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DISTANCE BETWEEN CONTINUOUS FRAMES IN HILBERT SPACE

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Abstract. In this paper, we study some equivalence relations between continuous frames in a Hilbert space $\mathcal{H}$. In particular, we seek two necessary and sufficient conditions under which two continuous frames are near. Moreover, we investigate a distance between continuous frames in order to acquire the closest and nearest tight continuous frame to a given continuous frame. Finally, we implement these results for shearlet and wavelet frames in two examples.

1. Introduction

Discrete and continuous frames arise in many applications in both pure and applied mathematics. Specially, they have important roles in digital processing and scientific computations. Discrete frames in a Hilbert space have been introduced by Duffin and Schaeffer in 1952 to study some deep problems in nonharmonic Fourier series [6]. The concept of generalization of frames to a family indexed by a locally compact spaces endowed with a Radon measure was proposed by Kaiser [10]. These frames are known as continuous frames. Gabardo and Han called these frames "frames associated with measurable spaces" in [8]. In mathematical physics they are referred to as coherent states [1]. Frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames. In [4] Balan has started with definition of distance measure for discrete frames. By inspiring of distance between discrete frames we want to investigate the approximation of continuous frames of a Hilbert space $\mathcal{H}$ by tight ones. This theory plays an important role in many areas, specially in wavelet and shearlet frames. To this end, we need to study some equivalence relations between continuous frames and by using them, define a distance between continuous frames. Also, we aim to establish the geometric meaning of this metric and find the nearest continuous tight frame to a given continuous frame.

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2. Preliminaries and notations

Let (Ω, μ) be a measure space and $\mathcal{H}$ be a separable Hilbert space. Recall (cf. [2, 1, 3]) that a mapping $F : \Omega \to \mathcal{H}$ (or $\{F(\omega)\}_{\omega\in\Omega}$) is called a continuous frame if it is weakly-measurable and there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \int_\Omega |\langle\, F(\omega), f\rangle|^2\, d\mu(\omega) \le B\|\, f\|^2, \quad (f \in \mathcal{H}). \tag{2.1}$$

A continuous frame is said to be tight, if $A = B$ and parseval if $A = B = 1$. The mapping F is called a Bessel map, if the second inequality in (2.1) holds. By definition, if F is a Bessel map, then $T_F : L^2(\Omega, \mu) \to \mathcal{H}$ weakly defined by

$$T_F(\varphi) = \int_\Omega \varphi(\omega)F(\omega)d\mu(\omega),$$

is a bounded linear operator. It is surjective and bounded if and only if F is a frame. This operator is called the synthesis operator. The adjoint of T_F is given by

$$T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \quad T_F^*(f)(\omega) = \langle f, F(\omega)\rangle, \quad \omega \in \Omega,$$

is called the analysis operator. The continuous frame operator is defined to be $S_F := T_F T_F^*$. It is invertible and positive.

It is easily shown that a Bessel map F is a frame if and only if there exists a Bessel map G such that

$$\langle f, g\rangle = \int_\Omega \langle f, G(\omega)\rangle\langle g, F(\omega)\rangle, \quad f, g \in \mathcal{H}.$$

We call such as G, a dual frame for $F$and $S_F^{-1}F$ known as the standard dual of F. It is certainly possible of a continuous frames F to have only one dual. In this case we call F a Riesz-type frame. It is known that F is Riesz-type if and only if $Rang(T_F^*) = L^2(\Omega, \mu)$ [3, 8]. We denote by $L^2(\Omega, \mu, \mathcal{H})$ the set of all mappings $F : \Omega \to \mathcal{H}$ such that for all $f \in \mathcal{H}$, the functions $\omega \to \langle f, F(\omega)\rangle$ defined almost everywhere on Ω, belong to $L^2(\Omega, \mu)$. We shorten $L^2(\Omega, \mu, \mathcal{H})$ to $L^2(\Omega, \mathcal{H})$.

In this note we shall discuss mainly the relations between two continuous frames. Consider two continuous frames F and G on Hilbert space $\mathcal{H}$, we investigate the partial equivalent, similarity, unitary equivalent and partial isometric equivalent. For a definition see Section 3.

3. Geometry of continuous frames

In this section, we are mainly concerned with the relations between two continuous frames.

