{aw} (see [20] and [10, Theorem 3.21]). A necessary and sufficient condition for $\sigma_w(A) = \sigma_b(A)$ for an

Received December 2, 2015.
2010 *Mathematics Subject Classification.* 47A10, 47A55, 47A53, 47B40.
Key words and phrases. Banach space operator, compact perturbation, SVEP, polaroid, left polaroid, B-Fredholm spectrum, Browder's theorem, Weyl's theorem, abstract shift condition.
The second named author was supported by the National Research Foundation of Korea, Grant funded by the Korean Government (No. 2013R1A1A2009365).

operator $A \in B(\mathcal{X})$ is that A has SVEP, *the single-valued extension property*, on the complement of $\sigma_w(A)$ (in $\mathbb{C}$, the set of complex numbers) [1, Theorem 3.52]. The perturbation of an operator in $B(\mathcal{X})$ by a commuting Riesz operator in $B(\mathcal{X})$ preserves SVEP at points, i.e., if $[A, R] = 0$ for an operator $A \in B(\mathcal{X})$ and a Riesz operator $R \in B(\mathcal{X})$, then A has SVEP at a point λ if and only if $A + R$ has SVEP at λ [2]. Choosing $R = K \in B(\mathcal{X})$ to be a compact operator it thus follows that if $[A, K] = 0$ and A has SVEP on $\mathbb{C} \setminus \sigma_w(A)$, then $\sigma_w(A + K) = \sigma_b(A + K)$. Here the hypothesis $[A, K] = 0$ is essential. For example, if $U \in B(\mathcal{X})$ (resp., $V \in B(\mathcal{X})$), $\mathcal{X} = \ell_p$ or c_0, $1 \le p < \infty$, is the forward unilateral shift (resp., backward unilateral shift), then the unitary operator $A = \left(\begin{smallmatrix} U & 1-UV \\ 0 & V \end{smallmatrix}\right)$ (has SVEP everywhere, and) satisfies $\sigma_w(A) = \sigma_b(A) = \partial\mathcal{D}$ (= the boundary of the unit disc in $\mathbb{C}$). The compact operator $K = \left(\begin{smallmatrix} 0 & 1-UV \\ 0 & 0 \end{smallmatrix}\right) \in B(\mathcal{X} \oplus \mathcal{X})$ does not commute with A; clearly, $\sigma_w(A - K) = \partial\mathcal{D} \ne \sigma_b(A - K) = \mathcal{D}$. In contrast (to the operator A above), the operator $B = (U + 1) \oplus (V - 1)$ has SVEP at points in the complement in $\mathbb{C}$ of $\sigma(B) = \sigma_w(B) = \{\lambda : |\lambda - 1| \le 1\} \cup \{\lambda : |\lambda + 1| \le 1\} = \sigma_b(B)$. Additionally, since $\sigma_w(B + K) = \sigma_w(B)$ for every compact operator K, $\sigma_w(B + K) = \sigma_b(B + K)$ for every compact operator $K \in B(\mathcal{X} \oplus \mathcal{X})$. Observe here that whereas $\mathbb{C}\setminus\sigma_w(B)$ is connected, $\mathbb{C}\setminus\sigma_w(A)$ is not connected. We prove in the following that this behaviour is typical in the sense that: "Given $A \in B(\mathcal{X})$ and a compact operator $K \in B(\mathcal{X})$, a sufficient condition for $\sigma_w(A+K) = \sigma_b(A+K)$ is that $\sigma_w(A)$ has no *holes*; furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then this condition is necessary too."

$A \in B(\mathcal{X})$ is said to be *polaroid* (resp., *left polaroid*) if points $\lambda \in \mathrm{iso}\sigma(A)$, the isolated points of the spectrum of A, are poles of the resolvent of A (resp., points $\lambda \in \mathrm{iso}\sigma_a(A)$, the isolated points of the approximate point spectrum of A, are left poles of A) [3]. A problem related to, but distinct from, the problem of the permanence of the equality of Browder and Weyl spectra under compact perturbations is that of the permanence of the polaroid (resp., left polaroid) property under compact perturbations. For example, if $U \in B(\mathcal{X})$ is the forward unilateral shift, then $U + K$ is polaroid (resp., left polaroid) for every compact operator $K \in B(\mathcal{X})$. Again, if $B \in B(\mathcal{H})$ is such that $\mathrm{iso}\sigma(\mathrm{B}) = \mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{B})$, then there exists a compact operator $K_0 \in B(\mathcal{H})$ such that $\sigma(B + K_0) = \sigma_w(B + K_0) = \sigma_w(B)$ [12, Theorem 3.48]. Let $A = B + K_0$. Then $A+K$ is not polaroid (also, not left polaroid) for every compact operator $K \in B(\mathcal{H})$. We prove in the following that these (two) examples are typical in the sense that: "Given $A \in B(\mathcal{X})$ and a compact operator $K \in B(\mathcal{X})$, a sufficient condition for $A + K$ to be polaroid (resp., left polaroid) is that $\mathrm{iso}\sigma_w(A) = \emptyset$ (resp., $\mathrm{iso}\sigma_{aw}(A) = \emptyset$). Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then this condition is necessary too."

The results we obtain have applications in Browder and Weyl type theorems (see [1, 4, 6, 8, 9] for information on Browder, Weyl type theorems). Let $\sigma_{uBw}(A)$ denote the upper B-Weyl spectrum of A [3, 4, 6, 9], and let $E(A)$ denote the set of isolated eigenvalues of A. (Our notation, mostly standard, is explained in the following sections.) We prove in the following that: "If $\text{iso}\sigma_{aw}(A) = \emptyset$ for an operator $A \in B(\mathcal{X})$, then a sufficient condition for $\sigma_a(A+K) \setminus \sigma_{uBw}(A+K) = E(A+K)$ for every compact operator $K \in B(\mathcal{X})$ is that the set of $\lambda \in \mathbb{C}$ for which $A-\lambda$ is upper semi-Fredholm with index $\text{ind}(A-\lambda) \leq 0$ is connected and A^* has SVEP on $\sigma_a(A) \setminus \sigma_{aw}(A)$. Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then this condition is necessary too." We consider applications to "weighted right shifts" and to "operators satisfying the abstract shift condition".

2. Preliminaries

In keeping with standard terminology, we shall denote the spectrum, the approximate point spectrum, the surjectivity spectrum and the isolated points of the spectrum of an operator $A \in B(\mathcal{X})$ by $\sigma(A)$, $\sigma_a(A)$, $\sigma_s(A)$ and $\text{iso}\sigma(\text{A})$, respectively. The boundary of a subset S of the set $\mathbb{C}$ of complex numbers will be denoted by ∂S and the interior of S will denoted by int(S). An operator $A \in B(\mathcal{X})$ has SVEP (= *the single-valued extension property*) at a point $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \longrightarrow \mathcal{X}$ satisfying $(A-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$. (Here we have shortened $A - \lambda I$ to $A - \lambda$.) Evidently, every A has SVEP at points in the resolvent $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.* The *ascent of* A, asc(A) (resp. *descent of* A, dsc(A)), is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$ (resp., $A^n(\mathcal{X}) = A^{n+1}(\mathcal{X})$): If no such integer exists, then asc(A) (resp. dsc(A))$= \infty$. It is well known that $\text{asc}(A) < \infty$ implies A has SVEP at 0, $\text{dsc}(A) < \infty$ implies A^* (= the dual operator) has SVEP at 0, finite ascent and descent for an operator implies their equality, and that a point $\lambda \in \sigma(A)$ is a pole (of the resolvent) of A if and only if $\text{asc}(A-\lambda) = \text{dsc}(A-\lambda) < \infty$ (see [1, 15, 18]).

An operator $A \in B(\mathcal{X})$ is: *upper semi–Fredholm at* $\lambda \in \mathbb{C}$, $\lambda \in \Phi_{uf}(A)$ or $A-\lambda \in \Phi_{uf}(\mathcal{X})$, if $A(\mathcal{X})$ is closed and the deficiency index $\alpha(A-\lambda) = \dim(\text{A}-\lambda)^{-1}(0) < \infty$; *lower semi–Fredholm at* $\lambda \in \mathbb{C}$, $\lambda \in \Phi_{lf}(A)$ or $A-\lambda \in \Phi_{lf}(\mathcal{X})$, if $\beta(A-\lambda) = \dim(\mathcal{X}/(\text{A}-\lambda)(\mathcal{X})) < \infty$; A is semi–Fredholm, $\lambda \in \Phi_{sf}(A)$ or $A-\lambda \in \Phi_{sf}(\mathcal{X})$, if $A-\lambda$ is either upper or lower semi–Fredholm, and A is Fredholm, $\lambda \in \Phi(A)$ or $A-\lambda \in \Phi(\mathcal{X})$, if $A-\lambda$ is both upper and lower semi–Fredholm. The index of a semi–Fredholm operator is the integer $\text{ind}(A) = \alpha(A) - \beta(A)$. Corresponding to these classes of one sided Fredholm operators, we have the following spectra: The *upper Fredholm spectrum* $\sigma_{uf}(A)$ of A defined by $\sigma_{uf}(A) = \{\lambda \in \sigma(A) : A-\lambda \notin \Phi_{uf}(\mathcal{X})\}$, and the *lower Fredholm spectrum* $\sigma_{lf}(A)$ of A defined by $\sigma_{lf}(A) = \{\lambda \in \sigma(A) : A-\lambda \notin \Phi_{lf}(\mathcal{X})\}$. The *Fredholm spectrum* $\sigma_f(A)$ of A is the set $\sigma_f(A) = \sigma_{uf}(A) \cup \sigma_{lf}(A)$, and the *Wolf spectrum* $\sigma_{ulf}(A)$ of A is the set $\sigma_{ulf}(A) = \sigma_{uf}(A) \cap \sigma_{lf}(A)$. $A \in B(\mathcal{X})$ is upper Weyl (resp., lower Weyl, (simply) Weyl) at 0 if it is upper Fredholm with $\text{ind}(A) \leq 0$ (resp., lower Fredholm with $\text{ind}(A) \geq 0$, Fredholm with $\operatorname{ind}(A)=0$). The upper (or, approximate) Weyl spectrum, the lower (or, surjectivity) Weyl spectrum and the Weyl spectrum of A are respectively the sets $\sigma_{aw}(A)=\{\lambda\in\sigma_a(A):\lambda\notin\Phi_{uf}(A)$ or $\operatorname{ind}(A-\lambda)\not\leq 0\}$, $\sigma_{sw}(A)=\{\lambda\in\sigma_s(A):\lambda\notin\Phi_{lf}(A)$ or $\operatorname{ind}(A-\lambda)\not\geq 0\}$ and $\sigma_w(A)=\sigma_{aw}(A)\cup\sigma_{sf}(A)$. It is well known that a semi- Fredholm operator A (resp., its conjugate operator A^*) has SVEP at a point λ if and only if $\operatorname{asc}(A-\lambda)<\infty$ (resp., $\operatorname{dsc}(A-\lambda)<\infty$) [1, Theorems 3.16, 3.17]; furthermore, if $A-\lambda$ is Weyl (resp., upper Weyl), i.e., if $\lambda\in\Phi(A)$ and $\operatorname{ind}(A-\lambda)=0$ (resp., $\lambda\in\Phi_{uf}(A)$ and $\operatorname{ind}(A-\lambda)\leq 0$), then A has SVEP at λ implies $\lambda\in\mathrm{iso}\sigma(\mathrm{A})$ with $\operatorname{asc}(A-\lambda)=\operatorname{dsc}(A-\lambda)<\infty$ (resp., $\lambda\in\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A})$ with $\operatorname{asc}(A-\lambda)<\infty$). Let

$$\sigma_{ab}(A)=\{\lambda\in\sigma_a(A):\lambda\notin\Phi_{uf}(A)\ \text{ or }\ \mathrm{asc(A-\lambda)}\not<\infty\}\ \text{ and}$$
$$\sigma_{sb}(A)=\{\lambda\in\sigma_s(A):\lambda\notin\Phi_{lf}(A)\ \text{ or }\ \mathrm{des(A-\lambda)}\not<\infty\}$$

denote, respectively, the approximate (or upper) and the surjectivity (or lower) Browder spectrum of A. Then $\sigma_{sb}(A)=\sigma_{ab}(A^*)$ (resp., $\sigma_{sb}(A)=\overline{\sigma_{ab}(A^*)}$ if $\mathcal{X}=\mathcal{H}$ is a complex Hilbert space), and $\sigma_b(A)=\sigma_{ab}(A)\cup\sigma_{sb}(A)$ is the Browder spectrum of A.

