J. Korean Math. Soc. **54** (2017), No. 1, pp. 59–68
https://doi.org/10.4134/JKMS.j150591
pISSN: 0304-9914 / eISSN: 2234-3008

CHARACTERIZATION OF PRIME SUBMODULES OF A FREE MODULE OF FINITE RANK OVER A VALUATION DOMAIN

Fatemeh Mirzaei and Reza Nekooei

Abstract. Let $F = R^{(n)}$ be a free R-module of finite rank $n \geq 2$. In this paper, we characterize the prime submodules of F with at most n generators when R is a Prüfer domain. We also introduce the notion of prime matrix and we show that when R is a valuation domain, every finitely generated prime submodule of F with at least n generators is the row space of a prime matrix.

0. Introduction

Prime submodules of a module over a commutative ring have been studied in [1, 7, 8, 9, 10] and prime submodules of a finitely generated free module over a PID have been studied in [5]. The authors in [5], have described prime submodules of a free module of finite rank n ($n \geq 2$) and with at most n generators over a UFD. They have characterized the prime submodules of a free module of finite rank over a PID. In [9] we have extended some results obtained in [4] to a Dedekind domain. In this paper we extend these results to a Prüfer domain. Moreover, we define the notion of prime matrix and show that when R is a valuation domain, every finitely generated prime submodule of a free R-module of finite rank n ($n \geq 2$), with at least n generators is the row space of a prime matrix.

Throughout this paper all rings are assumed to be commutative with identity and F denotes a free R-module of finite rank n ($n \geq 2$). We use the notation $R^{(n)}$ for $\underbrace{R \oplus \cdots \oplus R}_{n\text{-times}}$. Let M be a unitary R-module. A proper submodule N of M is called P-prime if $rm \in N$ for some $r \in R$ and $m \in M$ implies $m \in N$ or $r \in P = (N : M)$, where $(N : M) = \{r \in R \mid rM \subseteq N\}$.

Let R be a commutative domain and K be the quotient field of R. Then R is a valuation domain if for every $x \in K$, either $x \in R$ or $x^{-1} \in R$. Equivalently, the set of all ideals of R is totally ordered by inclusion. Let R be a commutative

Received October 1, 2015.
2010 *Mathematics Subject Classification.* 13F05, 13C99.
Key words and phrases. Dedekind domains, Prüfer domains, prime submodules.

domain and I be an ideal of R. Let $I^{-1} = (R :_K I) = \{r \in K \mid rI \subseteq R\}$. Then I is invertible if $II^{-1} = R$. An integral domain R is a Prüfer domain if each non-zero finitely generated ideal of R is invertible. It can be shown that an integral domain R is a Prüfer domain if and only if R_P is a valuation domain for every maximal ideal P of R (see [4]).

1. Prime submodules of $F = R^{(n)}$

Let $X_i = (x_{i1}, \ldots, x_{in}) \in F = R^{(n)}$ for some $x_{ij} \in R$, $1 \le i \le m$, $1 \le j \le n$. We put

$$B_{m\times n} = [X_1 \ldots X_m] = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ & \ddots & & \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix} \in M_{m\times n}(R).$$

Thus the jth row of the matrix $[X_1 \ldots X_m]$ consists of the components of X_j in F. We use $N = \langle B \rangle$ to denote a non-zero submodule of F generated by the rows of B. Also $B(j_1, \ldots, j_k) \in M_{m\times k}(R)$ will denote a submatrix of B consisting of the columns $j_1, \ldots, j_k \in \{1, \ldots, n\}$ of B.

