

ON HIGHER ORDER IRREGULAR SETS

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ABSTRACT. To indicate the statistical complexity of dynamical systems, we introduce the notions of higher order irregular set and higher order maximal Birkhoff average oscillation in this paper. We prove that, in the setting of topologically mixing Markov chain, the set consisting of those points having maximal k -order Birkhoff average oscillation for all positive integers k is as large as the whole space from the topological point of view. As applications, we discuss the corresponding results on a repeller.

1. Introduction and statement of results

Let σ be the shift map on the space of one-side sequence $\Sigma = \{1, 2, \dots, b\}^{\mathbb{N}}$, where $b \geq 2$ is an integer. We equip Σ with the distance

$$d(\omega, \omega') = b^{-n}, \quad \omega = (\omega_i)_{i \in \mathbb{N}}, \quad \omega' = (\omega'_i)_{i \in \mathbb{N}},$$

where n is the smallest integer such that $\omega_n \neq \omega'_n$. It is well known that (Σ, d) is a compact metric space.

Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be a continuous function. In multifractal analysis one is often interested in the “size” of the following level sets:

$$\Sigma_\varphi(\alpha) = \left\{ \omega \in \Sigma : \lim_{n \rightarrow \infty} B_\varphi(\omega, n) = \alpha \right\}, \quad \alpha \in \mathbb{R},$$

where

$$B_\varphi(\omega, n) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\sigma^i(\omega)).$$

However, the limit $\lim_{n \rightarrow \infty} B_\varphi(\omega, n)$ may not exist. The set consisting of those points for which the above limit does not exist is called the *irregular set* (or the set of divergence points) for φ and it is denoted by Σ_φ . More precisely,

$$(1.1) \quad \Sigma_\varphi = \left\{ \omega \in \Sigma : \liminf_{n \rightarrow \infty} B_\varphi(\omega, n) < \limsup_{n \rightarrow \infty} B_\varphi(\omega, n) \right\}.$$

As a consequence of Birkhoff’s ergodic theorem, the irregular set has zero measure with respect to any invariant measure. Therefore, at least from the

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point of view of ergodic theory, the set Σ_φ can be discarded. Remarkably, from the point of view of dimension theory the set Σ_φ (and the similar sets in more general setting, for example, in systems satisfying the specification property) can be as large as the whole space, see [5, 7, 10, 11, 14, 18, 20, 24, 25, 27] and references therein.

On the other hand, the set Σ_φ can also be “large” from the topological point of view. The notion of residual set is usually used to describe a set being “large” in the topological sense. Recall that in a metric space X , a set R is called *residual* if its complement is of the first category. Moreover, in a complete metric space a set is residual if it contains a dense G_δ set, see [23]. We say that a set is large from the topological point of view if it is residual. Barreira, Li and Valls [3] proved that the irregular set Σ_φ defined in (1.1) is either residual or empty (in fact, they obtained the result in more general setting). To state the result, we need to introduce some notation. Let

$$\mathcal{L}_\varphi = \left\{ \alpha \in \mathbb{R} : \lim_{n \rightarrow \infty} B_\varphi(\omega, n) = \alpha \text{ for some } \omega \in \Sigma \right\}.$$

We remark that the set \mathcal{L}_φ is a nonempty closed interval, see [9]. Finally, for $\omega \in \Sigma$, let $A_\varphi(\omega)$ be the set of accumulation points of the sequence $n \rightarrow B_\varphi(\omega, n)$. Following [8], a point $\omega \in \Sigma$ is said to have *maximal Birkhoff average oscillation* if $A_\varphi(\omega) = \mathcal{L}_\varphi$. The following result shows that the set of points having maximal Birkhoff average oscillation is large in the topological point of view.

Theorem A (See [3, 8]). *Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be a continuous function. The set*

$$\Sigma_{\max} = \{ \omega \in \Sigma : A_\varphi(\omega) = \mathcal{L}_\varphi \}$$

is residual.

We recall that φ is said to be cohomologous to a constant if there exist a bounded function ψ and a constant c such that

$$\varphi = \psi - \psi \circ \sigma + c \quad \text{on } \Sigma.$$

It is easy to check that \mathcal{L}_φ is a singleton and $\Sigma_{\max} = \Sigma$ if φ is cohomologous to a constant. Therefore, the irregular set Σ_φ defined as in (1.1) is empty if φ is cohomologous to a constant.

Clearly, it follows from Theorem A that the irregular set Σ_φ is residual if it is not empty since $\Sigma_{\max} \subset \Sigma_\varphi$.