Let F and G be two continuous frames in Hilbert space $\mathcal{H}$. We recall that F is called *partial equivalent* with G if there is a bounded linear operator $J : \mathcal{H} \to \mathcal{H}$ with $JF = G$. Moreover, if J is also invertible, F and G are called *similar*. In particular, if F and G are similar via a unitary operator J, they are called *unitary equivalent*. Also F and G are called *partial isometric* if they are partial equivalent via a partial isometry J. Note that in this case J should satisfy $JJ^* = 1$.

We want to describe similarity property via concept nearness. For a definition see Remark 3.3. To this end, we need the following lemmas.

Lemma 3.1. *Let F and G be two continuous parseval frames in $\mathcal{H}$. Then*

(1) *$RangT_G^* \subset RangT_F^*$ if and only if the continuous frame F is partial isometric equivalent with continuous frame G, via J.*

(2) *$RangT_G^* = RangT_F^*$ if and only if the two continuous frames F and G are unitary equivalent.*

Proof. (1) Suppose F is partial isometric equivalent with G. Then $JF = G$ and

$$\begin{aligned} JT_F(\varphi) &= J(\int_\Omega \varphi(\omega)F(\omega)d\mu(\omega)) \\ &= \int_\Omega \varphi(\omega)JF(\omega)d\mu(\omega) \\ &= \int_\Omega \varphi(\omega)G(\omega)d\mu(\omega) \\ &= T_G(\varphi). \end{aligned}$$

So $T_G^* = T_F^*J^*$ for some partial isometry J and $RangT_G^* \subset RangT_F^*$. Conversely, suppose $RangT_G^* \subset RangT_F^*$. Consider two projections $P_F = T_F^*T_F$ onto $RangT_F^*$ and $P_G = T_G^*T_G$ onto $RangT_G^*$. Since $RangT_G^* \subset RangT_F^*$, we have $P_FT_G^* = T_G^*$. Define $J : \mathcal{H} \to \mathcal{H}$, by $J = T_GT_F^*$. So

$$JJ^* = T_GT_F^*T_FT_G = T_GP_FT_G^* = T_GT_G^* = S_G = 1.$$

To show $JF = G$, we have

$$JT_F(\varphi) = T_GT_F^*T_F = T_GP_F(\varphi) = T_G(\varphi) \text{ for all } \varphi \in L^2(\Omega, \mu).$$

Thus

$$\int_\Omega \varphi(\omega)(JF(\omega) - G(\omega))d\mu(\omega) = 0 \text{ for all } \varphi \in L^2(\Omega, \mu).$$

Hence $JF = G$.

Proof of (2) follows immediately form (1). □

Lemma 3.2. *Let F and G be two continuous frames in Hilbert space $\mathcal{H}$. Denote by T_F^* and T_G^* respectively their analysis operators. Then*

(1) *$RangT_G^* \subset RangT_F^*$ if and only if the continuous frame F is partial equivalent with the continuous frame G.*

(2) *$RangT_G^* = RangT_F^*$ if and only if the two continuous frames F and G are similar.*

Proof. (1) If F is partial equivalent with G via J, then $T_G^* = T_F^*J^*$ and obviously $RangT_G^* \subset RangT_F^*$. Conversely, let us denote by S_F and S_G the frames operators. Suppose $RangT_G^* \subset RangT_F^*$, we have F is equivalent with $F_1 = S_F^{-\frac{1}{2}}F$. By Lemma 3.1, F_1 is partial equivalent with $G_1 = S_G^{-\frac{1}{2}}$ via $J_1 = T_{G_1}T_{F_1}^*$ which is a partial isometry. Also G_1 is partial isometry equivalent with G. By composing we get F is partial equivalent with G via $J = S_G^{-\frac{1}{2}}J_1S_F^{-\frac{1}{2}}$.

The proof of (2) is immediate from (1). □

Now we are going to describe some concept, such as closeness and nearness for continuous frames. Let F and G be continuous frames in Hilbert space $\mathcal{H}$. We say that G is *close* to F if there exists a non-negative number $\lambda \geq 0$ such that

$$\|(T_G - T_F)f\| \leq \lambda \|T_F f\|$$

for all $f \in L^2(\Omega, \mu)$. The infimum of λ will be called the *closeness bound* of the frame G to the frame F and denoted by $c(G, F)$. Note that the closeness relation is transitive, but not reflexive, generally.