Lemma 1.1. *Let R be a domain. Let $B \in M_{n\times n}(R)$, $\det B \neq 0$ and $B' = (b'_{ij})$ be the adjoint matrix of B. Then $(x_1, \ldots, x_n) \in \langle B \rangle$ for some $x_i \in R$ ($1 \le i \le n$) if and only if $\sum_{i=1}^n x_i b'_{ij} \in \langle \det B \rangle$ for every j, $1 \le j \le n$.*

Proof.

$$\begin{aligned} (x_1, \ldots, x_n) \in \langle B \rangle &\iff (x_1, \ldots, x_n) = (r_1, \ldots, r_n)B;\ \exists r_i \in R \\ &\iff (x_1, \ldots, x_n)B' = (r_1, \ldots, r_n)(\det B)I_n \\ &\iff \sum_{i=1}^n x_i b'_{ij} = (\det B)r_j;\ \forall j (j = 1, \ldots, n) \\ &\iff \sum_{i=1}^n x_i b'_{ij} \in \langle \det B \rangle;\ \forall j (j = 1, \ldots, n). \end{aligned}$$

□

Proposition 1.2. *Let R be an integral domain and $F = R^{(n)}$ ($n \ge 2$). Let $B = [X_1 \ldots X_m]$ for some $X_i \in F$ ($1 \le i \le m, m < n$) and rank $B = m$. If the ideal J of R generated by determinants of all $m \times m$ submatrices of B is R, then $N = \langle B \rangle$ is a prime submodule of F.*

Proof. Assume that $J = R$. It follows that

$$1 = \sum_{i_1,\ldots,i_m \in \{1,\ldots,n\}} r_{i_1\ldots i_m} \det B(i_1, \ldots, i_m)$$

for some $r_{i_1,\ldots,i_m} \in R$ and $1 \le i_j \le n$, $1 \le j \le m$. Put

$$M = \{X \in F \mid \det \beta(i_1, \ldots, i_{m+1}) = 0 \text{ for every } i_1, \ldots, i_{m+1} \in \{1, \ldots, n\}\}$$

where $\beta = [XX_1 \cdots X_m]$. Since $X_i \in M$ $(1 \le i \le m)$, then $N \subseteq M$. Now suppose that $X \in M$. Then by [9, Lemma 1.5], we have $(\det B(i_1, \ldots, i_m))X \in N$ for every $i_1, \ldots, i_m \in \{1, \ldots, n\}$. So

$$X = \sum_{i_1,\ldots,i_m \in \{1,\ldots,n\}} (r_{i_1 \cdots i_m} \det B(i_1, \ldots, i_m))X \in N.$$

Thus $N = M$ and N is a prime submodule of F [9, Corollary 1.9]. □

Proposition 1.3. *Suppose R is a domain and $F = R^{(n)}$ $(n \ge 2)$. Let $B \in M_{n\times n}(R)$ and rank $B = n$. If there exist a maximal ideal P of R and a positive integer α such that $\langle \det B \rangle = P^\alpha$ and the ideal J' of R generated by entries of B' is $P^{\alpha-1}$, where B' is the adjoint matrix of B, then $N = \langle B \rangle$ is a prime submodule of F.*

Proof. Suppose there exist a maximal ideal P of R and a positive integer α such that $\langle \det B \rangle = P^\alpha$ and $J' = P^{\alpha-1}$. Let $B' = (b'_{ij})$ and $r(x_1, \ldots, x_n) \in N$ for some $r, x_i \in R$ $(1 \le i \le n)$. Thus by Lemma 1.1, $r \sum_{i=1}^n x_i b'_{ij} \in \langle \det B \rangle, 1 \le j \le n$. If $\sum_{i=1}^n x_i b'_{ij} \in \langle \det B \rangle$ for every $1 \le j \le n$, then by Lemma 1.1, $(x_1, \ldots, x_n) \in N$. Now let $\sum_{i=1}^n x_i b'_{ij} \notin \langle \det B \rangle$ for some $1 \le j \le n$. Since $\langle \det B \rangle$ is P-primary, $r \in P$. But $b'_{ij} \in P^{\alpha-1}$, $1 \le i, j \le n$. So $r b'_{ij} \in \langle \det B \rangle$, $1 \le i, j \le n$. It follows that $(0, \ldots, 0, r, 0, \ldots, 0) \in N$, with r as the ith component $(1 \le i \le n)$. Thus $rF \subseteq N$ and so N is a prime submodule of F. □

2. Characterization of finitely generated prime submodules of $F = R^{(n)}$ over a valuation domain R

In this section we characterize the finitely generated prime submodules of $F = R^{(n)}$ $(n \ge 2)$, when R is a valuation domain.