In fact, more and more results showed that irregular sets can be large from the topological point of view. For example, Albeverio, Pratsiovytyi and Torbin [1], Hyde et al. [12] and Olsen [21] proved that some kinds of irregular sets associated with integer expansion are residual, and the result in [21] was generalized to iterated function systems by Baek and Olsen [2]. Very recently, Madritsch [19] discussed the set of extremely non-normal points associated with Markov partition from the topological point of view and his result generalized the results in [12] and [21]. Moreover, Li and Wu [15] proved that the

set of divergence points of self-similar measure with the open set condition is either residual or empty. For two-sided topological Markov chains, Barreira, Li and Valls [4] showed that the set of points for which the two-sided Birkhoff averages of a continuous function diverge is residual and the set of points for which the Birkhoff averages have a given set of accumulation points other than a singleton is residual. Li and Wu [16, 17] proved that the irregular set and the set consisting of all points having maximal Birkhoff average oscillation in the system satisfying the specification property are residual if they are not empty.

However, the points in the irregular set are often regarded as “bad” ones in ergodic theory. One often expect that the irregular set is as small as possible. It is well known that given a divergent sequence, forming its Cesàro averages may succeed in producing a convergent sequence. In the nice book [6], it has been proposed that in order to study cases where the Birkhoff average does not converge, one might consider the higher order Birkhoff averages. And it is suggested that they might provide a stratification of dynamical systems or orbits of such, indicating their statistical complexity. Therefore, it is interesting to study the higher order Birkhoff average. In this paper we will prove that the set consisting of those points for which the sets of accumulation points of all higher order Birkhoff average equal to \mathcal{L}_φ is also residual. In particular, we show that the higher order irregular set is still “large” in the topological point of view. We would like to point out that our study is also inspired by a paper by Olsen [22]. To state our result precisely, we need to introduce some notation. Let

$$B_\varphi^{(1)}(\omega, n) = B_\varphi(\omega, n),$$

and for $k \geq 2$, let

$$B_\varphi^{(k)}(\omega, n) = \frac{\sum_{j=1}^n B_\varphi^{(k-1)}(\omega, j)}{n}.$$

We call $B_\varphi^{(k)}(\omega, n)$ *k-order Birkhoff average*, and the set consisting of points ω for which the limit $\lim_{n \rightarrow \infty} B_\varphi^{(k)}(\omega, n)$ does not exist *k-order irregular set*. All *k-order irregular sets* with $k \geq 2$ are called *higher order irregular sets*.

We further consider more refined irregular sets. For $\omega \in \Sigma$ and $k \in \mathbb{N}$ let $A_\varphi^{(k)}(\omega)$ be the set of accumulation points of the sequence $n \rightarrow B_\varphi^{(k)}(\omega, n)$. A point $\omega \in \Sigma$ is said to have *maximal k-order Birkhoff average oscillation* if $A_\varphi^{(k)}(\omega) = \mathcal{L}_\varphi$. Roughly speaking, the points that have maximal *k-order Birkhoff average oscillation* are the “worst” divergence points for *k-order Birkhoff average*.

We can now state our main theorem which tells us that the set consisting of those points having maximal *k-order Birkhoff average oscillation* for all $k \in \mathbb{N}$ is still large from the topological point of view.

Theorem 1.1. *Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be a continuous function. The set*

$$\Sigma_{\varphi, \max}^\infty = \left\{ \omega \in \Sigma : A_\varphi^{(k)}(\omega) = \mathcal{L}_\varphi \text{ for all } k \in \mathbb{N} \right\}$$

is residual.

Remark 1.1. Using the fact that $\frac{1}{n} \sum_{k=1}^n a_k \rightarrow a$ if $a_n \rightarrow a$ as $n \rightarrow \infty$, we can check that $\Sigma_{\varphi, \max}^{\infty} = \Sigma$ if φ is cohomologous to a constant.

We would like to remark that we constructed a desired dense G_{δ} set directly in the proof of Theorem A. Unfortunately, the approach from [3] cannot be applied in this higher order case. To prove Theorem 1.1, we will use an approach inspired by the idea in [12].

Clearly, Theorem 1.1 strengthens Theorem A considerably and the following result follows immediately from it.

Corollary 1.2. *Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be a continuous function. The set*

$$\Sigma_{\varphi}^{\infty} = \left\{ \omega \in \Sigma : \liminf_{n \rightarrow \infty} B_{\varphi}^{(k)}(\omega, n) < \limsup_{n \rightarrow \infty} B_{\varphi}^{(k)}(\omega, n) \text{ for all } k \in \mathbb{N} \right\}$$

is residual if it is not empty.