Remark 3.3. If the continuous frame G is close to continuous frame F with a closeness bound less than 1, then F is also close to G but, in general the closeness bound is different. Indeed, we have $\|(T_G - T_F)f\| \leq \lambda \|T_F f\|$ and $\|T_F f\| \leq \|(T_G - T_F)f\| + \|T_G f\|$, thus for all $f \in L^2(\Omega, \mu)$,

$$\|(T_G - T_F)f\| \leq \frac{\lambda}{1-\lambda}\|T_G f\|.$$

Now we want to correct the nonreflexivity of the closeness relation. We say that two continuous frames F and G are *near* if F is close to G and G is close to F. In this case we define the predistance between F and G, denoted by $r^0(F, G)$ as the maximum of the two closeness bounds, that is

$$r^0(G, F) = \max(c(F, G), c(G, F)).$$

It is shown that $r^0(G, F)$ is positive and symmetric, but does not satisfy the triangle inequality.

We now present the connection between the closeness relation and partial equivalence.

Theorem 3.4. *Let F and G be two continuous frames in a Hilbert space $\mathcal{H}$. Then, they are near if and only if they are similar via some invertible operator J. Moreover, $r^0(G, F) = \max(\|J - 1\|, \|1 - J^{-1}\|)$.*

Proof. Suppose F and G are near. By the definition, F is close to G and G is close to F. Since F is close to G we have

$$\|(T_F - T_G)\varphi\| \leq \lambda \|T_G \varphi\|$$

for $\lambda = c(F, G)$. If $\varphi \in kerT_G$, then necessarily $\varphi \in kerT_F$. Therefore, $kerT_G \subset kerT_F$ or $RangT_F^* = (kerT_F)^\perp \subset (kerT_G)^\perp = RangT_G^*$. Since G is close to F, similarity we obtain $RangT_G^* \subset RangT_F^*$. Now applying Lemma 3.2, we get that G and F are similar for some invertible operator J. Hence,

$F = JG$ and if we put $\nu = T_G(\varphi)$ we have $\|(J-1)\nu\| \leq \lambda\|\nu\|$. The smallest value $\lambda \geq 0$ that satisfies this inequality for any $\nu \in \mathcal{H}$ is $\|J-1\|$. Therefore $c(F,G) = \|J-1\|$. Also if we put $\upsilon = T_F(\varphi)$ we have $\|(1-J^{-1})\upsilon\| \leq \lambda\|\upsilon\|$. The smallest value $\lambda \geq 0$ that satisfies this inequality for any $\upsilon \in \mathcal{H}$ is $\|(1-J^{-1})\|$. Therefore $r^0(G,F) = \max(\|J-1\|, \|1-J^{-1}\|)$.

Conversely, suppose G and F are similar via invertible operator J. Then, it is easy to check that $r^0(G,F) = \max(\|J-1\|, \|1-J^{-1}\|)$ and then F and G are near. □

As some consequences of Theorem 3.4, we have:

Corollary 3.5. *Consider F and G as two continuous frames in Hilbert space $\mathcal{H}$. Then, F is close to G if and only if G is partial equivalent with F via some bounded operator J. Moreover, $c(F,G) = \|J-1\|$.*

Corollary 3.6. *Let F be a continuous frame for $\mathcal{H}$. Then F and the standard dual of F are near and*

$$c(S_F^{-1}F, F) = \max(\|S_F - 1\|, \|1 - S_F^{-1}\|).$$

The following result make a connection between the Paley and Wiener Theorems given by Gabardo and Han in [8] and the relations introduced so far.