Theorem 2.1. *Let R be a valuation domain and $F = R^{(n)}$ $(n \ge 2)$. Let $B = [X_1 \cdots X_m] \in M_{m\times n}(R)$ for some $X_i \in F$ $(1 \le i \le m, m < n)$ and rank $B = m$. Then $N = \langle B \rangle$ is a prime submodule of F if and only if the determinant of one of the $m \times m$ submatrices of B is a unit.*

Proof. Let N be a prime submodule of F and J be the ideal of R generated by determinants of all $m \times m$ submatrices of B. Since R is a valuation domain, there exists a $m \times m$ submatrix $A = B(j_1, \ldots, j_m)$ of B for some $j_1 < j_2 < \cdots < j_m$ of $\{1, \ldots, n\}$ such that $J = \langle \det A \rangle$.

By [5, Lemma 2.2], $\det A \ne 0$. Let $A' = (a'_{ij})$ be the adjoint matrix of A. For the moment, fix $1 \le i \le m$. Consider the element $(x_1, \ldots, x_n) = (a'_{i1}, \ldots, a'_{im})B \in N$. Since $A'A = (\det A)I_m$, then $x_{j_i} = \det A$ and $x_{j_k} = 0$ $(1 \le k \le m, k \ne i)$. Also, if $C_j = B(j_1, \ldots, j_{i-1}, j, j_{i+1}, \ldots, j_m)$, then $x_j = \pm \det C_j$ and so $x_j \in \langle \det A \rangle$ for all $j \in \{1, \ldots, n\} \backslash \{j_1, \ldots, j_m\}$. Hence $\left(\frac{x_1}{\det A}, \ldots, \frac{x_n}{\det A}\right) \in F$. Note that $\det A \left(\frac{x_1}{\det A}, \ldots, \frac{x_n}{\det A}\right) \in N$. Since N is prime, $(\det A)F \subseteq N$ or $\left(\frac{x_1}{\det A}, \ldots, \frac{x_n}{\det A}\right) \in N$. If $(\det A)F \subseteq N$, then for

$j_0 \in \{1, \ldots, n\} \backslash \{j_1, \ldots, j_m\}$, we have $(0, \ldots, 0, \det A, 0, \ldots, 0) \in N$ with $\det A$ as the j_0th component. Hence there are $r_j \in R$ $(1 \leq j \leq m)$ such that $(0, \ldots, 0, \det A, 0, \ldots, 0) = (r_1, \ldots, r_m)A$.

It follows that $r_j \det A = 0 (1 \leq j \leq m)$ and hence $r_j = 0 (1 \leq j \leq m)$. Thus $\det A = 0$, which is a contradiction. So $\left(\frac{x_1}{\det A}, \ldots, \frac{x_n}{\det A}\right) \in N$, i.e., there are $s_j \in R$ $(1 \leq j \leq m)$ such that $\left(\frac{x_1}{\det A}, \ldots, \frac{x_n}{\det A}\right) = (s_1, \ldots, s_m)B$. We conclude that $(a'_{i1}, \ldots, a'_{im})B = (x_1, \ldots, x_n) = (\det A)(s_1, \ldots, s_m)B$, so that $(a'_{i1}, \ldots, a'_{im})A = (\det A)(s_1, \ldots, s_m)A$. Thus $a'_{ij} \det A = s_j (\det A)^2$ and hence $a'_{ij} = s_j \det A$ $(1 \leq j \leq m)$. Thus $\det A' = (\det A)^m s$ for some $0 \neq s \in R$. But $\det A' = (\det A)^{m-1}$. It follows that $\det A$ is a unit. Conversely, let the determinant of one of the $m \times m$ submatrices of B be a unit. Then the ideal J of R generated by determinants of all $m \times m$ submatrices of B is R. So, by Proposition 1.2, N is prime. □

Proposition 2.2. *Let R be a Prüfer domain and $F = R^{(n)}$ $(n \geq 2)$. Let $l \geq n$ be a positive integer and $\Psi \subseteq F$ be a finite subset of F with $|\Psi| = l$. If $N = \langle \Psi \rangle$ is a prime submodule of F, then $P = (N : F)$ is a finitely generated ideal of R.*

Proof. For $N = P^{(n)}$, the assertion is clear. Now suppose that $N \neq P^{(n)}$. Then by [9, Theorem 1.6], there exist a positive integer $k < n$ and a matrix $B = [X_1 \ldots X_k] \in M_{k \times n}(R), X_i \in \Psi, 1 \leq i \leq k$ such that determinant of one of its $k \times k$ submatrices is not in P.