Let us remark that the above results hold in more general setting. Given a $b \times b$ matrix $A = (a_{ij})$ with entries in $\{0, 1\}$, let

$$\Sigma_A = \{(\omega_1 \omega_2 \cdots) \in \Sigma : a_{\omega_n \omega_{n+1}} = 1 \text{ for } n \in \mathbb{N}\}.$$

Clearly, $\sigma(\Sigma_A) \subset \Sigma_A$. The restriction $\sigma|_{\Sigma_A} : \Sigma_A \rightarrow \Sigma_A$ is called the (one-sided) topological Markov chain or subshift of finite type with transition matrix A . We recall that $\sigma : \Sigma_A \rightarrow \Sigma_A$ is topologically mixing if and only if some power of A has only positive entries.

Theorem 1.1 can be generalized to the case of topologically mixing topological Markov chain.

Theorem 1.3. *Let $\sigma : \Sigma_A \rightarrow \Sigma_A$ be a topologically mixing topological Markov chain and let $\varphi : \Sigma_A \rightarrow \mathbb{R}$ be a continuous function. The set*

$$\Sigma_{A, \varphi, \max}^{\infty} = \left\{ \omega \in \Sigma_A : A_{\varphi}^{(k)}(\omega) = \mathcal{L}_{\varphi} \text{ for all } k \in \mathbb{N} \right\}$$

is residual.

Note that, unlike the case of full shift, the concatenation of two admissible words is not necessarily admissible in the topological Markov chain case. However, for topologically mixing topological Markov chain, it is well known that there exists $k_0 \in \mathbb{N}$ such that for any admissible words $\omega_1 \cdots \omega_n$ and $\omega'_1 \cdots \omega'_n$ there exists a word $\tau_1 \cdots \tau_{k_0}$ such that the concatenated word

$$\omega_1 \cdots \omega_n \tau_1 \cdots \tau_{k_0} \omega'_1 \cdots \omega'_n$$

is admissible. Therefore, with minor modifications of the proof of Theorem 1.1, we can obtain Theorem 1.3. To avoid tedious notation, we only present the proof for the full shift case.

Also, Theorem 1.3 implies the following result.

Corollary 1.4. *Let $\sigma : \Sigma_A \rightarrow \Sigma_A$ be a topologically mixing topological Markov chain and let $\varphi : \Sigma_A \rightarrow \mathbb{R}$ be a continuous function. The set*

$$\Sigma_{A,\varphi}^\infty = \left\{ \omega \in \Sigma_A : \liminf_{n \rightarrow \infty} B_\varphi^{(k)}(\omega, n) < \limsup_{n \rightarrow \infty} B_\varphi^{(k)}(\omega, n) \text{ for all } k \in \mathbb{N} \right\}$$

is residual if it is not empty.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Due to Remark 1.1, we can assume that φ is not cohomologous to a constant. We separate the proof into a sequence of lemmas.

First we introduce some notation. For $n \in \mathbb{N}$, let $\Sigma^n = \{1, \dots, b\}^n$ and $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n$, that is, Σ^n is the family of all words of length n and Σ^* is the family of all finite words. When $\omega = (\omega_1 \omega_2 \dots) \in \Sigma$ and $m \in \mathbb{N}$ or when $\omega = (\omega_1 \dots \omega_n) \in \Sigma^n$ and $m \in \mathbb{N}$ with $m \leq n$, we write

$$\omega|_m = \omega_1 \dots \omega_m.$$

Given words $\omega = (\omega_1 \dots \omega_n) \in \Sigma^n$ and $\omega' = (\omega'_1 \dots \omega'_m) \in \Sigma^m$, let

$$\omega\omega' = \omega_1 \dots \omega_n \omega'_1 \dots \omega'_m$$

be the concatenation of ω and ω' . Moreover, given $W \subset \Sigma^*$ and $\omega \in \Sigma^*$, we write

$$\omega W = \{\omega\eta : \eta \in W\}$$

and

$$W^\infty = \{\eta_1 \eta_2 \dots : \eta_i \in W, i \in \mathbb{N}\}.$$

Let D be the set of rational numbers in the interval \mathcal{L}_φ , i.e., $D = \mathbb{Q} \cap \mathcal{L}_\varphi$. We claim that $D \neq \emptyset$. In fact, since φ is not cohomologous to a constant it follows from Lemma 1.6 in [27] that $B_\varphi(\omega, n)$ does not converge uniformly (or even pointwise) to a constant. On the other hand, it is not difficult to check that $\mathcal{L}_\varphi = \bigcup_{\eta \in \Sigma} A_\varphi^{(1)}(\eta)$ since for each $a \in A_\varphi^{(1)}(\eta)$ there exists $\omega' \in \Sigma$ such that $\lim_{n \rightarrow \infty} B_\varphi^{(1)}(\omega', n) = a$, see Lemma 6.5 in [11]. Therefore, $|\mathcal{L}_\varphi| \geq 2$, where $|A|$ denotes the cardinality of the set A . Note that \mathcal{L}_φ is a nonempty closed interval, we have $D \neq \emptyset$.