Lemma 3.7. *Let F be a continuous frame for Hilbert space $\mathcal{H}$ over the measure space Ω, $G : \Omega \to \mathcal{H}$ be a vector-valued mapping and there exist constants $\lambda_1, \lambda_2 > 0$ such that $\max(\lambda_1, \lambda_2) < 1$, and*

$$\|(T_F - T_G)\varphi\| \leq \lambda_1\|T_F\varphi\| + \lambda_2\|T_G\varphi\|$$

for all $\varphi \in L^2(\Omega, \mathcal{H})$ with $(\mu\{\varphi \neq 0\}) < \infty$. Then G is a frame for $\mathcal{H}$ and

(1) *$RangT_F^* = RangT_G^*$,*
(2) *G and F are near,*
(3) *$r^0(G,F) < \infty$.*

Proof. The conclusion that G is a frame follows from a stability result proved by Gabardo and Han in [8]. If $\varphi \in kerT_F$, then

$$\|T_G\varphi\| \leq \lambda_2\|T_G\varphi\|.$$

Since $\lambda_2 < 1$, $T_G\varphi = 0$. Therefore, $kerT_F \subset kerT_G$. Similarly it is easy to show $kerT_G \subset kerT_F$. Hence, $RangT_F^* = RangT_G^*$. Now applying Theorem 3.4, the proof is complete. □

Corollary 3.8. *Suppose F is a Rize-type frame for Hilbert space $\mathcal{H}$. Let $G : \Omega \to \mathcal{H}$ be a vector-valued mapping and there exists constant $0 \leq \lambda < 1$, such that*

$$\|(T_F - T_G)\varphi\| \leq \lambda\|T_F\varphi\|.$$

Then G is a Rize-type frame for $\mathcal{H}$ and

(1) *G is similar to F,*
(2) *$c(F,G) \leq \lambda < 1$ and $r^0(G,F) < \infty$.*

Proof. By Remark 3.3, from $c(G,F)<1$, we get $c(F,G)\leq\frac{\lambda}{1-\lambda}<\infty$. Therefore G and F are near. Now Theorem 3.4 show that G is similar to F and G is a Rize-type frame for $\mathcal{H}$. □

Theorem 3.4 allows us to partition the set of all frames on Hilbert space $\mathcal{H}$, denote by $\mathbf{F}(\mathcal{H})$, into equivalent classes, $\{\varepsilon_\alpha\}_{\alpha\in A}$ where $\varepsilon_\alpha\subset\mathbf{F}(\mathcal{H})$ is the set of all frames which are near to each others. Therefore, for each index $\alpha\in A$, the function $r^o:\varepsilon_\alpha\times\varepsilon_\alpha\to\mathbb{R}_+$ is well-defined and finite. As mentioned earlier, $r^0(G,F)$ is positive and symmetric, but does not satisfy the triangle inequality. This inconvenience can be removed if we define the distance between F and G by

$$r:\varepsilon_\alpha\times\varepsilon_\alpha\to\mathbb{R}_+,\qquad r(F,G)=\log(r^0(F,G)+1).$$

Now we show that $r(F,G)$ is really a distance (a metric) on the set of all frames which are near. By Theorem 3.4, the function $r^0(F,G)$ is well defined on the set of all frames which are near to one another. So the function $r(F,G)$ on this set is well defined. To prove that $r^0(F,G)$ is a distance we need to check only the triangle inequality. Let F, G and K be frames which are near to each other. Then, there exist invertible bounded operators Q and R on $\mathcal{H}$ such that $G=QF$, $K=RG$ and therefore $K=RQF$. We have

$$\begin{aligned}r(F,G)&=\log(1+\max(Q-1,Q^{-1}-1)),\\ r(G,K)&=\log(1+\max(R-1,R^{-1}-1)),\\ r(F,K)&=\log(1+\max(RQ-1,Q^{-1}R^{-1}-1)),\end{aligned}$$

and

$$\begin{aligned}\|RQ-1\|&=\|(R-1)(Q-1)+R+Q-2\|\\ &<\|R-1\|\|Q-1\|+\|R-1\|+\|Q-1\|\\ &=(\|R-1\|+1)(\|Q-1\|+1)-1.\end{aligned}$$

Hence,

$$\log(\|RQ-1\|+1)\leq\log(\|R-1\|+1)+\log(\|Q-1\|+1).$$

Similarly

$$\log(\|Q^{-1}R^{-1}-1\|+1)\leq\log(\|R^{-1}-1\|+1)+\log(\|Q^{-1}-1\|+1),$$

and therefore $r(F,K)\leq r(F,G)+r(G,K)$.