If $\mathcal{X}=\mathcal{H}$ is a Hilbert space, and $A\in B(\mathcal{H})$ is such that $\lambda\in\Phi_{sf}(A)$, then *the minimal index of* $A-\lambda$ is the integer

$$\min\{\alpha(\mathrm{A}-\lambda),\beta(\mathrm{A}-\lambda)\}=\min\{\alpha(\mathrm{A}-\lambda),\alpha(\mathrm{A}-\lambda)^*\}.$$

It is well known that the function $\lambda\to\mathrm{min.ind(A-\lambda)}$ is constant on every component of $\Phi_{sf}(A)$ (except perhaps for a denumerable subset without limit points in $\Phi_{sf}(A)$) [12, Corollary 1.14].

3. Complementary results

Given a Banach space operator $A\in B(\mathcal{X})$, let

$$\Pi(A)=\{\lambda\in\mathrm{iso(A)}:\mathrm{asc(A-\lambda)}=\mathrm{des(A-\lambda)}<\infty\}$$

denote the set of poles (of the resolvent) of A, and let $\Pi_0(A)$ denote *the Riesz points* (i.e., isolated points of the spectrum of A which are finite rank poles of the resolvent) of A. For a given $\epsilon>0$, let

$$\sigma_0(A)=\{\lambda\in\Pi_0(A):\mathrm{dist}(\lambda,\partial\Phi_{\mathrm{sf}}(\mathrm{A}))\geq\epsilon/2\}.$$

Then $\sigma_0(A)$ is a clopen subset of $\sigma(A)$, and it follows from the Riesz decomposition theorem that there exists a decomposition $\mathcal{X}=\mathcal{X}_1\oplus\mathcal{X}_2$ of $\mathcal{X}$, and a corresponding upper triangular matrix representation

$$A=\begin{pmatrix}A_1 & *\\ 0 & A_2\end{pmatrix}\in B(\mathcal{X}_1\oplus\mathcal{X}_2) \tag{1}$$

of A such that $\sigma(A_1)=\sigma_0(A)$ and $\sigma(A_2)=\sigma(A)\setminus\sigma_0(A)$. Evidently, A is similar to $A_1\oplus A_2$; hence if Γ denotes either of $\mathrm{iso}\sigma_w(A)$, $\mathrm{iso}\sigma_{aw}(A)$ and $\Phi^-_{sf}(A)=\{\lambda\in\mathbb{C}:\lambda\in\Phi_{sf}(A),\operatorname{ind}(A-\lambda)<0\}$, then $\Gamma\neq\emptyset$ if and only if (the corresponding set) $\mathrm{iso}\sigma_w(A_2)$, $\mathrm{iso}\sigma_{aw}(A_2)$ and $\Phi^-_{sf}(A_2)$ is not empty.

It is easily seen that iso$\sigma_w(A) \subseteq$ iso$\sigma_{aw}(A) \subseteq \sigma_{ulf}(A)$; hence if a complex number $\lambda \in$ iso$\sigma_w(A) \cup$ iso$\sigma_{aw}(A)$, then $\lambda \in \sigma_{ulf}(A)$. Again, if $\Phi^-_{sf}(A) \neq \emptyset$, then there exists a λ ($\in \Phi^-_{sf}(A)$, and hence) $\in \sigma_{aw}(A)$. The following lemma lies at the heart of a number of our arguments below.

Lemma 3.1 ([16, Lemma 2.10]). *If* $(\emptyset \neq)\ \Gamma \subset \sigma_{uf}(A)$, *in particular if* $\emptyset \neq \Gamma = \mathrm{iso}\sigma_w(A) \cup \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A})$ *for an operator* $A \in B(\mathcal{H})$, *then for every* $\epsilon > 0$ *there exists a compact operator* $K \in B(\mathcal{H})$ *with* $||K|| < \epsilon$ *and a decomposition* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ *of* $\mathcal{H}$ *such that*

$$A + K = \begin{pmatrix} N & * \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

where N *is a diagonal normal operator of uniform infinite multiplicity,* $\sigma(N) = \sigma_{ulf}(N) = \Gamma$, $\sigma(A_2) = \sigma(A)$, $\sigma_{ulf}(A_2) = \sigma_{ulf}(A)$ *and* $ind(A_2 - \lambda) = ind(A - \lambda)$ *for all* $\lambda \in \Phi_{sf}(A)$.

Recall from [12, Proposition 3.47] that given an operator $A \in B(\mathcal{H})$ and an $\epsilon > 0$ there exists a compact operator $K \in B(\mathcal{H})$, $||K|| < \epsilon/2$, such that $\Pi_0(A) = \Pi_0(A + K)$. For an operator $A \in B(\mathcal{H})$ such that

$$d = \max\{\mathrm{dist}(\lambda, \partial\Phi_{sf}(A)) : \lambda \in \Pi_0(A)\}$$

is arbitrarily small, we have the following:

Lemma 3.2 ([12, Theorem 3.48]). *If* $d < \epsilon/2$ *for an arbitrary* $\epsilon > 0$, *then there exists a compact operator* $K \in B(\mathcal{H})$, $||K|| \leq \epsilon/2 + d < \epsilon$, *such that* $\min.ind(\mathrm{A} + \mathrm{K} - \lambda) = 0$ *for all* $\lambda \in \Phi_{sf}(A)$ *and* $\sigma(A + K) = \sigma_w(A)$ $(= \sigma_w(A + K))$.

It is clear from Lemma 3.2 that the operator $A + K - \mu$ is left invertible for every $\mu \in \Phi^-_{sf}(A + K)$. More generally one has:

Corollary 3.3. *Given an operator* $A \in B(\mathcal{H})$, *there exists a compact operator* $K \in B(\mathcal{H})$ *such that* $A + K - \mu$ *is left invertible for every* $\mu \in \Phi^-_{sf}(A)$.

Proof. Starting with the upper triangular representation (1) above for A, we use Lemma 3.2 to construct a compact operator $K \in B(\mathcal{H}_2)$ for $A_2 \in B(\mathcal{H}_2)$ such that $\sigma(A_2 + K_2) = \sigma_w(A_2)$ and $\min.\mathrm{ind}(\mathrm{A}_2 + \mathrm{K}_2 - \mu) = 0$ for all $\mu \in \Phi_{sf}(A_2)$. Now let $K = 0 \oplus K_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and let $\mu \in \Phi^-_{sf}(A + K) = \Phi^-_{sf}(A)$. Then

$$\mathrm{ind}(A - \mu) = \mathrm{ind}(A_2 + K_2 - \mu) < 0, \text{ and } \min.\mathrm{ind}(\mathrm{A}_2 + \mathrm{K}_2 - \mu) = 0.$$

Evidently, $\mu \in \sigma(A + K)$, $\mu \notin \sigma(A_1)$ and $A_2 + K_2 - \mu$ is bounded below (equivalently, left invertible). Hence $A + K - \mu$ is left invertible. □

A duality argument proves the following.

Corollary 3.4. *For every* $A \in B(\mathcal{H})$, *there exists a compact operator* $K \in B(\mathcal{H})$ *such that* $A + K - \mu$ *is right invertible for every* $\mu \in \Phi^+_{sf}(A)$.

4. Spectral equalities $\sigma_w(A+K)=\sigma_b(A+K)$ and $\sigma_{aw}(A+K)=\sigma_{ab}(A+K)$

For an operator $A \in B(\mathcal{X})$ and a subset $\mathcal{S}$ of $\mathbb{C}$, let $\Xi_A(\mathcal{S}) = \{\lambda \in \mathcal{S} : A$ does not have SVEP at $\lambda\}$. Then

$$\begin{aligned}
&\sigma_w(A) = \sigma_b(A) \Longleftrightarrow \Xi_A(\sigma(A) \setminus \sigma_w(A)) = \emptyset,\\
&\sigma_{aw}(A) = \sigma_{ab}(A) \Longleftrightarrow \Xi_A(\sigma_a(A) \setminus \sigma_{aw}(A)) = \emptyset,\\
&\sigma_{sw}(A) = \sigma_{sb}(A) \Longleftrightarrow \Xi_A(\sigma_s(A) \setminus \sigma_{sw}(A)) = \emptyset
\end{aligned}$$

[1, Theorem 3.52]. The equalities $\sigma_w(A) = \sigma_b(A)$ and $\sigma_{aw}(A) = \sigma_{ab}(A)$ imply, respectively, that

$$\sigma(A) \setminus \sigma_w(A) = \Pi_0(A) \text{ and } \sigma_a(A) \setminus \sigma_{aw}(A) = \Pi_0^a(A), \tag{2}$$

where ($\Pi^a(A)$ denotes the set of left poles of A and) $\Pi_0^a(A)$ denotes the set of finite rank left poles of A. Following current terminology, *we say that operators A satisfying identities* (2) *satisfy Browder's theorem* (*or, Bt for short*) *and, respectively, a-Browder's theorem* (*or, $a-Bt$ for short*). Thus, an operator $A \in B(\mathcal{X})$ satisfies

$$Bt \Longleftrightarrow A \text{ has SVEP on } \sigma(A) \setminus \sigma_w(A); \tag{3}$$

$$a - Bt \Longleftrightarrow A \text{ has SVEP on } \sigma_a(A) \setminus \sigma_{aw}(A). \tag{4}$$

Perturbation of an operator by a commuting compact (indeed, Riesz) operator preserves SVEP at points [2]. This, however, fails for non-commuting compact operators. For example: (i) The operator $A = \begin{pmatrix} U & I-UV \\ 0 & V \end{pmatrix} \in B(\ell^p(\mathbb{N}) \oplus \ell^p(\mathbb{N}))$, where $1 \le p < \infty$, U is the forward unilateral shift and V is the backward unilateral shift on $\ell^p(\mathbb{N})$, has SVEP (everywhere) but its perturbation by the compact operator $K = \begin{pmatrix} 0 & -I+UV \\ 0 & 0 \end{pmatrix}$ does not have SVEP at points λ such that $|\lambda| < 1$; (ii) if $A \in B(\mathcal{H})$ is such that $\Phi_{sf}^+(A) = \{\lambda \in \Phi_{sf}(A) : \mathrm{ind}(A-\lambda) > 0\} = \emptyset$, then there exists a compact operator $K \in B(\mathcal{H})$ such that $A+K$ has (empty point spectrum [14, Proposition 3.4] and hence) SVEP irrespective of whether A has SVEP or not.