Inspired by the idea in [12], we define the property P as follows. Let $g_1(x) = 2^x$ and $g_m(x) = g_1(g_{m-1}(x))$ for $m \geq 2$. We say that a sequence $\{x_n\}_n$ of real numbers has property P if for all $q \in D, m \in \mathbb{N}, i \in \mathbb{N}$ and $\varepsilon > 0$, there exists $j \in \mathbb{N}$ satisfying:

- (i) $j \geq i$;
- (ii) $\frac{j \|\varphi\|}{2^j} < \varepsilon$, where $\|\varphi\| = \max_{\omega \in \Sigma} |\varphi(\omega)|$;
- (iii) if $j < n < g_m(2^j)$, then $|x_n - q| < \varepsilon$.

Write

$$E = \left\{ \omega \in \Sigma : (B_\varphi^{(1)}(\omega, n))_{n=1}^\infty \text{ has property } P \right\}.$$

Lemma 2.1. *The set E is residual.*

Proof. For fixed $h, m, i \in \mathbb{N}$ and $q \in D$, we define property $P_{h,m,q,i}$ as follows. We say that a sequence $(x_n)_n$ has property $P_{h,m,q,i}$ if for every $\varepsilon < 1/h$, there exists $j \in \mathbb{N}$ satisfying:

- (i) $j \geq i$;
- (ii) $\frac{j\|\varphi\|}{2^j} < \varepsilon$;
- (iii) if $j < n < g_m(2^j)$, then $|x_n - q| < \varepsilon$.

Let

$$E_{h,m,q,i} = \left\{ \omega \in \Sigma : (B_\varphi^{(1)}(\omega, n))_{n=1}^\infty \text{ has property } P_{h,m,q,i} \right\}.$$

It is not difficult to check that

$$E = \bigcap_{h \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcap_{q \in D} \bigcap_{i \in \mathbb{N}} E_{h,m,q,i}.$$

Further, we claim that

$$(2.1) \quad E_{h,m+1,q,i+1} \subset E_{h,m,q,i}.$$

In fact, let $\omega \in E_{h,m+1,q,i+1}$, then there exists $j \in \mathbb{N}$ such that $j \geq i + 1$, $(j\|\varphi\|)/2^j < \varepsilon$, and if $j < n < g_{m+1}(2^j)$, then $|B_\varphi^{(1)}(\omega, n) - q| < \varepsilon$. On the other hand, it is easy to check that $j \geq i$, $(j\|\varphi\|)/2^j < \varepsilon$, and if $j < n < g_m(2^j)$, then $|B_\varphi^{(1)}(\omega, n) - q| < \varepsilon$, that is, $\omega \in E_{h,m,q,i}$.

For $n \in \mathbb{N}$, we introduce the numbers

$$\text{Var}_n(\varphi) = \sup \{ |\varphi(\omega) - \varphi(\omega')| : \omega, \omega' \in \Sigma, \omega|_n = \omega'|_n \}$$

and

$$V_n(\varphi) = \sum_{j=1}^n \text{Var}_j(\varphi).$$

Let us remark that $V_n(\varphi)/n \rightarrow 0$ when $n \rightarrow \infty$ since φ is continuous on the compact metric space Σ . Therefore, for $\varepsilon < 1/h$ there exists $N \in \mathbb{N}$ such that

$$(2.2) \quad \frac{V_n(\varphi)}{n} < \frac{\varepsilon}{2} \text{ for any } n > N.$$

It follows from (2.1) that

$$E = \bigcap_{h \in \mathbb{N}} \bigcap_{m > N} \bigcap_{q \in D} \bigcap_{i > N} E_{h,m,q,i}.$$

To complete the proof, it is sufficient to show that $E_{h,m,q,i}$ is open and dense in Σ for $m > N$ and $i > N$.

First, we show that $E_{h,m,q,i}$ is open for $m > N$ and $i > N$. Now let $\omega \in E_{h,m,q,i}$. For $\varepsilon < 1/h$, it follows from the definition of $E_{h,m,q,i}$ that there exists $j \in \mathbb{N}$ such that $j \geq i$, $(j\|\varphi\|)/2^j < \varepsilon/2$ and if $j < n < g_m(2^j)$, then $|B_\varphi^{(1)}(\omega, n) - q| < \varepsilon/2$.