Without loss of generality, we can assume that $d = \det B(1, \ldots, k) \notin P$. Put

$$N = \{X \in F | \det \beta(i_1, \ldots, i_{k+1}) \in P \text{ for every } i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}\},$$

where $\beta = [X X_1 \ldots X_k]$. By [9, Lemma 1.5], $dX_t = \sum_{i=1}^{k} r_{ti} X_i + Y_t$ for some $r_{ti} \in R$ $(k+1 \leq t \leq l,\ 1 \leq i \leq k)$ and $Y_t = (0, \ldots, 0, y_{tk+1}, \ldots, y_{tn}) \in P^{(n)}$.

Let M be the submodule of F generated by the set $\{X_i, Y_j; 1 \leq i \leq k, k+1 \leq j \leq l\}$. Then $dN \subseteq M$. Now fix $p \in P$. Then $d(0, \ldots, p) = \sum_{i=1}^{k} r_i X_i + \sum_{j=k+1}^{l} l_j Y_j$ for some $r_i, l_j \in R$ $(1 \leq i \leq k, k+1 \leq j \leq l)$. Thus $(r_1, \ldots, r_k)B(1, \ldots, k) = (0, \ldots, 0)$. It follows that $r_i \det B(1, \ldots, k) = 0$ and hence $r_i = 0$ $(1 \leq i \leq k)$. Let I be the ideal of R generated by the set $\{y_{in} \in P, k+1 \leq i \leq l\}$. Then $dP \subseteq I$. Since R_P is a valuation domain, $PR_P = IR_P = \langle \frac{y_{tn}}{1} \rangle_P$ for some $k+1 \leq t \leq l$. So $s_i y_{in} \in \langle y_{tn} \rangle$ for some $s_i \in R - P$, $k+1 \leq i \leq l$. Let $s = \prod_{i=k+1}^{l} s_i$, then $sdP \subseteq sI \subseteq \langle y_{tn} \rangle$. Thus $\frac{sd}{y_{tn}} \in (R :_K P)$. If P is not finitely generated, it is not an invertible ideal and so by [3, Corollary 3.1.8], $(R :_K P) = (P :_K P)$. Hence $\frac{sd}{y_{tn}} \in (P :_K P)$. It follows that $sdP \subseteq P^2$. Now, since R is a Prüfer domain, by [4, Theorem 4.23.3], $P = P[P + \langle sd \rangle]$. It follows that $P = P^2$ and hence $PR_P = P^2 R_P$, which is a contradiction. Thus P is finitely generated and by [4, Proposition 4.23.3], it is maximal. □

Corollary 2.3. *Suppose R is a valuation domain and $F = R^{(n)}$ $(n \geq 2)$. Let $l \geq n$ be a positive integer and $\Psi \subseteq F$ a finite subset of F with $|\Psi| = l$. If*

$N = \langle\Psi\rangle$ is a prime submodule of F then $P = (N : F)$ is a finitely generated ideal of R and $N = \langle B\rangle$ for some matrix $B \in M_{n\times n}(R)$.