For a frame G, we denote the set of all tight frames which are close to G by Λ_1 and the set of all tight frames that G is close to them by Λ_2. More precisely,

$$\Lambda_1=\{F:F\text{is a tight frame and }c(G,F)<\infty\},$$

and

$$\Lambda_2=\{F:F\text{ is a tight frame and }c(F,G)<\infty\}.$$

Let $r^1:\Lambda_1\to\mathbb{R}_+$, $r^2:\Lambda_1\to\mathbb{R}_+$ be maps denote the map from each F to the associated closeness bound, i.e., $r^1(F)=c(G,F)$ and $r^2(F)=c(F,G)$.

Obviously, if G is a tight frame, then $G \in \Lambda_1 \cap \Lambda_2$ and $\min r^1 = \min r^2 = 0$. Now consider the intersection between these two sets:

$$\Lambda = \Lambda_1 \cap \Lambda_2 = \{F : F \text{ is a tight frame and } r(G,F) < \infty\}.$$

We are looking for the minimum of the functions r^1, r^2 and $r\mid_\Lambda$. By applying Theorem 3.4, we obtain the following lemma:

Lemma 3.9. *Let G be a continuous frame in $\mathcal{H}$ and Λ as above. Then Λ is parametrized in the following way*:

$$\Lambda = \{F : F = \gamma U(S_G^{-\frac{1}{2}} G) \quad \textit{where } \gamma > 0,\ U \textit{ is unitary}\}.$$

Proof. Let $\gamma \geq 0$ and U be unitary. Then $F = \gamma U(S_G^{-\frac{1}{2}} G)$ is a tight frame with bound $\gamma^{\frac{1}{2}}$. Indeed, we have

$$F(\omega) = \int_\Omega \langle F(\omega), S_G^{-\frac{1}{2}}(G(\omega))\rangle S_G^{-\frac{1}{2}}(G(\omega)) d\mu(\omega).$$

Hence,

$$\gamma^2 \|F(\omega)\|^2 = \int_\Omega \|\langle F(\omega), \gamma U S_G^{-\frac{1}{2}} G(\omega)\rangle\|^2 d\mu(\omega).$$

Conversely, for $F \in \Lambda$ from Theorem 3.4, we get $F = J(S_G^{-\frac{1}{2}} G)$ for some invertible operator J. Now its frame operator:

$$\begin{aligned} S_F(f) &= \int_\Omega \langle f, F(\omega)\rangle F(\omega) d\mu(\omega) \\ &= J(\int_\Omega \langle J^* f, S_G^{-\frac{1}{2}} G(\omega)\rangle S_G^{-\frac{1}{2}} G(\omega) d\mu(\omega)) \\ &= JJ^*(f). \end{aligned}$$

Therefore $JJ^* = A.1$, which means that $\frac{1}{A}J$ is unitary and $J = \sqrt{A}U$ for some unitary operator U. □

Now we are concerned here with the closeness and distance functions r^1, r^2 and $r\mid_\Lambda$ which introduced earlier. In fact, we would like to characterize the minimum of these distances.

Theorem 3.10. *Let G be a continuous frame in $\mathcal{H}$ with optimal frame bounds A, B. Then* $\min(r^1) = \min(r^2) = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}}$ *and* $\min(r\mid_\Lambda) = \frac{1}{4}\log\frac{B}{A}$.

Proof. We are looking for the infimum of the functions r^1, r^2. We know that $\frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}} < 1$, we may then restrict our attention to only the tight frames $F \in \Lambda_1$ such that $r^1(F) < 1$. But this implies also that $r^2(F) < \infty$ (or for $F \in \Lambda_2$ such that $r^2(F) < 1$. This implies that $r^1(F) < \infty$). Therefore, we may restrict our attention only to tight frames in $\Lambda_1 \cap \Lambda_2 = \Lambda$. Lemma 3.9, tells us that these frames must have the form $F = \sqrt{\gamma} U(S_G^{-\frac{1}{2}} G)$ for some $\gamma > o$ and U unitary. So

(1) $r^1(F) = \|1 - \frac{1}{\sqrt{\gamma}} S^{\frac{1}{2}} U^{-1}\| = \|\frac{1}{\sqrt{\gamma}} S^{\frac{1}{2}} - U\|$,
(2) $r^2(F) = \|\sqrt{\gamma} S^{-\frac{1}{2}} U - 1\| = \|\sqrt{\gamma} S^{-\frac{1}{2}} - U\|$,
(3) $r(F,G) = \max(\|\frac{1}{\sqrt{\gamma}} S^{\frac{1}{2}} - U\|, \|\sqrt{\gamma} S^{-\frac{1}{2}} - U\|)$.