Choose $\delta = 1/k^{g_m(2^j)}$ (note that m and j are fixed). Let $B(\omega, \delta)$ denote the ball centered at ω with radius δ . Then, for any $\omega' \in B(\omega, \delta)$ and $j < n < g_m(2^j)$ we have $\omega|_n = \omega'|_n$. Therefore,

$$\begin{aligned} \left| B_\varphi^{(1)}(\omega', n) - q \right| &\leq \left| B_\varphi^{(1)}(\omega', n) - B_\varphi^{(1)}(\omega, n) \right| + \left| B_\varphi^{(1)}(\omega, n) - q \right| \\ &\leq \frac{V_n(\varphi)}{n} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad (\text{by (2.2) and the fact that } i > N) \\ &< \varepsilon. \end{aligned}$$

This implies that $B(\omega, \delta) \subset E_{h,m,q,i}$, and therefore $E_{h,m,q,i}$ is open for $m > N$ and $i > N$.

Next we show that $E_{h,m,q,i}$ is dense in Σ . Let $\omega \in \Sigma$ and $r > 0$. We must find $\omega' \in B(\omega, r) \cap E_{h,m,q,i}$.

For each $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, write

$$G(\alpha, n, \varepsilon) = \left\{ \omega|_n : \omega \in \Sigma \text{ and } |B_\varphi^{(1)}(\omega, n) - \alpha| < \varepsilon \right\}.$$

Choose $t \in \mathbb{N}$ such that $1/k^t < r$. For $\varepsilon < 1/h$, choose a positive integer $n_0 \geq 2$ such that

$$(2.3) \quad \frac{V_{n_0}(\varphi)}{n_0} < \frac{\varepsilon}{4}.$$

Then, for $q \in D$ let

$$(2.4) \quad \omega' \in \omega_1 \omega_2 \cdots \omega_t G(q, n_0, \varepsilon/4)^\infty.$$

Clearly, $\omega' \in B(\omega, r)$ since $\omega'|_t = \omega|_t$.

Next, choose $j \in \mathbb{N}$ such that $(j \|\varphi\|)/2^j < \varepsilon$, and

$$(2.5) \quad j \geq \max \left\{ i, t + n_0, \frac{8t \|\varphi\|}{\varepsilon}, \frac{8n_0 \|\varphi\|}{\varepsilon} \right\}.$$

Fix a positive integer n with $j < n < g_m(2^j)$. There exist $\ell, p \in \mathbb{N}$ with $0 \leq p < n_0$ such that $n = t + \ell n_0 + p$. Observe that $|q| \leq \|\varphi\|$ since $q \in D$, we have

$$\begin{aligned} \left| \sum_{e=0}^{n-1} \varphi(\sigma^e(\omega')) - nq \right| &\leq \left| \sum_{e=0}^{t-1} \varphi(\sigma^e(\omega')) - tq \right| \\ &\quad + \left| \sum_{e=t}^{t+\ell n_0-1} \varphi(\sigma^e(\omega')) - \ell n_0 q \right| + \left| \sum_{e=t+\ell n_0}^{n-1} \varphi(\sigma^e(\omega')) - pq \right| \\ &\leq 2t \|\varphi\| + \sum_{s=0}^{\ell-1} \left| \sum_{v=0}^{n_0-1} \varphi(\sigma^v(\sigma^{t+s n_0}(\omega'))) - n_0 q \right| + 2p \|\varphi\|. \end{aligned}$$

By (2.4) and the definition of the set $G(q, n_0, \varepsilon/4)$, one can choose sequences $\bar{\omega}^0, \dots, \bar{\omega}^{\ell-1} \in \Sigma$ such that

$$(2.6) \quad \sigma^{t+sn_0}(\omega')|_{n_0} = \bar{\omega}^s|_{n_0}$$

and

$$(2.7) \quad \left| B_\varphi^{(1)}(\bar{\omega}^s, n_0) - q \right| < \frac{\varepsilon}{4} \text{ for } s = 0, \dots, \ell - 1.$$

It follows from (2.6) and (2.7) that

$$\begin{aligned} & \left| \sum_{v=0}^{n_0-1} \varphi(\sigma^v(\sigma^{t+sn_0}(\omega'))) - n_0q \right| \\ & \leq \left| \sum_{v=0}^{n_0-1} \varphi(\sigma^v(\sigma^{t+sn_0}(\omega'))) - \sum_{v=0}^{n_0-1} \varphi(\sigma^v(\bar{\omega}^s)) \right| + \left| \sum_{v=0}^{n_0-1} \varphi(\sigma^v(\bar{\omega}^s)) - n_0q \right| \\ & \leq V_{n_0}(\varphi) + \frac{n_0\varepsilon}{4}. \end{aligned}$$

Hence,

$$\left| \sum_{e=0}^{n-1} \varphi(\sigma^e(\omega')) - nq \right| \leq 2t\|\varphi\| + \ell V_{n_0}(\varphi) + \frac{n_0\ell\varepsilon}{4} + 2p\|\varphi\|.$$