Proof. By Proposition 2.2, P is a finitely generated ideal of R. Since R is a valuation domain, $P = \langle p\rangle$ for some $p \in R$. If $N = P^{(n)}$, then $N = \langle B\rangle$, where $B = pI_n$. Now let $P^{(n)} \subset N$. Then by the proof of Proposition 2.2, there exist a positive integer $k < n$ and $X_i \in \Psi(1 \le i \le k)$, $Y_t = (0, \dots, 0, y_{tk+1}, \dots, y_{tn}) \in P^{(n)}(k+1 \le t \le l)$, such that $N = \langle\{X_i, Y_t \mid 1 \le i \le k, k+1 \le t \le l\}\rangle$. Let $X_i = (0, \dots, p, \dots, 0)$ with p as ith component, $k+1 \le i \le n$. We show that the submodule M_1 of F generated by $\{Y_t \mid k+1 \le t \le l\}$ is equal to the submodule M_2 of F generated by $\{X_i \mid k+1 \le i \le n\}$. Since $Y_t \in M_2$, $k+1 \le t \le n$, hence $M_1 \subseteq M_2$. Now since $X_i \in N$, $k+1 \le i \le n$, we have $X_i = \sum_{j=1}^{k} r_{ij}X_j + \sum_{t=k+1}^{l} l_{it}Y_t$ for some $r_{ij}, l_{it} \in R$, $1 \le j \le k$, $k+1 \le t \le l$, $k+1 \le i \le n$. By an argument similar to that in the proof of Proposition 2.2, $r_{ij} = 0$, $1 \le j \le k$, $k+1 \le i \le n$. So $X_i \in M_1$, $k+1 \le i \le n$ and $M_2 \subseteq M_1$. Now let $B = [X_1 \dots X_n]$, then $N = \langle B\rangle$. □

Theorem 2.4. *Suppose R is a valuation domain with maximal ideal m and $F = R^{(n)}$ $(n \ge 2)$. Let $l \ge n$ be a positive integer and $\Psi \subseteq F$ a finite subset of F with $\mid \Psi \mid = l$. Let $N = \langle\Psi\rangle$. Then N is a prime submodule of F if and only if there exist a matrix $B \in M_{n\times n}(R)$ and a positive integer $\alpha \le n$ such that $N = \langle B\rangle$, $m^\alpha = \langle\det B\rangle$ and the ideal J' of R generated by entries of B' is $m^{\alpha-1}$, where B' is the adjoint matrix of B.*

Proof. Let $N = \langle\Psi\rangle$ be a prime submodule of F. By Corollary 2.3, $N = \langle B\rangle$ for some matrix $B \in M_{n\times n}(R)$ and $(N : F)$ is a finitely generated ideal of R. By [4, Theorem 4.23.3], $m = (N : F)$ is principal. Assume that $m = \langle p\rangle$ for some $p \in R$. By [9, Lemma 1.1], $\langle\det B\rangle \subseteq m$. If $\langle\det B\rangle \subseteq m^k$ for every positive integer $k \ge 1$, then $\langle\det B\rangle \subseteq \bigcap_{k=1}^{\infty} m^k$. So by [4, Theorem 3.17.1] and [9, Corollary 1.3], $m = \bigcap_{k=1}^{\infty} m^k$. Hence $m^2 = m$, which is a contradiction. Thus there exist a positive integer α and a unit $u \in R$ such that $\det B = up^\alpha$. So $\langle\det B\rangle = m^\alpha$. Now since $p \in (N : F)$, by Lemma 1.1, $pb'_{ij} \in \langle p^\alpha\rangle$ and hence $b'_{ij} \in \langle p^{\alpha-1}\rangle$ for every $1 \le i, j \le n$. Thus $\det B' = (\det B)^{n-1} \in \langle p^{n(\alpha-1)}\rangle$. Therefore $(up^\alpha)^{n-1} = sp^{n(\alpha-1)}$ for some $s \in R$. Since p is not a unit, $n(\alpha-1) \le \alpha(n-1)$ and so $\alpha \le n$. Let J' be the ideal of R generated by the entries of B'. Then $J' = \langle b'_{ij}\rangle$ for some $1 \le i, j \le n$. Since $\langle p^\alpha\rangle \subseteq \langle b'_{ij}\rangle \subseteq \langle p^{\alpha-1}\rangle$, hence J' is m-primary and since $m \ne m^2$, then $J' = m^t$ for $t = \alpha$ or $\alpha - 1$ [4, Theorem 3.17.3]. If $J' = m^\alpha$, then $\det B' = (\det B)^{n-1} \in \langle p^{\alpha n}\rangle$. Hence p is a unit, which is a contradiction. So $J' = m^{\alpha-1}$. □

In the following we assume that (R, m) is a valuation domain with principal maximal ideal m. We introduce the notion of prime matrix and show that every finitely generated prime submodule of $R^{(n)}$ $(n \ge 2)$, with at least n generators is the row space of a prime matrix. Note that, R is not necessarily a PID.