A *hole* of a compact subset of $\mathbb{C}$ (more generally, of a subset of a topological space) is any bounded component of its complement [17]. Thus, a hole of $\sigma_w(A)$ (resp., $\sigma_{aw}(A)$) for an operator A is a bounded maximal connected subset of $\mathbb{C} \setminus \sigma_w(A)$ (resp., $\mathbb{C} \setminus \sigma_{aw}(A)$). Given an operator A, define the sets $\Omega(A)$ and $\Omega_a(A)$ by

$$\begin{aligned}
\Omega(A) &= \{\lambda \in \Phi_{sf}(A) : \mathrm{ind}(A-\lambda) = 0\} \text{ and}\\
\Omega_a(A) &= \{\lambda \in \Phi_{uf}(A) : \mathrm{ind}(A-\lambda) \le 0\}.
\end{aligned}$$

It is then clear that for an operator $A \in B(\mathcal{X})$ such that $\Omega(A)$ is connected, the resolvent $\rho(A)$ intersects $\Omega(A)$ and hence both A and A^* have SVEP on $\Omega(A)$ [1, Theorem 3.36]. Observe that $\Omega(A)$ is connected whenever $\sigma_w(A)$ has no holes. The following theorem says that, given an operator $A \in B(\mathcal{X})$, the hypothesis $\sigma_w(A)$ (resp., $\sigma_{aw}(A)$) has no holes is sufficient to ensure SVEP at points in $\sigma(A+K)\setminus\sigma_w(A+K)$ (resp., $\sigma_a(A+K)\setminus\sigma_{aw}(A+K)$) for the operator $A+K$ for every compact operator $K\in B(\mathcal{X})$; furthermore, if $\mathcal{X}=\mathcal{H}$ is a Hilbert space, then this condition is necessary too.

Theorem 4.1. *Let $A\in B(\mathcal{X})$. A sufficient condition for $\sigma_w(A+K)=\sigma_b(A+K)$ (resp., $\sigma_{aw}(A+K)=\sigma_{ab}(A+K)$) for every compact operator $K\in B(\mathcal{X})$ is that $\sigma_w(A)$ (resp., $\sigma_{aw}(A)$) has no holes. Furthermore, if our Banach space is a Hilbert space, then this condition is necessary too.*

Proof. We prove by contradiction.
Sufficiency. In view of our observations above, to prove the sufficiency of the conditions it would suffice to prove that if $\Omega(A)$ (resp., $\Omega_a(A)$) is connected then $A+K$ has SVEP for every compact operator K on $\sigma(A+K)\setminus\sigma_w(A+K)$ (resp., $\sigma_a(A+K)\setminus\sigma_{aw}(A+K)$). Suppose then that $\Omega(A)$ (resp., $\Omega_a(A)$) has just one component, namely itself. Suppose further that there exists a compact operator $K\in B(\mathcal{X})$ such that $A+K$ fails to have SVEP on $\Omega(A)$ (resp., $\Omega_a(A)$). Then, since the conjugate operator $(A+K)^*$ has SVEP and $A+K$ fails to have SVEP at a point $\lambda\in\Phi_{sf}(A+K)=\Phi_{sf}(A)$ (resp., point $\lambda\in\Phi_{uf}(A+K)=\Phi_{uf}(A)$) implies $\mathrm{ind}(A-\lambda)>0$, we must have that both $A+K$ and $(A+K)^*$ fail to have SVEP on $\Omega(A)$ (resp., $\Omega_a(A)$). Hence $\mathrm{asc(A+K-\lambda)}=\mathrm{des(A+K-\lambda)}=\infty$ for all $\lambda\in\Omega(A)$ (resp., $\lambda\in\Omega_a(A)$). On the other hand, since the resolvent set $\rho(A+K)\subset\Omega(A)$ (resp., $\rho(A+K)\subset\Omega_a(A)$), the continuity of the index at points $\lambda\in\Omega(A)$ (resp., $\lambda\in\Omega_a(A)$) implies that $\mathrm{ind}(A+K-\lambda)=0$. But then $\alpha(A+K-\lambda)=\beta(A+K-\lambda)=0$ (resp., $\alpha(A+K-\lambda)=0$) at every $\lambda\in\Omega(A)$ (resp., $\lambda\in\Omega_a(A)$) except perhaps for a countable subset. Since $\lambda\in\Phi_{sf}(A)$ and $\alpha(A+K-\lambda)=\beta(A+K-\lambda)=0$ implies $\mathrm{asc(A+K-\lambda)}=\mathrm{des(A+K-\lambda)}=0$ (hence that $A+K-\lambda$ is invertible), and since $\lambda\in\Phi_{uf}(A)$ and $\alpha(A+K-\lambda)=0$ implies $\mathrm{asc(A+K-\lambda)}=0$ (hence that $A+K-\lambda$ is bounded below), we have a contradiction.
Necessity. Suppose now that $A\in B(\mathcal{H})$, and $A+K$ has SVEP for every compact operator $K\in B(\mathcal{H})$ on the complement of $\sigma_w(A+K)$ (resp., on the complement of $\sigma_{aw}(A+K)$) in $\mathbb{C}$. Suppose further that $\Omega(A)$ (resp., $\Omega_a(A)$) has a bounded component $\Omega_0(A)$. Then $\Gamma=\partial\Omega_0(A)\subset\sigma_{ulf}(A)$, and there exists (see Lemma 3.1) a compact operator $K_1\in B(\mathcal{H})$ such that $A+K_1$ has the upper triangular matrix representation

$$A+K_1=\begin{pmatrix} N & * \\ 0 & A_2\end{pmatrix}\in B(\mathcal{H}_1\oplus\mathcal{H}_2),\ \ \dim(\mathcal{H}_1)=\infty,$$

with respect to some decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ of $\mathcal{H}$. Here N is a normal diagonal operator of uniform infinite multiplicity, $\sigma(N)=\sigma_{uf}(N)=\Gamma$, $\sigma_{uf}(A_2)=\sigma_{uf}(A)$ and $\mathrm{ind}(A_2-\lambda)=\mathrm{ind}(A-\lambda)$ for all $\lambda\in\Phi_{sf}(A)$. The spectrum $\sigma(N)=\Gamma$ of N being the boundary of a bounded connected open subset of $\mathbb{C}$, we can find a compact operator $K_2\in B(\mathcal{H}_1)$ such that $\sigma(N+K_2)$ equals the closure $\overline{\Omega_0(A)}$: This follows from [13, Theorem 3.1]. Define the compact operator $K \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by $K = K_1 + (K_2 \oplus 0)$. Then

$$A + K = \begin{pmatrix} N + K_2 & * \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{H}),$$

where for every $\mu \in \Omega_0(A)$ we have $\mu \in \Phi_{sf}(N + K_2) = \Phi_{sf}(N)$ with $\operatorname{ind}(N + K_2 - \mu) = 0$. It is easily seen that SVEP for $A + K$ at a point implies SVEP for $N + K_2$ at the point. Hence every $\mu \in \Omega_0(A)$ is an isolated point − a contradiction. Conclusion: $\Omega(A)$, resp. $\Omega_a(A)$, has no bounded component. □

Given $A \in B(\mathcal{X})$, a necessary condition for $A + K$ to have SVEP for every compact $K \in B(\mathcal{X})$ is that $\Phi^+_{sf}(A) = \{\lambda \in \mathbb{C} : \lambda \in \Phi_{sf}(A), \operatorname{ind}(A-\lambda) > 0\} = \emptyset$. This is seen as follows. If $\Phi^+_{sf}(A) = \Phi^+_{sf}(A + K) \neq \emptyset$, then there exists a complex number λ such that $\lambda \in \Phi_{sf}(A + K)$ and $\operatorname{ind}(A + K - \lambda) > 0$. Since $\lambda \in \Phi_{sf}(T)$ for an operator $T \in B(\mathcal{X})$ with SVEP at λ implies $\operatorname{ind}(T - \lambda) \leq 0$ [1, Corllary 3.19], we have a contradiction. The hypotheses of Theorem 4.1 do not guarantee "SVEP everywhere" for $A + K$ for all compact operators K. SVEP everywhere requires more.

Theorem 4.2. *Let $A \in B(\mathcal{X})$ and let $K \in B(\mathcal{X})$ be a compact operator. Then*

(a) *A sufficient condition for $A + K$ and $(A + K)^*$ to have SVEP is that*

(i) *$\Omega(A)$ is connected*;

(ii) $\operatorname{int}(\sigma_w(A)) = \emptyset$.

Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then this condition is necessary too.

(b) *A sufficient condition for $A + K$ to have SVEP is that*

(i) *$\Omega_a(A)$ is connected*;

(ii) $\operatorname{int}(\sigma_{aw}(A)) = \emptyset$.

Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then this condition is necessary too.

Proof. Sufficiency. (a) In view of the observations that the adjoint of a compact operator $K \in B(\mathcal{X})$ is again a compact operator, $\Phi_{sf}(A) = \Phi_{uf}(A) \cup \Phi_{lf}(A) = \Phi_{lf}(A^*) \cup \Phi_{uf}(A^*) = \Phi_{sf}(A^*)$ and $\operatorname{ind}(A-\lambda) = 0$ if and only if $\operatorname{ind}(A^* - \lambda) = 0$ for an operator $A \in B(\mathcal{X})$, to prove the sufficiency of the conditions in part (a) of the theorem it would suffice to consider simply the operator A. As observed in the proof of Theorem 4.1, condition (i) implies ($\operatorname{asc}(A-\lambda) = \operatorname{dsc}(A-\lambda) < \infty$ for every $\lambda \in \Phi_{sf}(A)$, and hence)

$$\begin{aligned} \Omega(A) &= \Omega(A + K) = \rho(A + K) \cup \Pi_0(A + K) \\ &= \{\mathbb{C} \setminus \sigma(A + K)\} \cup \Pi_0(A + K) \\ \Longrightarrow \qquad \sigma_w(A + K) &= \sigma(A + K) \setminus \Pi_0(A + K) \end{aligned}$$

for every compact operator $K \in B(\mathcal{X})$. Now if also (ii) is satisfied, then

$$\begin{aligned} &\emptyset = \operatorname{int}(\sigma_w(A)) = \operatorname{int}(\sigma_w(A + K)) = \operatorname{int}(\sigma(A + K) \setminus \Pi_0(A + K)) \\ \Longrightarrow \quad &\operatorname{int}(\sigma(A + K)) = \emptyset \end{aligned}$$

for every compact operator $K \in B(\mathcal{X})$. Thus $\sigma(A+K) = \partial\sigma(A+K)$; consequently, $A+K$ has SVEP everywhere.

(b) If $\Omega_a(A)$ is connected, then (it follows from the proof of Theorem 4.1 that) $\mathrm{asc}(A+K-\lambda) < \infty$ for every $\lambda \in \Phi_{uf}(A) = \Phi_{uf}(A+K)$. Hence, for every compact operator $K \in B(\mathcal{H})$,

$$\Omega_a(A+K) = \rho_l(A+K) \cup \Pi_0^a(A+K) = \mathbb{C} \setminus \sigma_a(A+K) \cup \Pi_0^a(A+K)$$

(where $\rho_l(A+K) = \{\lambda \in \Phi_{uf}(A+K) : \alpha(A+K-\lambda) = 0\} = \{\lambda \notin \sigma_a(A+K)\}$). Now if also condition (b)(ii) is satisfied, then

$$\begin{aligned} &\emptyset = \mathrm{int}(\sigma_{aw}(A)) = \mathrm{int}(\sigma_{aw}(A+K)) = \mathrm{int}(\sigma_a(A+K) \setminus \Pi_0^a(A+K)) \\ \implies\ &\mathrm{int}(\sigma_a(A+K)) = \emptyset. \end{aligned}$$

Hence $A+K$ has SVEP (everywhere).