Finally, it follows from (2.3) and (2.5) that

$$\begin{aligned} \left| B_\varphi^{(1)}(\omega', n) - q \right| & \leq \frac{2t\|\varphi\|}{n} + \frac{\ell V_{n_0}(\varphi)}{n} + \frac{n_0\ell\varepsilon}{4n} + \frac{2p\|\varphi\|}{n} \\ & \leq \frac{2t\|\varphi\|}{j} + \frac{V_{n_0}(\varphi)}{n_0} + \frac{n_0\ell\varepsilon}{4n} + \frac{2p\|\varphi\|}{j} \\ & < \frac{2t\|\varphi\|}{j} + \frac{V_{n_0}(\varphi)}{n_0} + \frac{\varepsilon}{4} + \frac{2n_0\|\varphi\|}{j} \\ & < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \quad (\text{by (2.3) and (2.5)}) \\ & < \varepsilon. \end{aligned}$$

This implies that $\omega' \in E_{h,m,q,i}$. The proof of Lemma 2.1 is completed. \square

Lemma 2.2. *For $\omega \in \Sigma$ and $k \in \mathbb{N}$, if the sequence $(B_\varphi^{(k)}(\omega, n))_{n=1}^\infty$ has property P , then the sequence $(B_\varphi^{(k+1)}(\omega, n))_{n=1}^\infty$ also has property P .*

Proof. Fix $\varepsilon > 0, q \in D, i \in \mathbb{N}$ and $m \in \mathbb{N}$. Since the sequence $(B_\varphi^{(k)}(\omega, n))_{n=1}^\infty$ has property P , there exists $j' \in \mathbb{N}$ such that $j' \geq i, (j'\|\varphi\|)/2^{j'} < \varepsilon/3$, and if $j' < n < g_{m+1}(2^{j'})$, then $|B_\varphi^{(k)}(\omega, n) - q| < \varepsilon/3$.

Let $j = 2^{j'}$. It is easy to check that $j \geq i$, and

$$\frac{j\|\varphi\|}{2^j} = \frac{2^{j'}\|\varphi\|}{2^{2^{j'}}} < \frac{j'\|\varphi\|}{2^{j'}} < \varepsilon.$$

Moreover, for any $j < n < g_m(2^j)$, since $|B_\varphi^{(k)}(\omega, n)| \leq \|\varphi\|$ for any $k \in \mathbb{N}$ and $|q| \leq \|\varphi\|$ for any $q \in D$, we have

$$\begin{aligned}
& \left| B_\varphi^{(k+1)}(\omega, n) - q \right| \\
&= \left| \frac{B_\varphi^{(k)}(\omega, 1) + B_\varphi^{(k)}(\omega, 2) + \cdots + B_\varphi^{(k)}(\omega, n)}{n} - q \right| \\
&= \left| \frac{B_\varphi^{(k)}(\omega, 1) + \cdots + B_\varphi^{(k)}(\omega, j')}{n} \right. \\
&\quad \left. + \frac{B_\varphi^{(k)}(\omega, j'+1) + \cdots + B_\varphi^{(k)}(\omega, n) - (n-j')q}{n} - \frac{j'q}{n} \right| \\
&\leq \left| \frac{B_\varphi^{(k)}(\omega, 1) + \cdots + B_\varphi^{(k)}(\omega, j')}{n} \right| + \frac{\sum_{v=j'+1}^n |B_\varphi^{(k)}(\omega, v) - q|}{n} + \left| \frac{j'q}{n} \right| \\
&\leq \frac{j' \|\varphi\|}{n} + \frac{n-j'}{n} \cdot \frac{\varepsilon}{3} + \frac{j' \|\varphi\|}{n} \\
&\leq \frac{j' \|\varphi\|}{2^{j'}} + \frac{\varepsilon}{3} + \frac{j' \|\varphi\|}{2^{j'}} \\
&< \varepsilon.
\end{aligned}$$

The proof of Lemma 2.2 is completed. \square

Lemma 2.3. *The set E is a subset of $\Sigma_{\varphi, \max}^\infty$.*

Proof. Let $\omega \in E$, then it follows from Lemma 2.2 that $(B_\varphi^{(k)}(\omega, n))_{n=1}^\infty$ has property P for all $k \in \mathbb{N}$. In order to prove $\omega \in \Sigma_{\varphi, \max}^\infty$, we must show that $A_\varphi^{(k)}(\omega) = \mathcal{L}_\varphi$ for all $k \in \mathbb{N}$.