Now we want to solve these norm problems. For r^1 and r^2 we apply Lemma 3.2 in [4], that assert for a bounded invertible selfadjoint operator T on Hilbert space $\mathcal{H}$. Put $a = \|T^{-1}\|^{-1}$ and $b = \|T\|$, the solution of the following inf-problem

$$\mu = \inf_{\alpha>0,\ U \text{unitary}} \|\alpha T - U\|$$

is given by $\mu = \frac{b-a}{b+a}$ and $\alpha = \frac{2}{a+b}$.

For part (1), apply this lemma with $T = S^{\frac{1}{2}}$, $\alpha = \frac{1}{\sqrt{\gamma}}$ and $a = \sqrt{A}, b = \sqrt{B}$, then we get $\mu = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$ and $\alpha = \frac{2}{\sqrt{B}+\sqrt{A}}$. For part (2), we apply the lemma 3.2 in [4], with $T = S^{-\frac{1}{2}}$, $\alpha = \sqrt{\gamma}$ and $a = \frac{1}{\sqrt{B}}, b = \frac{1}{\sqrt{A}}$. We get $\mu = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}}$ and $\alpha = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}}$.

For $r(F,G)$ we apply Lemma 3.3 in [4], that for a bounded invertible self-adjoint operator T on $\mathcal{H}$ with $a = \|T^{-1}\|^{-1}$ and $b = \|T\|$, the solution of the following optimization problem:

$$\mu = \inf_{\alpha>0,\ U \text{unitary}} \max(\|\alpha T - U\|, \|\frac{1}{\alpha} T^{-1} - U\|)$$

is given by $\mu = \sqrt{\frac{b}{a}} - 1$, $\alpha = \frac{1}{\sqrt{ab}}$.

The solution for $r(F,G)$ is now straightforward. If we apply this lemma with $T = S^{\frac{1}{2}}$, $\alpha = \frac{1}{\sqrt{\gamma}}$ and $a = \sqrt{A}$, $b = \sqrt{B}$. We get $\mu = (\frac{B}{A})^{\frac{1}{4}} - 1$ and $\alpha = \frac{1}{(AB)^4}$. □

The values in Theorem 3.10 are achieved by the following scaling of associated tight frames of G.

Proposition 3.11. *Let G be a continuous frame in $\mathcal{H}$ with optimal frame bounds A, B. If*

$$F_1 = \frac{\sqrt{A}+\sqrt{B}}{2} S_G^{-\frac{1}{2}} G,$$
$$F_2 = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} S_G^{-\frac{1}{2}} G,$$
$$F_3 = (AB)^{\frac{1}{4}} S_G^{-\frac{1}{2}} G,$$

then $r^1(F_1) = r^2(F_2) = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$ and $r(G, F_3) = \frac{1}{4}\log\frac{B}{A}$.

Proof. . For $F_1 = JG$ with $J = \frac{\sqrt{A}+\sqrt{B}}{2} S_G^{-\frac{1}{2}}$ we have $r^1(F_1) = c(G, F_1) = \|1 - J^{-1}\|$. Now $\sqrt{A} \leq S_G^{\frac{1}{2}} \leq \sqrt{B}$, therefore

$$-\frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}} \leq 1 - J^{-1} \leq \frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}},$$

which mean that $\|1 - J^{-1}\| = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}}$.

For $F_2 = JG$ with $J = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} S_G^{-\frac{1}{2}}$, we have $r^2(F_2) = c(F_2, G) = \|J - 1\|$ and a similar calculations show that

$$-\frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}} \leq J - 1 \leq \frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}},$$

thus $r^2(F_2) = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{A}+\sqrt{B}}$. For $F_3 = JG$ with $J = (AB)^{\frac{1}{4}} S_G^{-\frac{1}{2}}$, we have

$$r(G, F_3) = \log(1 + \max(\|J - 1\|, \|1 - J^{-1}\|)).$$

Since $\sqrt{A} \leq S_G^{\frac{1}{2}} \leq \sqrt{B}$ we can deduce $(\frac{A}{B})^{\frac{1}{4}} \leq J \leq (\frac{B}{A})^{\frac{1}{4}}$ and therefore

$$\|J - 1\| = \|1 - J^{-1}\| = \max\left((\frac{B}{A})^{\frac{1}{4}} - 1, 1 - (\frac{A}{B})^{\frac{1}{4}}\right) = (\frac{B}{A})^{\frac{1}{4}} - 1.$$