Example. Take $Z \oplus Z = Z^{(2)}$ with lexicographic order. Let K be a field and define the valuation $v : K[x, y] \to Z^{(2)}$ with $v(x) = (1, 0) \leq v(y) = (0, 1)$ and take the value of a polynomial as the minimal value among those of its monomials. Then by [4, Proposition 3.18.1], $v' : K(x, y) \to Z^{(2)}$ with $v'(\frac{f}{g}) = v(f) - v(g)$; $f, g \in K[x, y]$ is a valuation on $K(x, y)$. In this case, the maximal ideal consists of all the elements whose valuations are strictly greater than $(0, 0)$. But the valuation of any such element is at least $(0, 1)$ and therefore any element of value $(0, 1)$ gives a generator of the maximal ideal. Also, since the value group is $Z^{(2)}$, the valuation ring is not a DVR.

Definition. Suppose R is a valuation domain with principal maximal ideal m and $m = \langle p \rangle$ for some $p \in R$. Let $J = \{j_1, \ldots, j_\alpha\}$ be a subset of $\{1, \ldots, n\}$. A matrix $B = (b_{ij}) \in M_{n\times n}(R)$ is said to be a p-prime matrix if it satisfies the following conditions:

i) B is upper triangular.

ii) For all i, $1 \leq i \leq n$, $a_{ii} = p$, if $i \in J$ and $a_{ii} = 1$, if $i \notin J$.

iii) For all $i, j \in \{1, \ldots, n\}$, $a_{ij} = 0$ except possibly when $i \notin J$ and $j \in J$.

Sometimes we call J the set of integers associated with B and denote it by J_B. By (i) and (ii), it is clear that $\det(B) = p^\alpha$.

Lemma 2.5. *Suppose R is a valuation domain with principal maximal ideal $m = \langle p \rangle$ and $r_i \in R$, $1 \leq i \leq n$. Let $J = \{j_1, \ldots, j_\alpha\}$ be a subset of $\{1, \ldots, n\}$ and $J_k = \{0, 1, \ldots, j_k\} - J$, $1 \leq k \leq \alpha$. Then $(r_1, \ldots, r_n) \in \langle B \rangle$ for some p-prime matrix $B \in M_{n\times n}(R)$ with $J_B = J$ if and only if for every k, $1 \leq k \leq \alpha$ the equation $\sum_{j\in J_k} r_j x_j \equiv r_{j_k} (\mathrm{mod}\ p)$ has a solution.*

Proof. Let $B = (b_{ij})$ be a p-prime matrix with $J_B = \{j_1, \ldots, j_\alpha\}$ and let $B' = (b'_{ij})$. For all $1 \leq i, j \leq n$, it is easy to see that $b'_{ii} = p^{\alpha-1}$ if $i \in J_B$, $b'_{ii} = p^\alpha$ if $i \notin J_B$ and $b'_{ij} = -p^{\alpha-1} b_{ij}$ if $i \neq j$. Hence by Lemma 1.1,

$$
\begin{aligned}
(r_1, \ldots, r_n) \in \langle B \rangle \quad &\Longleftrightarrow \quad p^\alpha \Big| \sum_{j=1}^{n} r_j b'_{j\ell}, \quad 1 \leq \ell \leq n \\
&\Longleftrightarrow \quad p^\alpha \Big| \sum_{j=0}^{\ell-1} r_j(-p^{\alpha-1} b_{j\ell}) + p^{\alpha-1} r_\ell \text{ for every } \ell \in J_B \\
&\Longleftrightarrow \quad p \Big| \sum_{j\in J_k} -r_j b_{jj_k} + r_{j_k}, \quad 1 \leq k \leq \alpha \\
&\Longleftrightarrow \quad \sum_{j\in J_k} r_j b_{jj_k} \equiv r_{j_k} (\mathrm{mod}\ p) \quad \text{for every } k, 1 \leq k \leq \alpha.
\end{aligned}
$$