Necessity. The necessity of conditions (a)(i) and (b)(i) is immediate from Theorem 4.1; the necessity of conditions (a)(ii) and (b)(ii) follows from the following argument. Assume that both $A+K$ and $(A+K)^*$ have SVEP (resp., $A+K$ has SVEP) for every compact operator K. Let $\Delta(A) = \sigma_w(A)$ in the case in which $\mathrm{int}(\sigma_w(A)) \neq \emptyset$ and let $\Delta(A) = \sigma_{aw}(A)$ in the case in which $\mathrm{int}(\sigma_{aw}(A)) \neq \emptyset$. If either of the conditions (a)(ii) and (b)(ii) fails, then $\mathrm{int}(\Delta(A)) \neq \emptyset$. Let $\lambda \in \mathrm{int}(\Delta(A))$, and let, for an arbitrary $\epsilon > 0$, $B_\epsilon(\lambda)$ be a ball of radius ϵ centered at λ such that $\Gamma = \partial B_\epsilon(\lambda) \subset \Delta(A)$. Then, see Lemma 3.1, there exists a compact operator $K_0 \in B(\mathcal{H})$ and a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$ such that $A+K_0 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ has a representation $A+K_0 = \left(\begin{smallmatrix} N & * \\ 0 & A_2 \end{smallmatrix}\right)$ for some normal operator N (of uniform infinite multiplicity) such that $\sigma(N) = \sigma_{uf}(N) = \Gamma$. (Observe that points $\mu \in \Gamma$ belong to $\sigma_{ulf}(A)$ (resp., $\sigma_{uf}(A)$) : for if such a $\mu \in \Phi_{sf}(A)$, then our SVEP assumption implies that $\mu \in \Phi(A)$ satisfies $\mathrm{ind}(A-\mu) = 0$ (resp., $\mu \in \Phi_{uf}(A)$ satisfies $\mathrm{ind}(A-\mu) \leq 0$), i.e., $\mu \notin \sigma_w(A)$ (resp., $\mu \notin \sigma_{aw}(A)$).) Recall from [12, Theorem 3.1] that there exists a compact operator $K_{11} \in\ B(\mathcal{H}_1)$ such that $\sigma(N+K_{11}) = \Gamma \cup B_\epsilon(\lambda) = \overline{B_\epsilon(\lambda)}$ (= the closure of $B_\epsilon(\lambda)$). The operator $N+K_{11}$ does not have SVEP on $B_\epsilon(\lambda)$: for if $N+K_{11}$ has SVEP at a point $\mu \in B_\epsilon(\lambda)$, then $\mu \in \Phi_{sf}(N) = \Phi_{sf}(N+K_{11})$ with $\mathrm{ind}(N-\mu) = \mathrm{ind}(N+K_{11}-\mu) = 0$, and hence (in the presence of SVEP) $\mu \in \mathrm{iso}\sigma(\mathrm{N}+\mathrm{K}_{11})$. Define $K_1 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by $K_1 = K_{11} \oplus 0$, and let $K = K_0 + K_1$. Then the operator $A+K$ has SVEP implies $N+K_{11}$ has SVEP− a contradiction. Hence $A+K$ does not have SVEP− again a contradiction. Conclusion: $\mathrm{int}(\Delta(\mathrm{A})) = \emptyset$. □

The following corollary follows from Theorems 4.1 and 4.2 by a duality argument.

Corollary 4.3. *Given an operator $A \in B(\mathcal{H})$, $\sigma_{sw}(A+K) = \sigma_{sb}(A+K)$ for every compact operator $K \in B(\mathcal{H})$ if and only if $\sigma_{sw}(A) = \overline{\sigma_{aw}(A^*)}$ (= the complex conjugate of $\sigma_{aw}(A^*)$) has no holes. Furthermore A^*+K has SVEP (everywhere) for all compact operators $K \in B(\mathcal{H})$ if and only if* (i) $\Omega_s(A) = \{\lambda \in \Phi_{lf}(A) : \mathit{ind}(A-\lambda) \geq 0\}$ *is connected and* (ii) $\mathrm{int}(\sigma_{\mathrm{sw}}(\mathrm{A})) = \emptyset$.

Remark 4.4. (i) Zhu and Li [22, Theorem 1.3] prove that a necessary and sufficient condition for an operator $A \in B(\mathcal{H})$ to satisfy the property that $A + K$ has SVEP (everywhere) for every compact operator $K \in B(\mathcal{H})$ is that $\Phi_{sf}(A)$ is connected and $\mathrm{int}(\sigma_{ulf}(A)) = \emptyset$. Theorem 4.2 subsumes this result, as the following argument shows. Since $\Phi_{sf}(A)$ is the disjoint union of the (possible empty) open sets $\{\Phi^n_{sf}(A)\} \cup \Phi^{\infty}_{sf}(A) \cup \Phi^{-\infty}_{sf}(A)$, $\Phi^n_{sf}(A) = \{\lambda \in \mathbb{C} : \lambda \in \Phi_{sf}(A), \mathrm{ind}(A - \lambda) = n\}$, $\Phi_{sf}(A)$ is connected implies $\Phi^0_{sf}(A) = \Omega(A)$ is connected, and this in turn implies $\sigma_w(A+K) = \sigma(A+K) \setminus \Pi_0(A+K)$. Since $\rho(A + K) \subseteq \Phi^0_{sf}(A + K)$, we also have $\sigma_{ulf}(A + K) = \sigma(A + K) \setminus \Pi_0(A + K)$ (see [12, Corollary 1.14(v)]). Hence $\mathrm{int}(\sigma_{ulf}(A)) = \emptyset$ implies $\mathrm{int}(\sigma_w(A)) = \emptyset$.

(ii) It is clear from the hypotheses of Theorem 4.2 that the class of operators A for which $A + K$ has SVEP for every compact operator K is indeed very small. A better proposition here might be to start with an operator A such that $A + K$ satisfies Bt (or a-Bt) for a compact operator K, and then look for additional hypotheses guaranteeing SVEP for $A + K$. For example, let $A + K \in B(\mathcal{X})$ satisfy Bt. Then A has an upper triangular representation $A + K = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{X}_1 \oplus \mathcal{X}_2)$ such that $\sigma(A_1) = \Pi_0(A + K)$ and $\sigma(A_2) = \sigma_w(A) = \sigma_w(A_2) = \sigma(A) \setminus \Pi_0(A)$. Evidently, A_1 has SVEP. If we now assume that the set $\sigma_w(A_2) \cap \sigma_p(A_2)$ is at best countable, then A_2 has SVEP. This then implies that $A + K$ has SVEP.

5. B-Fredholm operators: Spectral equalities $\boldsymbol{\sigma_{Bw}(A + K) = \sigma_{Bb}(A + K)}$ and $\boldsymbol{\sigma_{aBw}(A + K) = \sigma_{aBb}(A + K)}$

An operator $T \in B(\mathcal{X})$ is semi B-Fredholm, $T \in \Phi_{sBf}(\mathcal{X})$, if there exists an integer $n \geq 1$ such that $T^n(\mathcal{X})$ is closed and the induced operator $T_{[n]} = T|_{T^n(\mathcal{X})}$, $T_{[0]} = T$, is semi Fredholm. It is seen that if $T_{[n]} \in \Phi_{sf}(\mathcal{X})$ for an integer $n \geq 1$, then $T_{[m]} \in \Phi_{sf}(\mathcal{X})$ for all integers $m \geq n$, and one may (unambiguously) define the index of T by $\mathrm{ind}(T) = \alpha(T) - \beta(T)$ $(= \mathrm{ind}(T_{[n]}))$ [6]. Upper semi B-Fredholm, lower semi B-Fredholm and B-Fredholm spectra of T are the sets

$$\begin{aligned}\sigma_{uBf}(T) &= \{\lambda \in \sigma(T) : \lambda \notin \Phi_{uBf}(T)\} \\ &= \{\lambda \in \sigma(T) : T - \lambda \text{ is not upper semi B-Fredholm}\},\end{aligned}$$

$$\begin{aligned}\sigma_{lBf}(T) &= \{\lambda \in \sigma(T) : \lambda \notin \Phi_{lBf}(T)\} \\ &= \{\lambda \in \sigma(T) : T - \lambda \text{ is not lower semi B-Fredholm}\}\end{aligned}$$

and

$$\sigma_{Bf}(T) = \sigma_{uBf}(T) \cup \sigma_{lBf}(T).$$

If we let

$$\begin{aligned}&\sigma_{Bw}(T = \{\lambda \in \sigma(T) : \lambda \in \sigma_{Bf}(T) \text{ or } \mathrm{ind}(T - \lambda) \neq 0\}, \\ &\sigma_{uBw}(T) = \{\lambda \in \sigma_a(T) : \lambda \in \sigma_{uBf}(T) \text{ or } \mathrm{ind}(T - \lambda) \not\leq 0\}, \\ &\sigma_{lBw}(T) = \{\lambda \in \sigma_s(T) : \lambda \in \sigma_{lBf}(T) \text{ or } \mathrm{ind}(T - \lambda) \not\geq 0\},\end{aligned}$$

$$\sigma_{Bb}(T = \{\lambda \in \sigma(T) : \lambda \in \sigma_{Bf}(T) \text{ or } \operatorname{asc}(T-\lambda) \neq \operatorname{dsc}(T-\lambda)\},$$
$$\sigma_{uBb}(T) = \{\lambda \in \sigma_a(T) : \lambda \in \sigma_{uBf}(T) \text{ or } \operatorname{asc}(T-\lambda) = \infty\}$$

and

$$\sigma_{lBb}(T) = \{\lambda \in \sigma_s(T) : \lambda \in \sigma_{lBf}(T) \text{ or } \operatorname{dsc}(T-\lambda) = \infty\}$$

denote, respectively, the *the B-Weyl, the upper B-Weyl, the lower B-Weyl, the B-Browder, the upper B-Browder* and *the lower B-Browder spectrum* of T, then $\sigma_{Bw}(T) = \sigma_{uBw}(T) \cup \sigma_{lBw}(T)$, $\sigma_{Bb}(T) = \sigma_{uBb}(T) \cup \sigma_{lBb}(T)$, $\sigma_{uBw}(T) = \sigma_{lBw}(T^*)$ and $\sigma_{lBb}(T) = \sigma_{uBb}(T^*)$. Let

$$\Pi^a(T) = \{\lambda \in \sigma_a(T) : \operatorname{asc}(T-\lambda) = d < \infty \text{ and } (T-\lambda)^{d+1}(\mathcal{X}) \text{ is closed}\}$$

denote, the set of left poles of T, and let $\Pi_0^a(T)$ denote the set of finite rank left poles of T. The following implications are well known [4, Theorems 2.1 and 2.2]:

$$\begin{aligned}\sigma_w(T) = \sigma_b(T) &\Longleftrightarrow \sigma_{Bw}(T) = \sigma_{Bb}(T) \Longleftrightarrow \sigma(T) \setminus \sigma_{Bw}(T) = \Pi(T) \\ &\Longleftrightarrow T \text{ has SVEP at points in } \sigma(T) \setminus \sigma_{Bw}(T)\end{aligned} \tag{4}$$

and

$$\begin{aligned}\sigma_{aw}(T) = \sigma_{ab}(T) &\Longleftrightarrow \sigma_{uBw}(T) = \sigma_{uBb}(T) \\ &\Longleftrightarrow \sigma_a(T) \setminus \sigma_{uBw}(T) = \Pi^a(T) \\ &\Longleftrightarrow T \text{ has SVEP at points in } \sigma_a(T) \setminus \sigma_{uBw}(T).\end{aligned} \tag{5}$$

Evidently, $\sigma_{aw}(T) \subseteq \sigma_w(T)$ and $\sigma_{uBw}(T) \subseteq \sigma_{Bw}(T)$; hence

$$\begin{aligned}\sigma_{uBw}(T) = \sigma_{uBb}(T) &\Longleftrightarrow \sigma_{aw}(T) = \sigma_{ab}(T) \\ \Longrightarrow \sigma_w(T) = \sigma_b(T) &\Longleftrightarrow \sigma_{Bw}(T) = \sigma_{Bb}(T)\end{aligned} \tag{6}$$

(where the one way implications are strict). Combining this with Theorem 4.1, we have:

Theorem 5.1. *Given an operator $A \in B(\mathcal{X})$, the condition $\sigma_w(A)$ (resp., $\sigma_{aw}(A)$) has no holes is sufficient for $\sigma_{Bw}(A+K) = \sigma_{Bb}(A+K)$ (resp., $\sigma_{uBw}(A+K) = \sigma_{uBb}(A+K)$) for every compact operator $K \in B(\mathcal{X})$. Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then the condition is necessary too.*

In keeping with current terminology [4, 6, 8], we say in the following that an operator

A satisfies generalized Browder's theorem, or gBt, if $\sigma_{Bw}(A) = \sigma_{Bb}(A)$;
A satisfies generalized a-Browder's theorem, or a-gBt, if $\sigma_{uBw}(A) = \sigma_{uBb}(A)$.