First, we show that $A_\varphi^{(k)}(\omega) \subset \mathcal{L}_\varphi$ for all $k \in \mathbb{N}$. Since $\mathcal{L}_\varphi = \cup_{\eta \in \Sigma} A_\varphi^{(1)}(\eta)$ we have $A_\varphi^{(1)}(\omega) \subset \mathcal{L}_\varphi$. On the other hand, it is not difficult to check that $A_\varphi^{(k)}(\omega) \subset A_\varphi^{(1)}(\omega)$ for all $k \geq 2$, see [13].

Next, we show that $\mathcal{L}_\varphi \subset A_\varphi^{(k)}(\omega)$ for all $k \in \mathbb{N}$. Let $p \in \mathcal{L}_\varphi$. Fix $\ell \in \mathbb{N}$ and $q \in D$ such that $|p - q| < 1/\ell$. Since $\omega \in E$, it follows from Lemma 2.2 that $(B_\varphi^{(k)}(\omega, n))_{n=1}^\infty$ has property P . In particular, there exists $j \in \mathbb{N}$ with $j \geq \ell$ such that if $j < n < g_m(2^j)$, then $|B_\varphi^{(k)}(\omega, n) - q| < 1/\ell$. Choose any integer $n_\ell \in (j, g_m(2^j))$ then $|B_\varphi^{(k)}(\omega, n_\ell) - q| < 1/\ell$. Therefore, we get a sequence of integers $(n_\ell)_\ell$ with $n_\ell > \ell$ such that

$$|B_\varphi^{(k)}(\omega, n_\ell) - p| \leq |B_\varphi^{(k)}(\omega, n_\ell) - q| + |q - p| \leq 2/\ell.$$

Moreover, since $n_\ell > \ell$ we can extract an increasing subsequence (n_{ℓ_u}) of (n_ℓ) such that $B_\varphi^{(k)}(\omega, n_{\ell_u}) \rightarrow p$ when $u \rightarrow \infty$. This implies that $p \in A_\varphi^{(k)}(\omega)$ and therefore $\mathcal{L}_\varphi \subset A_\varphi^{(k)}(\omega)$ for all $k \in \mathbb{N}$. The proof of Lemma 2.3 is completed. \square

Finally, Theorem 1.1 follows from Lemmas 2.1, 2.2 and 2.3 immediately.

3. Applications

In this section we give applications of Theorem 1.3 and Corollary 1.4. More precisely, we obtain corresponding results for the Birkhoff averages of a continuous function on a repeller.

Let $f: M \rightarrow M$ be a C^1 map on a smooth manifold and let $J \subset M$ be a compact f -invariant set. We say that f is *expanding* on J and that J is a *repeller* for f if there exist $c > 0$ and $\tau > 1$ such that

$$\|d_x f^n v\| \geq c\tau^n \|v\|$$

for $x \in J$, $v \in T_x M$ and $n \in \mathbb{N}$. Given a continuous function $\varphi: J \rightarrow \mathbb{R}$, we consider the higher order irregular set

$$J_\varphi^\infty = \left\{ x \in J : \liminf_{n \rightarrow \infty} B_\varphi^{(k)}(x, n) < \limsup_{n \rightarrow \infty} B_\varphi^{(k)}(x, n) \text{ for all } k \in \mathbb{N} \right\},$$

where $B_\varphi^{(k)}(x, n)$ are defined inductively by

$$B_\varphi^{(k)}(x, n) = \frac{1}{n} \sum_{j=1}^n B_\varphi^{(k-1)}(x, j) \text{ with } B_\varphi^{(1)}(x, n) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)).$$

Also, let

$$\mathcal{R}_\varphi = \left\{ \alpha \in \mathbb{R} : \lim_{n \rightarrow \infty} B_\varphi^{(1)}(x, n) = \alpha \text{ for some } x \in J \right\}.$$

Moreover, let

$$J_{\varphi, \max}^\infty = \left\{ x \in J : A_\varphi^{(k)}(x) = \mathcal{R}_\varphi \text{ for all } k \in \mathbb{N} \right\},$$

where $A_\varphi^{(k)}(x)$ is the set of accumulation points of the sequence $n \mapsto B_\varphi^{(k)}(x, n)$.

The following is a version of Theorem 1.3 for the Birkhoff average of a continuous function on a repeller. The proof is similar to that in [3]. However, we present the details for the reader's convenience.

Theorem 3.1. *Let J be a repeller for a topologically mixing C^1 map and let $\varphi: J \rightarrow \mathbb{R}$ be a continuous function. The set $J_{\varphi, \max}^\infty$ is residual.*

Proof. Recall that a collection of closed sets $R_1, \dots, R_k \subset J$ is called a Markov partition of J (with respect to f) if:

- (1) $J = \bigcup_{i=1}^k R_i$ and $R_i = \overline{\text{int } R_i}$ for each i ;
- (2) $\text{int } R_i \cap \text{int } R_j = \emptyset$ whenever $i \neq j$;
- (3) if $\text{int } f(R_i) \cap \text{int } R_j \neq \emptyset$, then $f(R_i) \supset R_j$.