□

Lemma 2.6. *Suppose R is a valuation domain with principal maximal ideal $m = \langle p \rangle$ and s and n are positive integers such that $s < n$. Also, suppose that $A \in M_{n\times s}(R)$, $Y \in M_{n\times 1}(R)$ and $X = (x_1, \ldots, x_s) \in R^{(s)}$. Let $C \in M_{n\times(s+1)}(R)$ be the augmented matrix $[A : Y]$. If p does not divide the determinant of at least one $s \times s$ submatrix of A, then the system of equations*

$AX \equiv Y (\mathrm{mod}\ p)$ has a solution if and only if p divides the determinants of all $(s+1) \times (s+1)$ submatrices of C.

Proof. Suppose $AX \equiv Y \pmod p$ has a solution and C_0 is an $(s+1) \times (s+1)$ submatrix of C. If Y_0 is the last column of C_0 and A_0 consists of all columns of C_0 except for Y_0, then $A_0X \equiv Y_0 \pmod p$. So that $C_0'A_0X \equiv C_0'Y_0 \pmod p$. The last equation of this system is $0 \equiv \det(C_0) \pmod p$. Hence $p \mid \det(C_0)$. Conversely, let $X_1, \ldots, X_s \in M_{n\times 1}(R)$ be the columns of A. Then $A^t = [X_1^t \ldots X_s^t] \in M_{s\times n}(R)$ and $C^t = [X_1^t \ldots X_s^t Y^t] \in M_{(s+1)\times n}(R)$. Now let $p \nmid \det(A^t(i_1, \ldots, i_s))$. Then by [9, Lemma 1.5(ii)], $\det(A^t(i_1, \ldots, i_s))Y^t \in \langle p\rangle F + \langle A^t\rangle$. Since $\det(A^t(i_1, \ldots, i_s))$ is unit, $Y^t \in \langle p\rangle F + \langle A^t\rangle$ and so the system of equations $AX \equiv Y \pmod p$ has a solution. □

Theorem 2.7. *Suppose R is a valuation domain with principal maximal ideal $m = \langle p\rangle$. Let s, n and α be positive integers such that $s \le n$ and $1 \le \alpha \le n$ and $A \in M_{s\times n}(R)$. Then $\langle A\rangle \subseteq \langle B\rangle$ for some p-prime matrix $B \in M_{n\times n}(R)$ with $\det(B) = p^\alpha$ if and only if p divides the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrices of A.*

Proof. Let $\langle A\rangle \subseteq \langle B\rangle$ for some p-prime matrix B with $\det(B) = p^\alpha$. So there exists $C \in M_{s\times n}(R)$ such that $A = CB$. Let A_0 be an $(n-\alpha+1)\times(n-\alpha+1)$ submatrix of A. Thus there exists an $(n-\alpha+1)\times n$ submatrix C_0 of C and an $n \times (n-\alpha+1)$ submatrix B_0 of B such that $A_0 = C_0B_0$. Suppose that B_1 is an $(n-\alpha+1)\times(n-\alpha+1)$ submatrix consisting of rows $i_1, \ldots, i_{n-\alpha+1}$ of B_0. Since J_B has α elements, $i_k \in J_B$ for some k, $1 \le k \le n-\alpha+1$.

It follows that the entries of the row i_k of B_0 are 0 or p. Thus $p \mid \det(B_1)$. By the Binet-Cauchy formula [6, Theorem 1], $\det(A)$ may be expressed as a linear combination of the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrices of B_0, hence $p \mid \det(A_0)$. Conversely, assume that p divides the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrices of A. By adding some zero rows to A if necessary, we may suppose that $A \in M_{n\times n}(R)$. We use induction on α. By assumption for $\alpha = 1$, $p \mid \det(A)$. Let k be the smallest integer such that p divides the determinants of all $k \times k$ submatrices of A_k, where $A_k \in M_{n\times k}(R)$ consists of the first columns of A. If $A = (a_{ij})$ then by Lemma 2.6, the system of equations