It is clear that every pole of the resolvent of an operator $A \in B(\mathcal{X})$ is both a left and a right pole of the operator; hence $\Pi(A) \subseteq \Pi^a(A)$ (and $\Pi_0(A) \subseteq \Pi_0^a(A)$) for every $A \in B(\mathcal{X})$. A sufficient (indeed, also necessary) condition for $\lambda \in \Pi^a(A)$ to imply $\lambda \in \Pi(A)$ is that A^* has SVEP at λ. Observe from Theorems 4.1 and 4.2(b) that the hypothesis $\Omega_a(A)$ is connected does not guarantee SVEP for A^* on $\sigma_a(A) \setminus \sigma_{aw}(A)$. The following theorem says that the hypotheses $\Omega_a(A)$ is connected and A^* has SVEP on $\sigma_a(A)\setminus\sigma_{aw}(A)$ guarantee $\Pi^a(A+K)=\Pi(A+K)$ for all compact operators K.

Theorem 5.2. *Given an operator $A\in B(\mathcal{X})$ such that $\Omega_a(A)$ is connected, $\sigma(A+K)\setminus\sigma_{uBw}(A+K)=\Pi(A+K)$ for every compact operator $K\in B(\mathcal{X})$ if and only if A^* has SVEP on $\sigma_a(A)\setminus\sigma_{aw}(A)$.*

Proof. The hypothesis $\Omega_a(A)$ is connected implies $A+K$ satisfies a-gBt for every compact operator $K\in B(\mathcal{X})$, i.e.,

$$\sigma_a(A+K)\setminus\sigma_{uBw}(A+K)=\Pi^a(A+K)$$

for every compact operator $K\in B(\mathcal{X})$.
Sufficiency. We prove that $\Pi^a(A+K)\subseteq\Pi(A+K)$ for every compact operator $K\in B(\mathcal{X})$. Let $\lambda\in\Pi^a(A+K)$. Then $A+K-\lambda$ is upper semi B-Fredholm and $\mathrm{ind}(A+K-\lambda)\le 0$: There exists a (large enough) positive integer n such that $A+K-\lambda-\frac{1}{n}$ is upper semi Fredholm and $\mathrm{ind}(A+K-\lambda-\frac{1}{n})\le 0$ [11, Corollary 3.2]. Consequently $A-\lambda-\frac{1}{n}$ is upper semi Fredholm and $\mathrm{ind}(A-\lambda-\frac{1}{n})\le 0$. Since A^* has SVEP at points in $\sigma_a(A)\setminus\sigma_{aw}(A)$ implies $\mathrm{ind}(A-\lambda-\frac{1}{n})\ge 0$, it follows that $A-\lambda-\frac{1}{n}$ is Fredholm and $\mathrm{ind}(A-\lambda-\frac{1}{n})=0$. But then $A+K-\lambda-\frac{1}{n}$ is Fredholm and $\mathrm{ind}(A+K-\lambda-\frac{1}{n})=0$ for every compact operator $K\in B(\mathcal{X})$. Since $\lambda\in\Pi^a(A+K)$ implies $\lambda\in\mathrm{iso}(\sigma_a(A+K))$, $A+K$ has SVEP at $\lambda+\frac{1}{n}$ (for large enough n). Hence

$$\mathrm{asc}(A+K-\lambda-\frac{1}{n})=\mathrm{dsc}(A+K-\lambda-\frac{1}{n})<\infty,$$

and this (by [11, Corollary 4.8]) implies that

$$\mathrm{asc}(A+K-\lambda)=\mathrm{dsc}(A+K-\lambda)<\infty,$$

i.e., $\lambda\in\Pi(A+K)$.
Necessity. If we let K equal the 0 operator, then $\sigma_a(A)\setminus\sigma_{uBw}(A)=\Pi^a(A)=\Pi(A)$ implies A^* has SVEP at points in $\sigma_a(A)\setminus\sigma_{uBw}(A)$. Since $\sigma_{uBw}(A)\subseteq\sigma_{aw}(A)$, the necessity follows. $\square$

Remark 5.3. (i) The requirement that A^* has SVEP at points in $\sigma_a(A)\setminus\sigma_{aw}(A)$ in Theorem 5.1 may be replaced by the requirement that A^* has SVEP at points in $\sigma_a(A)\setminus\sigma_{uBw}(A)$. Indeed, as the following argument shows, "A^* has SVEP at points in $\sigma_a(A)\setminus\sigma_{aw}(A)$ if and only if A^* has SVEP at points in $\sigma_a(A)\setminus\sigma_{uBw}(A)$". As seen in the necessity of the proof of Theorem 5.1 above, $\sigma_a(A)\setminus\sigma_{aw}(A)\subseteq\sigma_a(A)\setminus\sigma_{uBw}(A)$, and hence A^* has SVEP at points in $\sigma_a(A)\setminus\sigma_{uBw}(A)$ implies A^* has SVEP at points in $\sigma_a(A)\setminus\sigma_{aw}(A)$. Conversely, let A^* have SVEP at points in $\sigma_a(A)\setminus\sigma_{aw}(A)$. If $\lambda\in\sigma_a(A)\setminus\sigma_{uBw}(A)$, then (as seen above) there exists a large enough integer n such that $\lambda+\frac{1}{n}\in\Phi_{uf}(A)$ with $\mathrm{ind}(A-\lambda-\frac{1}{n})\le 0$. But then $\lambda+\frac{1}{n}\in\sigma_a(A)\setminus\sigma_{aw}(A)$, and hence (since A^* has SVEP at $\lambda+\frac{1}{n}$) $\lambda+\frac{1}{n}\in\Phi(A)$, $\mathrm{ind}(A-\lambda-\frac{1}{n})=0$ and $\mathrm{asc}(A-\lambda-\frac{1}{n})=\mathrm{dsc}(A-\lambda-\frac{1}{n})<\infty$. This implies that $\mathrm{asc}(A-\lambda)=\mathrm{dsc}(A-\lambda)<\infty$. Trivially, A^* has SVEP at λ.

(ii) In contrast to Theorem 5.1, the hypothesis $\Omega(A)$ is connected for an $A \in B(\mathcal{X})$ is both necessary and sufficient for $\sigma(A+K) \setminus \sigma_{Bw}(A+K) = \Pi^a(A+K)$ for every compact operator $K \in B(\mathcal{X})$. This follows since $\lambda \in \Pi^a(A+K)$ implies $A+K$ has SVEP at λ, hence if also $\lambda \in \sigma(A+K) \setminus \sigma_{Bw}(A+K)$, then $\lambda \in \Pi(A+K)$. Thus $\Pi^a(A+K) = \Pi(A+K)$ for every $K \in B(\mathcal{X})$.

The following question arises naturally out of Theorem 5.1: "Given an operator $A \in B(\mathcal{X})$, is the condition $\sigma_{Bw}(A)$ (resp., $\sigma_{uBw}(A)$) has no holes sufficient and/or necessary for $\sigma_{Bw}(A+K) = \sigma_{Bb}(A+K)$ (resp., $\sigma_{uBw}(A+K) = \sigma_{uBb}(A+K)$) for every compact operator $K \in B(\mathcal{X})$?" The following argument proves the sufficiency of the condition for the case in which $\sigma_{Bw}(A)$ has no holes; the proof for the other case is similar. Given a complex number $\lambda \notin \sigma_{Bw}(A)$, we can find a large enough integer $n = n(\lambda)$ such that $A-(\lambda+\frac{1}{n})$ is semi B-Fredholm with $\operatorname{ind}(A-\lambda-\frac{1}{n}) = 0$. Hence if $\sigma_{Bw}(A)$ has no holes, then the set

$$\begin{aligned}
&\Omega_B(A)\\
&= \{\lambda : A-\lambda \text{ is semi B-Fredholm and } \operatorname{ind}(A-\lambda)=0\}\\
&= \{\lambda+\frac{1}{n} : A-\lambda-\frac{1}{n} \text{ is semi Fredholm and } \operatorname{ind}(A-\lambda-\frac{1}{n})=0\}\\
&= \{\lambda+\frac{1}{n} : A+K-\lambda-\frac{1}{n} \text{ is semi Fredholm and } \operatorname{ind}(A+K-\lambda-\frac{1}{n})=0\}
\end{aligned}$$

is connected for every compact operator $K \in B(\mathcal{X})$. Assuming that $A+K$ fails to have SVEP at λ, and arguing as in the sufficiency part of the proof of Theorem 4.1, it now follows that $\operatorname{asc}(A+K-\lambda-\frac{1}{n}) = \operatorname{dsc}(A+K-\lambda-\frac{1}{n}) = 0$, and hence that $\operatorname{asc}(A+K-\lambda) = \operatorname{dsc}(A+K-\lambda) = 0$ (except perhaps for a countable set of λ). This is a contradiction. We have proved:

Proposition 5.4. *A sufficient condition for $A+K$ to satisfy gBt (resp., a-gBt) for a given operator $A \in B(\mathcal{X})$ and every compact operator $K \in B(\mathcal{X})$ is that $\sigma_{Bw}(A)$ (resp., $\sigma_{uBw}(A)$) has no holes.*

Proposition 5.4 generalizes [6, Theorem 4.6] and other similar results.

6. Generalized Weyl's theorem and compact perturbations

For an operator $A \in B(\mathcal{X})$, let

$$\begin{aligned}
E(A) &= \{\lambda \in \operatorname{iso}\sigma(A) : 0 < \alpha(A-\lambda)\},\\
E_0(A) &= \{\lambda \in E(A) : \alpha(A-\lambda) < \infty\},\\
E^a(A) &= \{\lambda \in \operatorname{iso}\sigma_a(A) : 0 < \alpha(A-\lambda)\} \text{ and}\\
E_0^a(A) &= \{\lambda \in E^a(A) : \alpha(A-\lambda) < \infty\}.
\end{aligned}$$

Then

$$\Pi(A) \subseteq E(A), \quad \Pi_0(A) \subseteq E_0(A), \quad \Pi^a(A) \subseteq E^a(A), \quad \text{and} \quad \Pi_0^a(A) \subseteq E_0^a(A).$$

Observe that if $\lambda \in \Pi^a(A)$ (resp., $\lambda \in \Pi_0^a(A)$), then there exists a positive integer p such that $H_0(A-\lambda) = (A-\lambda)^{-p}(0)$ (resp., $H_0(A-\lambda) = (A-\lambda)^{-p}(0)$ with $\dim(\mathrm{H_0(A-\lambda)}) < \infty$). (Here, $H_0(A-\lambda) = \{x \in \mathcal{X} : \lim_{n\to\infty} ||(A-\lambda)^n x||^{\frac{1}{n}} = 0\}$ is the *quasi-nilpotent part* of $A-\lambda$ [1]. It is easily seen that a point $\lambda \in \mathrm{iso}\sigma(A)$ is a pole of the resolvent of A if and only if $H_0(A-\lambda) = (A-\lambda)^{-p}(0)$ for some integer $p > 0$.) If $\lambda \in \Pi_0^a(A)$, then the existence of an integer $n \geq 1$ such that $(A-\lambda)^n(\mathcal{X})$ is closed and $(\alpha(A-\lambda) < \infty \Longrightarrow)$ $\alpha(A-\lambda)^n < \infty$ imply $((A-\lambda)^n$, hence also) $A-\lambda$ is upper semi-Fredholm.