We note that the interiors are computed with respect to the induced topology on J . Any repeller J for a C^1 map f has Markov partitions of arbitrary small diameter (see for example [26]). Let $A = (a_{ij})$ be a $b \times b$ matrix with entries

$a_{ij} = 1$ if $\text{int } f(R_i) \cap \text{int } R_j \neq \emptyset$ and $a_{ij} = 0$ otherwise. We obtain a coding map $\pi: \Sigma_A \rightarrow J$ for the repeller J letting

$$\pi(\omega) = \bigcap_{n \in \mathbb{N}} f^{-n+1} R_{\omega_n}, \quad \omega = (\omega_1 \omega_2 \dots).$$

One can easily verify that π is continuous, onto and that $\pi \circ \sigma = f \circ \pi$. The last identity implies that $\mathcal{R}_\varphi = \mathcal{L}_{\varphi \circ \pi}$.

Now let

$$B = \bigcup_{n \geq 0} f^{-n} \bigcup_{i=1}^k \partial R_i,$$

where ∂R_i is the boundary of R_i . This is the set of points in J for which the coding is not unique. Since $f(C) \subset C$, where $C = \bigcup_{i=1}^k \partial R_i$, the sequence $f^{-n}C$ is increasing and hence the set B is invariant, that is, $(f|J)^{-1}B = B$. We define

$$S = \Sigma_A \setminus \pi^{-1}B \quad \text{and} \quad J^* = J \setminus B.$$

Clearly, the map $\pi: S \rightarrow J^*$ is bijective. Moreover, B is an F_σ set and since π is continuous, S is a G_δ set. In addition, it follows from the f -invariance of B that $(f|J)^{-1}J^* = J^*$ and hence $(\sigma|_{\Sigma_A})^{-1}S = S$. That is to say, S is a backward invariant set. It is not difficult to check that any nonempty backward invariant set of a one-sided shift $\sigma|_{\Sigma_A}$ is dense. Therefore, S is a dense G_δ set.

We note that $\psi = \varphi \circ \pi$ is a continuous function on Σ_A . It follows from Theorem 1.3 that there exists a dense G_δ set $E \subset \Sigma_{A, \varphi, \max}^\infty$. To complete the proof, it suffices to show that the set $F = \pi(E \cap S) \subset J^*$ satisfies the following properties:

- (1) $F \subset J_{\varphi, \max}^\infty$;
- (2) F is dense in J ;
- (3) F is a G_δ set.

It follows from the identity $\pi \circ \sigma = f \circ \pi$ that

$$F \subset \pi(E) \subset \pi(\Sigma_{A, \varphi, \max}^\infty) = J_{\varphi, \max}^\infty.$$

Moreover, $E \cap S$ is a dense G_δ set since both E and S are dense G_δ sets. In particular,

$$J = \pi(\Sigma_A) = \pi(\overline{E \cap S}) \subset \overline{\pi(E \cap S)} = \overline{F}$$

and F is dense in J . For the last property, we observe that

$$\begin{aligned} J \setminus F &= (B \cup J^*) \setminus F = B \cup (J^* \setminus F) \quad (\text{since } B \cap F = \emptyset) \\ &= B \cup (\pi(S) \setminus \pi(E \cap S)) \\ &= B \cup \pi(S \setminus (E \cap S)) \quad (\text{since } \pi \text{ is bijective on } S) \\ &= \pi(\Sigma_A \setminus S) \cup \pi(S \setminus (E \cap S)) \\ &= \pi((\Sigma_A \setminus S) \cup (S \setminus (E \cap S))) \\ &= \pi(\Sigma_A \setminus (E \cap S)). \end{aligned}$$

Finally, $\Sigma_A \setminus (E \cap S)$ is an F_σ set (since $E \cap S$ is a G_δ set) and writing $\Sigma_A \setminus (E \cap S) = \bigcup_i F_i$ as a countable union of closed sets $F_i \subset \Sigma_A$, we obtain

$$J \setminus F = \pi(\Sigma_A \setminus (E \cap S)) = \bigcup_i \pi(F_i),$$

where $\pi(F_i)$ is a closed set (since π is continuous and J is compact). This shows that F is a G_δ set and the proof of the theorem is completed. \square

Also, we can obtain a version of Corollary 1.4 for the Birkhoff average of a continuous function on a repeller.

Corollary 3.2. *Let J be a repeller for a topologically mixing C^1 map and let $\varphi: J \rightarrow \mathbb{R}$ be a continuous function. The set J_φ^∞ is residual if it is nonempty.*

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