Then $r(G, F_3) = \frac{1}{4} \log \frac{B}{A}$. □

Now we present two examples to illustrate how our results can be applied. Shearlet frames and wavelet frames are among the most important continuous frames.

The shearlet representation was first introduced by Kutyniok and Labate in [14]. Also they define the continuous shearlet frames and investigate some of their applications [9], [13], [15] . Recently Kamyabi-Gol and Atayi in [11] have introduced abstract locally compact shearlet group as a semidirect product group and studied some of its properties.

Example 3.12. Let $\psi \in L^2(\mathbb{R}^2)$ be admissible, i.e.,

$$c_\psi = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \frac{|\widehat{\psi}(\xi_1, \xi_2)|^2}{\xi_1^2} d\xi_2 d\xi_1 < \infty.$$

For shearlet group $\mathbb{S} = (\mathbb{R}^+ \times \mathbb{R}) \times_\lambda \mathbb{R}^2$ with the operation

$$(a, s, t).(a', s', t') = (aa', s' + s\sqrt{a'}, t' + S_{s'} A_{a'} t),$$

consider the continuous unitary representation

$$\sigma(a, s, t\psi f)(x) = D_{A_a^{-1} S_s^{-1}} T_t \psi(x), \quad x \in \mathbb{R}^2,$$

where

$$S_s = \begin{pmatrix} 1 & s \\ \circ & 1 \end{pmatrix}, \quad A_a = \begin{pmatrix} a & \circ \\ \circ & \sqrt{a} \end{pmatrix}.$$

We have

$$f = \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} (\mathcal{SH}_\psi f)(a,s,t)\sigma(a,s,t)\psi \, \frac{dadsdt}{a}, \quad f \in L^2(\mathbb{R}^2).$$

The family $\{\sigma(a,s,t\psi f)\psi\}_{a>0,s\in\mathbb{R},t\psi f\in\mathbb{R}^2}$ is a continuous shearlet frame for $L^2(\mathbb{R}^2)$, where $\mathcal{SH}_\psi$ is the continuous shearlet transform. (See Proposition 3.3 in [12].) The family $\{\sigma(s)\psi\}_{a>0,s\in\mathbb{R},t\psi f\in\mathbb{R}^2}$ is a tight frame and we have $\min(r^1) = \min(r^2) = \min(r\mid_\Lambda) = 0$.

Example 3.13. If $\psi \in L^2(\mathbb{R})$ is admissible, i.e.,

$$C_\psi = \int_{\mathbb{R}} \frac{|\widehat{\psi}(\gamma)|^2}{|\gamma|} d\gamma < \infty$$

and, for $a, b \in \mathbb{R}, a \neq 0$,

$$\psi^{a,b}(x) = (T_b D_a \psi)(x) = \frac{1}{|a|^{\frac{1}{2}}}\psi(\frac{x-b}{a}), \qquad \forall x \in \mathbb{R},$$

then $\{\psi^{a,b}\}_{a\neq 0, b\in\mathbb{R}}$ is a continuous frame for $L^2(\mathbb{R})$ with respect to $\mathbb{R}\setminus 0 \times \mathbb{R}$ equipped with the measure $\frac{dadb}{a^2}$ and, for all $f \in L^2(\mathbb{R})$,

$$f = \int_{\mathbb{R}}\int_{\mathbb{R}} W_\psi(f)(a,b)\psi^{a,b}\frac{dadb}{a^2},$$

where W_ψ is the continuous wavelet transform defined by

$$W_\psi(f)(a,b) = \int_{\mathbb{R}} f(x)\frac{1}{|a|^{\frac{1}{2}}}\overline{\psi(\frac{x-b}{a})}dx.$$

For details see [5], [7].

It is easily shown that $\min(r^1) = \min(r^2) = \min(r\mid_\Lambda) = 0$.

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