We say in the following that *an operator* $A \in B(\mathcal{X})$ *is polaroid* (resp., *left polaroid*) if $\lambda \in \mathrm{iso}\sigma(A)$ implies $\lambda \in \Pi(A)$ (resp., $\lambda \in \mathrm{iso}\sigma_a(A)$ implies $\lambda \in \Pi^a(A)$); A is *right polaroid* if A^* is left polaroid. The operator A is said to satisfy

Weyl's theorem, or Wt, if $\sigma(A) \setminus \sigma_w(A) = E_0(A)$;

generalized Weyl's theorem, or gWt, if $\sigma(A) \setminus \sigma_{Bw}(A) = E(A)$;

a-Weyl's theorem, or a-Wt, if $\sigma_a(A) \setminus \sigma_{aw}(A) = E_0^a(A)$;

generalized a-Weyl's theorem, or a-gWt, if $\sigma_a(A) \setminus \sigma_{uBw}(A) = E^a(A)$.

The following implications are well known [1, 4, 6, 8]:

$$\text{gWt} \Longrightarrow \text{Wt} \Longrightarrow \text{Bt} \Longleftrightarrow \text{gBt},\ \text{a-gWt} \Longrightarrow \text{gWt},$$

$$\text{a-gWt} \Longrightarrow \text{a-Wt} \Longrightarrow \text{a-Bt} \Longrightarrow \text{Bt},\ \text{a-gWt} \Longrightarrow \text{a-gBt} \Longleftrightarrow \text{a-Bt}.$$

It is clear from Theorem 5.1 that a sufficient condition for $A+K$, $A \in B(\mathcal{X})$ and $K \in B(\mathcal{X})$ compact, to satisfy gBt (resp., a-gBt) is that the set $\Omega(A)$ (resp., $\Omega_a(A)$) is connected. A sufficient condition for an operator A satisfying gBt (resp., a-gBt) to satisfy gWt (resp., a-gWt) is that A is polaroid (resp., left polaroid). The polaroid and left polaroid properties do not survive perturbation by compact operators: This is clear from the following theorem which proves that the condition of Theorem 5.1 is not sufficient for the perturbation of an $A \in B(\mathcal{H})$ by a compact operator to satisfy a-gWt or gWt.

Theorem 6.1. *Given an operator $A \in B(\mathcal{H})$ such that $\Omega(A)$ (resp., $\Omega_a(A)$) is connected, there exists a compact operator $K \in B(\mathcal{H})$ such that*:
(a) *$A+K$ satisfies gWt (resp., a-gWt).*
(b) *$A+K$ does not satisfy gWt (resp., a-gWt).*

Before going on to prove the theorem, we observe from representation (1) of operators $A \in B(\mathcal{H})$ that to prove the theorem it would suffice to consider operators $A \in B(\mathcal{H})$ for which $\max\{\mathrm{dist}(\lambda, \partial\Phi_{sf}(A)) : \lambda \in \Pi_0(A)\} < \epsilon/2$, $\epsilon > 0$ arbitrary. If A is such an operator, then Lemma 3.2 ensures the existence of a compact operator $K \in B(\mathcal{H})$ such that $\mathrm{min.ind}(A+K-\lambda) = 0$ for all

$\lambda \in \Phi_{sf}(A+K) = \Phi_{sf}(A)$ and $\sigma(A+K) = \sigma_w(A+K) = \sigma_w(A)$. Thus to prove the theorem we may restrict ourselves to the consideration of only those operators $A \in B(\mathcal{H})$ for which $\sigma(A) = \sigma_w(A)$ and min.ind$(A-\lambda) = 0$ for all $\lambda \in \Phi_{sf}(A)$. Let $A \in B(\mathcal{H})$ be such an operator, and let $(\emptyset \neq)\ \Gamma \subset \sigma_{uf}(A)$. Then, for every $\epsilon > 0$, there exists a compact operator $K_1 \in B(\mathcal{H})$ (with $||K_1|| < \epsilon$) and a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$ such that

$$A + K_1 = \begin{pmatrix} N & C \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad \dim(\mathcal{H}_1) = \infty, \tag{7}$$

N is a diagonal normal operator of uniform infinite multiplicity, $\sigma(N) = \sigma_{ulf}(N) = \Gamma$, $\sigma(A_2) = \sigma(A)$, $\sigma_{ulf}(A_2) = \sigma_{ulf}(A)$ and $\mathrm{ind}(A_2 - \lambda) = \mathrm{ind}(A-\lambda)$ for all $\lambda \in \Phi_{sf}(A)$ (see Lemma 3.1). The following lemma lies at the heart of our proof of Theorem 6.1.

Lemma 6.2. *If $\mathcal{H}_1 = \bigoplus_{i=1}^{\infty} \mathcal{H}_{1i}$, $\dim(\mathcal{H}_{1\mathrm{i}}) = \infty$ for all $i \geq 1$, and $N = \bigoplus_{i=1}^{\infty} \lambda_i I_{\mathcal{H}_{1i}}$, then there exists a compact operator $K_0 \in B(\mathcal{H}_1)$ such that*:

(i) *$K_0 = \bigoplus_{i=1}^{\infty} K_i \in B(\bigoplus_{i=1}^{\infty} \mathcal{H}_{1i})$, $N + K_0 = \bigoplus_{i=1}^{\infty}(\lambda_i I_{\mathcal{H}_{1i}} + K_i)$, each $\lambda_i I_{\mathcal{H}_{1i}} + K_i$ is a diagonal operator* diag$\{\lambda_{\mathrm{i}1}, \lambda_{\mathrm{i}2}, \ldots\}$, *$\sigma(N+K_0)$ is the closure of the set $\{\lambda_{ij} : 1 \leq i, j\}$ (consequently, none of the points λ_i is isolated in $\sigma(N+K_0)$), and $\sigma_{ulf}(N+K_0)$ is the closure of the set $\{\lambda_i : 1 \leq i\}$.*

(ii) *$K_0 = \bigoplus_{i=1}^{\infty} K_i \in B(\bigoplus_{i=1}^{\infty} \mathcal{H}_{1i})$, $\sigma(N+K_0)$ is the closure of the set $\{\mu_i : 1 \leq i\}$ and asc$(N + K_0 - \mu_i) = \infty$ for all $i \geq 1$. Here, $\mu_i \neq \lambda_i$ and, for some $\epsilon > 0$, $|\lambda_i - \mu_i| < \epsilon/2$ for all $i \geq 1$ (consequently, each of the points μ_i is isolated in $\sigma(N+K_0)$).*

Proof. Let $\{e_{ki}\}_{k=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}_{1i}$ (thus $\mathcal{H}_{1i} = \bigvee \{e_{ki}\}_{k=1}^{\infty}$).

(i) The points λ_i being isolated in $\sigma(N) = \sigma_{ulf}(N)$, there exists an $\epsilon > 0$, an ϵ-neighbourhood $\mathcal{N}_\epsilon(\lambda_i)$ of λ_i and a sequence $\{\lambda_{ij}\}_{j=1}^{\infty} \subset \mathcal{N}_\epsilon(\lambda_i)$ such that $|\lambda_{ij} - \lambda_i| < \epsilon/2^i$ for all $i \geq 1$. Then the (construction of the proof of [12, Theorem 3.49] leads us to the) compact operator

$$K_i = \sum_{j=1}^{\infty} (\lambda_{ij} - \lambda_i)(e_{ij} \otimes e_{ij}) \in B(\mathcal{H}_{1i})$$

such that $||K_i|| = \max_j |\lambda_{ij} - \lambda_i| < \epsilon/2^i$ for all $i \geq 1$. Define the compact operator K_0 by $K_0 = \bigoplus_{i=1}^{\infty} K_i \in B(\mathcal{H}_1)$ and let $A_1 = N + K_0 = \bigoplus_{i=1}^{\infty}(\lambda_i I_{\mathcal{H}_{1i}} + K_i) = \bigoplus_{i=1}^{\infty} N_i$. Then each N_i is a diagonal operator diag$\{\lambda_{\mathrm{i}1}, \lambda_{\mathrm{i}2}, \ldots\}$, $\sigma(A_1) = \bigcup_{i=1}^{\infty} \sigma(N_i)$ and $\sigma_{ulf}(A_1)$ is the closure of the set $\{\lambda_i : 1 \leq i\}$.

(ii) Again, the points λ_i being isolated in $\sigma(N) = \sigma_{ulf}(N)$, to each λ_i there corresponds a point μ_i $(\neq \lambda_i)$ in an $\epsilon/2^i$-neighbourhood of λ_i such that the operator

$$K_i = \begin{pmatrix} \nu_i & \epsilon/2^i & & \\ & \nu_i & \epsilon/3^i & \\ & & \nu_i & \epsilon/4^i \\ & & \ddots & \ddots \end{pmatrix} \in B(H_{1i}), \quad \nu_i = \mu_i - \lambda_i,$$

is compact with $||K_i|| \leq \epsilon/2^i$. Define $A_1 \in B(\mathcal{H}_1)$ by $A_1 = N + K_0 = N + \bigoplus_{i=1}^{\infty} K_i = \bigoplus_{i=1}^{\infty}(\lambda_i I_{\mathcal{H}_{1i}} + K_i)$. Then $\sigma(A_1)$ is the closure of the set $\{\mu_i : 1 \leq i\}$ and $\text{asc}(A_1 - \mu_i) = \infty$ for all $i \geq 1$. □

Proof of Theorem 6.1. The hypothesis $\Omega(A)$ (resp., $\Omega_a(A)$) is connected implies

$$\sigma(A+K)\backslash\sigma_{Bw}(A+K)=\Pi(A+K) \ \ (\text{resp., } \sigma_a(A+K)\backslash\sigma_{uBw}(A+K)=\Pi^a(A+K))$$

for every compact operator $K \in B(\mathcal{H})$. To prove parts (a) and (b) of the theorem it would thus suffice to prove the existence of a compact operator K such that: (a) $A+K$ is polaroid on $E(A+K)$ (resp., left polaroid on $E^a(A+K)$); (b) not every point of $E(A+K)$ is a pole of $A+K$ (resp., not every point of $E^a(A+K)$ is a left pole of $A+K$). Recall from [21, Exercise 7, Page 293] that if an operator $A+K$ has an upper triangular matrix representation $A+K = \left(\begin{smallmatrix} A_1 & * \\ 0 & A_2 \end{smallmatrix}\right) \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, then

$$\begin{aligned} &\text{asc}(A_1-\lambda) \leq \text{asc}(A+K-\lambda) \leq \text{asc}(A_1-\lambda) + \text{asc}(A_2-\lambda) \ \text{ and} \\ &\text{dsc}(A_2-\lambda) \leq \text{dsc}(A+K-\lambda) \leq \text{dsc}(A_1-\lambda) + \text{dsc}(A_2-\lambda) \end{aligned}$$

for every complex λ. Thus: To prove (a) above it would suffice to prove that A_1 is polaroid (resp., left polaroid) for a compact operator K such $A+K$ has the above triangular representation with the property that $\lambda \in E(A+K)$ if and only if $\lambda \in E(A_1)$ (resp., with the property that $\lambda \in E^a(A+K)$ if and only if $\lambda \in E^a(A_1)$), and to prove (b) above it would suffice to prove that there exists a $\lambda \in E(A+K)$ (resp., $\lambda \in E^a(A+K)$) such that $\text{asc}(A_1 - \lambda) = \infty$.

It is clear from our considerations preceding Lemma 6.2 that we may assume our (starter) operator $A \in B(\mathcal{H})$ to be such that $\sigma_w(A) = \sigma(A)$ and $\text{min.ind}(A-\lambda) = 0$ for all $\lambda \in \Phi_{sf}(A)$. Let $(\emptyset \neq)\ \Gamma = \text{iso}\sigma(A) \subset \sigma_{ulf}(A)$ (resp., $\Gamma = \text{iso}\sigma_a(A) \subset \sigma_{ulf}(A)$). Then ($\Gamma$ is at best a countable set and) there exists a compact operator K_1 such that $A + K_1$ has the upper triangular representation (7), the normal operator N satisfies the hypotheses of Lemma 6.2, $\sigma(A_2) = \sigma(A)$, $\sigma_{ulf}(A_2) = \sigma_{ulf}(A)$, $\text{ind}(A_2-\lambda) = \text{ind}(A-\lambda)$ and $\text{min.ind}(\text{A}_2 - \lambda) = 0$ for all $\lambda \in \Phi_{sf}(A)$. Define the operator $A_1 = N + K_0$ as in the proof of Lemma 6.2, let $K = K_0 \oplus 0 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and consider the operator $A+K$.

(a) Let K_0 be the compact operator of Lemma 6.2(i), and consider a point $\lambda \in E(A+K)$ (resp., $\lambda \in E^a(A+K)$). Then both $A+K$ and $(A+K)^*$ have SVEP at λ (resp., $A+K$ has SVEP at λ), and either $\lambda \notin \sigma_w(A+K)$ or $\lambda \in \text{iso}\sigma_w(A+K)$ (resp., either $\lambda \notin \sigma_{aw}(A+K)$ or $\lambda \in \text{iso}\sigma_{aw}(A+K)$). Since $\sigma_{Bw}(A+K) \subseteq \sigma_w(A+K)$ (resp., $\sigma_{uBw}(A+K) \subseteq \sigma_{aw}(A+K)$), $\lambda \notin \sigma_w(A+K)$ implies $\lambda \notin \sigma_{Bw}(A+K)$ (resp.,$\lambda \notin \sigma_{aw}(A+K)$ implies $\lambda \notin \sigma_{uBw}(A+K)$), and hence (SVEP at λ) implies $\lambda \in \Pi(A+K)$ (resp., $\lambda \in \Pi^a(A+K)$). (Observe that $\lambda \notin \sigma(A_2)$ in this case.) Consider now the case in which $\lambda \in \text{iso}\sigma_w(A+K) = \text{iso}\sigma_w(A)$ (resp., $\lambda \in \text{iso}\sigma_{aw}(A+K) = \text{iso}\sigma_{aw}(A)$). Then $\lambda \in \text{iso}\sigma_{ulf}(A) = \Gamma$, i.e., $\lambda = \lambda_i$ for some integer $i \geq 1$ (see Lemma 6.2), and hence $\lambda \in \sigma(A_1) = \sigma_a(A_1)$ is the limit of a sequence $\{\lambda_{ij}\} \subset \sigma(A+K)$ converging to λ. This contradicts $\lambda \in \text{iso}\sigma(A+K)$ (resp., $\lambda \in \text{iso}\sigma_a(A+K)$). Conclusion: $\lambda \in E(A+K)$ implies $\lambda \in \Pi(A+K)$ (resp., $\lambda \in E_a(A+K)$ implies $\lambda \in \Pi^a(A+K)$).

(b) In this case, let K_0 be the compact operator of Lemma 6.2(ii). Then $\mu_i \in \sigma(A+K)$ for all integers $i \geq 1$. Evidently, $\mu_i \in \text{iso}\sigma(A_1)$ is an eigenvalue of $A+K$. Furthermore, since each λ_i, $1 \leq i$, is isolated in $\sigma(A)$ (indeed, $\sigma_a(A)$), $\mu_i \notin \sigma(A_2)$ for all $i \geq 1$. Since $\text{asc}(A_1 - \mu_i) = \infty$, $\text{asc}(A+K-\mu_i) = \infty$. Consequently, $A+K$ is not left polaroid (hence also not polaroid). $\square$

Remark 6.3. We remark here that our choice of $\lambda \in E(A+K)$ (resp., $\lambda \in E^a(A+K)$) in the proof of Theorem 6.1(a) above is but incidental. The proof goes through with $\lambda \in \text{iso}\sigma(A+K)$ (resp., $\lambda \in \text{iso}\sigma_a(A+K)$). Hence the operator $A+K$ of the proof of Theorem 6.1(a) is polaroid (respectively, left polaroid).

Theorem 6.1 leads us to consider the following problem: "Suppose that an operator $A \in B(\mathcal{X})$ satisfies the property that $\sigma_{Bw}(A+K) = \sigma_{Bb}(A+K)$ (resp., $\sigma_{uBw}(A+K) = \sigma_{uBb}(A+K)$) for every compact operator $K \in B(\mathcal{X})$. What is a sufficient and/or necessary condition for $A+K$ to satisfy gWt (resp., a-gWt)? More generally, what is a sufficient and/or necessary condition for $A+K$ to be polaroid (resp., left polaroid) for every compact operator K?" Observe that $\text{iso}\sigma(A+K) \cap \sigma_w(A+K) = \emptyset$ (resp., $\text{iso}\sigma_a(A+K) \cap \sigma_{aw}(A+K) = \emptyset$) for the operators A and K of Theorem 6.1(a), and $\text{iso}\sigma(A+K) \cap \sigma_w(A+K) \neq \emptyset$ (resp., $\text{iso}\sigma_a(A+K) \cap \sigma_{aw}(A+K) \neq \emptyset$) for the operators A and K of Theorem 6.1(b). The absence, or the presence, of points $\text{iso}\sigma_w(A)$ (resp., $\text{iso}\sigma_{aw}(A)$) gives rise to the dichotomy of Theorem 6.1. Just as the hypothesis $\Omega(A)$ (resp., $\Omega_a(A)$) is connected is seen to be sufficient in Theorem 4.1, we prove in the following that the hypothesis $\sigma_w(A)$ (resp., $\sigma_{aw}(A)$) is connected is sufficient for $A+K$ to be polaroid (resp., left polaroid) for every compact operator $K \in B(\mathcal{X})$. (See also [19], where a similar result is proved, albeit using a somewhat different argument, for Hilbert space operators.)

Theorem 6.4. *Let $A \in B(\mathcal{X})$ and let $K \in B(\mathcal{X})$ be a compact operator. A sufficient condition for $A+K$ to be polaroid (resp., left polaroid) is that* $\text{iso}\sigma_w(A) = \emptyset$ *(resp.,* $\text{iso}\sigma_{aw}(A) = \emptyset$*). Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then this condition is necessary too.*

Proof. The proof is by contradiction.
Sufficiency. Suppose that $\text{iso}\sigma_w(A) = \emptyset$ (resp., $\text{iso}\sigma_{aw}(A) = \emptyset$), and suppose further that $A+K$ is not polaroid (resp., not left polaroid) for some compact operator $K \in B(\mathcal{X})$. Let $\lambda \in \text{iso}\sigma(A+K)$ (resp., $\lambda \in \text{iso}\sigma_a(A+K)$). Then there exists a positive integer $d = d(\lambda)$ such that $(A+K-\lambda)^d(\mathcal{X})$ is closed: For if $(A+K-\lambda)^d(\mathcal{X})$ is not closed for any positive integer d, then $\lambda \in \text{iso}\sigma(A+K)$ implies $\lambda \in \text{iso}\sigma_w(A+K)$ (resp., $\lambda \in \text{iso}\sigma_a(A+K)$ implies $\lambda \in \text{iso}\sigma_{aw}(A+K)$) – a contradiction. Since $\lambda \in \text{iso}\sigma(A+K)$ implies $A+K$, also $(A+K)^*$, has

SVEP at λ, and since SVEP at λ along with $\mathrm{dsc}(A+K-\lambda)<\infty$ (similarly, SVEP at λ for $(A+K)^*$ and $\mathrm{asc}(A+K-\lambda)<\infty$) implies λ is a pole of the resolvent of $A+K$ [1, Theorem 3.81], the fact that λ is a pole of the resolvent of $A+K$ if and only if $\mathrm{asc}(A+K-\lambda)=\mathrm{dsc}(A+K-\lambda)<\infty$ leads us to conclude that if $A+K$ is not polar at λ then we must have $\mathrm{asc}(A+K-\lambda)=\infty$. Again, since $\lambda \in \mathrm{iso}\sigma_a(A+K)$ implies $A+K$ has SVEP at λ and since (in the presence of $(A+K-\lambda)^d(\mathcal{X})$ is closed for some positive integer d) $\mathrm{asc}(A+K-\lambda)<\infty$ implies $A+K$ is left polar at λ, we must have that $\mathrm{asc}(A+K-\lambda)=\infty$ also in the case in which $\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{K})$. It is well known [8, Lemma 8.2.1] that a semi B-Fredholm operator T has SVEP at a point μ if and only if $\mathrm{asc}(T-\mu)<\infty$. Hence our operator $A+K$, which has SVEP at λ but satisfies $\mathrm{asc}(A+K-\lambda)=\infty$, must be such that $\lambda\in\sigma_w(A+K)$ (resp., $\lambda\in\sigma_{aw}(A+K)$). But then $\lambda\in\mathrm{iso}\sigma_w(A+K)$ (resp., $\lambda\in\mathrm{iso}\sigma_{aw}(A+K)$). This is a contradiction. Necessity. We show, using constructions from our earlier results, that assuming $\mathrm{iso}\sigma_w(A)\neq\emptyset$ (resp., $\mathrm{iso}\sigma_{aw}(A)\neq\emptyset$) leads to a contradiction of the hypothesis that $A+K$ is polaroid (resp., left polaroid) for every compact operator K. Assume thus that $A\in B(\mathcal{H})$ and $A+K$ is polaroid (resp., left polaroid) for every compact operator $K\in B(\mathcal{H})$. Assume further that there exists a $\lambda\in\mathrm{iso}\sigma_w(A)$ (resp., $\lambda\in\mathrm{iso}\sigma_{aw}(A)$). Then $\lambda\in\sigma_{ulf}(A)$. Choose (arbitrarily) an $\epsilon>0$. Then there exists a decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ of $\mathcal{H}$ such that $\dim(\mathcal{H}_1)<\infty$, $A=\begin{pmatrix} A_1 & * \\ 0 & A_2\end{pmatrix}\in B(\mathcal{H}_1\oplus\mathcal{H}_2)$, $\sigma(A_1)=\{\lambda\in\Pi_0(A):\mathrm{dist}(\lambda,\Phi_{\mathrm{sf}}(\mathrm{A}))\geq\epsilon/2\}$, $\sigma(A_2)=\sigma(A)\setminus\sigma(A_1)$ and $\lambda\in\sigma(A_2)$ (see (1)). Now apply Lemma 3.1 to $A_2\in B(\mathcal{H}_2)$ to obtain a compact operator $K_2=0\oplus K_{21}\in B(\mathcal{H}_1\oplus\mathcal{H}_2)$ such that

$$A+K_2=\begin{pmatrix} A_1 & * & * \\ 0 & N & * \\ 0 & 0 & A_{22}\end{pmatrix}\in B(\mathcal{H}_1\oplus(H_{21}\oplus\mathcal{H}_{22})),$$
$$\mathcal{H}_{21}\oplus\mathcal{H}_{22}=\mathcal{H}_2,\ \dim(\mathcal{H}_{21})=\infty.$$

Here $N=\lambda I_{\mathcal{H}_{21}}$, $\sigma(A_{22})=\sigma(A_2)$, $\sigma_{ulf}(A_{22})=\sigma_{ulf}(A_2)=\sigma_{ulf}(A)$ and $\mathrm{ind}(A_{22}-\mu)=\mathrm{ind}(A_2-\mu)$ for all $\mu\in\Phi_{sf}(A_2)$. Letting $\lambda=\lambda_1$ (say), define a compact operator $K_1\in B(\mathcal{H}_{21})$ as in the proof of Lemma 6.2(b). Then $A_{21}=N+K_1$ satisfies $\sigma(A_{21})=\{\mu_1\}$ and $\mathrm{asc}(A_{21}-\mu_1)=\infty$. Next use Lemma 3.2 to find a compact operator $K_{22}\in B(\mathcal{H}_{22})$ such that $\sigma(A_{22}+K_{22})=\sigma_w(A_{22}+K_{22})=\sigma_w(A_{22})=\sigma_w(A)$ and $\mathrm{min.ind}(\mathrm{A}_{22}+\mathrm{K}_{22}-\nu)=0$ for all $\nu\in\Phi_{sf}(A_{22})$. It is then clear that $\lambda\in\mathrm{iso}\sigma(\mathrm{A}_{22}+\mathrm{K}_{22})$, and hence (since μ_1 is in a deleted ϵ-neighbourhood of λ for some $\epsilon>0$) $\mu_1\notin\sigma(A_{22}+K_{22})$; again, since $\mathrm{min.ind}(\mathrm{A}_{22}+\mathrm{K}_{22}-\mu_1)=0$ whenever $\lambda\in\mathrm{iso}\sigma_{aw}(A)$, $\mu_1\notin\sigma_a(A_{22}+K_{22})$ in the case in which $\lambda\in\mathrm{iso}\sigma_{aw}(A)$. Thus, in either case, $\mathrm{asc}(A_{22}+K_{22}-\mu_1)=0$. Finally, define the compact operator $K\in B(\mathcal{H})$ by $K=K_2+(0\oplus K_1\oplus 0)+(0\oplus 0\oplus K_{22})$. Then $\mu_1\in\mathrm{iso}\sigma(\mathrm{A}+\mathrm{K})\cap\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{K})$. (Note here that μ_1 can always be chosen so that $\mu_1\notin\sigma(A_1)$.) Hence $\mathrm{asc}(A+K-\mu_1)=\mathrm{asc}(A_{21}-\mu_1)=\infty$ – a contradiction. □

Remark 6.5. An examination of the proof of the sufficiency part of the theorem above shows that one may replace the hypotheses $\mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A}) = \emptyset$ and $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$ by the hypotheses $\mathrm{iso}\sigma_{\mathrm{Bw}}(\mathrm{A}) = \emptyset$ and $\mathrm{iso}\sigma_{\mathrm{uBw}}(\mathrm{A}) = \emptyset$, respectively. We observe here that if there does not exist a positive integer d such that $(A + K - \lambda)^d(\mathcal{X})$ is closed for a $\lambda \in \sigma(A + K)$ then $\lambda \in \sigma_{uBw}(A + K)$ ($\subseteq \sigma_{Bw}(A+K)$), and a semi B-Fredholm operator T has SVEP at a point μ if and only if $\mathrm{asc}(T - \mu) < \infty$ [8, Lemma 8.2.1].

The following corollary follows from a duality argument.

Corollary 6.6. *Let $A \in B(\mathcal{X})$ and let $K \in B(\mathcal{X})$ be a compact operator. A sufficient condition for $A + K$ to be right polaroid is that $\mathrm{iso}\sigma_{sw}(A) = \emptyset$. Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then this condition is necessary too.*

It is clear from Theorems 4.1 and 6.4 that a sufficient condition for $A + K$, where A, $K \in B(\mathcal{X})$ and K is compact, to satisfy gWt (resp., a-gWt) is that $\Omega(A)$ and $\sigma_w(A)$ (resp., $\Omega_a(A)$ and $\sigma_{aw}(A)$) are connected. Indeed, if $\Omega(A)$ (resp., $\Omega_a(A)$) is connected, then $A + K$ satisfies gWt (resp., a-gWt) for every compact K such $E(A+K) \subseteq \Pi(A+K)$ (resp., $E^a(A+K) \subseteq \Pi^a(A+K)$. The following theorem extends Theorem 5.2.

Theorem 6.7. *If $\mathrm{iso}\sigma_{aw}(A) = \emptyset$ for an operator $A \in B(\mathcal{X})$, then a sufficient condition for $\sigma_a(A + K) \setminus \sigma_{uBw}(A + K) = E(A + K)$ for every compact operator K is that $\Omega_a(A)$ is connected and A^* has SVEP on $\sigma_a(A) \setminus \sigma_{aw}(A)$. Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then this condition is necessary too.*

Proof. If $\Omega_a(A)$ is connected and A^* has SVEP on $\sigma_a(A) \setminus \sigma_{aw}(A)$, then Theorem 5.2 implies

$$\sigma_a(A + K) \setminus \sigma_{uBw}(A + K) = \Pi^a(A + K) = \Pi(A + K) \quad (\subseteq E(A + K))$$

for every compact operator $K \in B(\mathcal{X})$. Consider a $\lambda \in E(A + K)$. Since $\mathrm{iso}\sigma_{aw}(A) = \emptyset$ implies $A + K$ is left polaroid, and since $\lambda \in E(A+K)$ trivially implies $\lambda \in \mathrm{iso}\sigma(\mathrm{A + K})$, $\lambda \in \Pi^a(A + K) = \Pi(A + K)$. Hence $\Pi(A + K) = E(A + K)$, proving thereby the sufficiency. To prove the necessity, start by observing that $\sigma_a(A + K) \setminus \sigma_{uBw}(A + K) = E(A + K)$ implies both $A + K$ and $(A + K)^*$ have SVEP at points in $\sigma_a(A + K) \setminus \sigma_{uBw}(A + K)$. Hence $\sigma_a(A+K) \setminus \sigma_{uBw}(A+K) = \Pi(A+K)$. Consequently, Theorem 5.2 implies A^* has SVEP on $\sigma_a(A) \setminus \sigma_{aw}(A)$ and Theorem 5.1 implies $\Omega_a(A)$ is connected. □

7. An application

Let `ASC` denote the class of Banach space operators $A \in B(\mathcal{X})$ which satisfy the *abstract shift condition* (that their hyper-range $\cap_{n=1}^{\infty} A^n(\mathcal{X}) =)\ A^\infty(\mathcal{X}) = \{0\}$. An important subclass of the class `ASC` is that of *weighted right shift* operators A, operator $A \in$ `WRS`, in $B(\ell^p(\mathtt{N}))$, $1 \leq p < \infty$. The following properties of operators $A \in$ `ASC` are well known (see [18, Section 1.6] and [1, Sections 2.5 and 3.10]):

(a) Operators $A \in \mathtt{ASC}$ have SVEP (hence $\sigma(A) = \sigma_s(A)$), are not surjective (hence $0 \in \sigma(A)$) and $\alpha(A - \mu) = 0$ for every non-zero $\mu \in \sigma(A)$.

(b) $\sigma(A)$ is connected (so that either isoσ(A) $= \emptyset$, or, $\sigma(A) = \{0\}$ and A is a quasinilpotent) and $\sigma(A) = \sigma_w(A) = \sigma_b(A)$ [1, Theorem 3.116].

Define $\kappa(A)$, $A \in B(\mathcal{X})$, by

$$\kappa(A) = \inf\{||\mathrm{Ax}|| : \mathrm{x} \in \mathcal{X}, ||\mathrm{x}|| = 1\}.$$

Let

$$i(A) = \lim_{n\to\infty} \{\kappa(A^n)\}^{\frac{1}{n}} = \sup_{n\to\infty} \{\kappa(A^n)\}^{\frac{1}{n}},$$

and let

$$r(A) = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}}$$

denote the spectral radius of A. Then

$$i(A) \leq r(A) \ \text{ and } \ \mathcal{D}(0, i(A)) \subseteq \sigma(A),$$

where $\mathcal{D}(0, i(A))$ denotes the closed disc centered at 0 of radius $i(A)$. If $A \in \mathtt{WRS}$, then

(c) $\sigma(A) = \mathcal{D}(0, r(A))$, and $\sigma_a(A) = \{\lambda : i(A) \leq |\lambda| \leq r(A)\} = \sigma_f(A)$ [1, Page 127].

If $A \in \mathtt{ASC}$ and $i(A) = r(A)$, then

(d) $\sigma(A) = \mathcal{D}(0, r(A))$ and $\sigma_a(A) = \partial\mathcal{D}(0, r(A))$.

Consider an operator A such that either $A \in \mathtt{WRS}$, or, $A \in \mathtt{ASC}$ and $i(A) = r(A)$.

(i) A has SVEP (everywhere) implies both A and A^* satisfy a-gBt (hence also gBt, a-Bt and Bt); see [7, 8] or [9, Corollary 8.3.5]. Since $\sigma_w(A) = \mathcal{D}(0, r(A))$ has no holes, $A + K$ satisfies gBt (hence also Bt) for every compact operator K. (We remark here that $A + K$ satisfies gBt if and only if $(A + K)^*$ satisfies gBT [9, Remark 8.3.9]; hence $A^* + K$ satisfies gBt for every compact operator $K \in B(\ell^q(\mathtt{N}))$, $\frac{1}{p} + \frac{1}{q} = 1$, and (respectively) $K \in B(\mathcal{X}^*)$). It is clear from the above that $\sigma_a(A) = \sigma_{aw}(A) = \{\lambda : i(A) \leq |\lambda| \leq r(A)\}$ if $A \in \mathtt{WRS}$ and $\sigma_a(A) = \sigma_{aw}(A) = \partial\mathcal{D}(0, r(A))$ otherwise. Hence $\Omega_a(A)$ is not connected. However, in view of the additional information that $\sigma_a(A) = \sigma_{aw}(A)$ has just one hole for operators $A \in \mathtt{WRS}$, it is reasonable to ask here the question: Does $A + K$ satisfy a-gBt for every compact operator K?

(ii) The fact that iso$\sigma_w(A) =$ iso$\sigma_{aw}(A) =$ iso$\sigma_{sw}(A) = \emptyset$ for operators $A \in \mathtt{ASC}$ implies that the operators $A \in \mathtt{ASC}$ are polaroid, left polaroid and right polaroid. For operators $A \in \mathtt{WRS}$, we have the following implications:

$$\begin{aligned} &A \ \text{ is polaroid} \Longleftrightarrow A \ \text{ is left polaroid} \\ \Longleftrightarrow \quad &A \ \text{ is right polaroid} \Longleftrightarrow A \ \text{ is not quasinilpotent.} \end{aligned}$$

More is true: If $A \in \mathtt{ASC}$, then $A + K$ is polaroid (also left polaroid and right polaroid) for every compact operator $K \in B(\mathcal{X})$, and if $A \in \mathtt{WRS}$, then

$$A + K \ \text{ is polaroid} \Longleftrightarrow A + K \ \text{ is left polaroid}$$

$$\iff \quad A + K \text{ is right polaroid} \iff A \text{ is not quasinilpotent}$$

for every compact operator $K \in B(\ell^p(\mathtt{N}))$.

Combining (i) and (ii) above, we have:

Theorem 7.1. *If an $A \in \mathtt{ASC}$ is such that $i(A) = r(A)$, or, $A \in \mathtt{WRS}$ is not quasinilpotent, then $A + K$ satisfies gWt (hence also Wt) for every compact operator $K \in B(\mathcal{X})$, respectively $K \in B(\ell^p(\mathtt{N}))$.*

The requirement that $A \in \mathtt{ASC}$ in Theorem 7.1 can be relaxed to $A \in B(\mathcal{X})$ is a non-invertible operator such that $i(A) = r(A)$: For such operators A it is known that A has SVEP and $\sigma(A) = \sigma_w(A) = \sigma_b(A) = \mathcal{D}(0, r(A))$ [1, Theorem 3.117]. Hence, for such operators A, $A + K$ satisfies gWt for every compact operator $K \in B(\mathcal{X})$.

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