$$\left\{ \sum_{j=0}^{k-1} a_{ij}x_j \equiv a_{ik} \pmod p \mid 1 \le i \le n \right\}$$

has a solution. Therefore by Lemma 2.5, there exists a prime matrix B with $J_B = \{k\}$ such that $\langle A\rangle \subseteq \langle B\rangle$. Now suppose that the assertion is true for some α, $1 \le \alpha \le n-1$. Assume that p divides the determinants of all $(n-\alpha)\times(n-\alpha)$ submatrices of $A = (a_{ij})$. Hence p divides the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrices of A. Therefore by the induction hypothesis, there exists a prime matrix B with $\det(B) = p^\alpha$ such that $\langle A\rangle \subseteq \langle B\rangle$. Let $J_B = \{j_1, \ldots, j_\alpha\}$ and $J_k = \{0, 1, \ldots, j_k\} - J_B$, $1 \le k \le \alpha$. Fix k for the moment. By Lemma 2.5, the system of equations

$$\left\{\sum_{j\in J_k} a_{ij}x_j \equiv a_{ij_k} \pmod p \mid 1\le i\le n\right\}$$

has a solution, say $x_j = r_j$ for some $r_j \in R$, $j \in J_k$. Thus we have

$$\sum_{j\in J_k} a_{ij}r_j \equiv a_{ij_k} \pmod p \ \forall i,\ 1\le i\le n. \tag{1}$$

Let A_0 be the $n\times(n-\alpha)$ submatrix obtained by deleting columns $j_1,\ldots,j_\alpha$ from A. Let ℓ be the smallest integer such that p divides the determinants of all $\ell\times\ell$ submatrices of $A_\ell \in M_{n\times\ell}(R)$ consisting of the first ℓ columns of A_0. Assume that j_0 is the integer such that column ℓ of A_0 is column j_0 of A. Clearly $j_0 \notin J_B$. Let $J_0 = \{0,\ldots,j_0-1\} - J_B$. By Lemma 2.6, the system of equations

$$\left\{\sum_{j\in J_0} a_{ij}x_j \equiv a_{ij_0} \pmod p \mid 1\le i\le n\right\}$$

has a solution, say $x_j = s_j$ for some $s_j \in R$, $j\in J_0$. Therefore we have

$$\sum_{j\in J_0} a_{ij}s_j \equiv a_{ij_0} \pmod p \ \forall i,\ 1\le i\le n. \tag{2}$$

Put $J' = \{j_1,\ldots,j_\alpha,j_0\}$ and let $J'_k = \{0,1,\ldots,j_k\} - J'$. If $j_k > j_0$, then combining (1) and (2) yields

$$a_{ij_k} \equiv \sum_{j\in J'_k} a_{ij}r_j + (\sum_{j\in J_0} a_{ij}s_j)r_{j_0} \pmod p,$$

for every i, $1\le i\le n$. Hence the system of equations

$$\left\{\sum_{j\in J'_k} a_{ij}x_j \equiv a_{ij_k} \pmod p \mid 1\le i\le n\right\}$$

has a solution. On the other hand, if $j_k \le j_0$, then obviously the above system has a solution by (1). Since k is arbitrary, by Lemma 2.5, there exists a prime matrix B_0 with $\det(B_0) = p^{\alpha+1}$ such that $\langle A\rangle \subseteq \langle B_0\rangle$ and $j_{B_0} = J'$. Thus the assertion is true for $\alpha+1$ and hence by induction for every α, $1\le\alpha\le n$. □

Proposition 2.8. *Suppose R is a valuation domain with principal maximal ideal $m = \langle p\rangle$ and n a positive integer. Let $A \in M_{n\times n}(R)$ and $1\le\alpha\le n$, be the greatest integer such that $p^\alpha | \det(A)$ and $p^{\alpha-1}$ divides all entries of A'. Then p divides the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrices of A.*

Proof. By [2, Lemma 4.4], there exist a diagonal matrix $C = (c_{ij})$ and invertible matrices $D, E \in M_{n\times n}(R)$ such that $AE = DC$, so that $E'A' = C'D'$. By hypothesis $p^{\alpha-1}$ divides all entries of A' and hence those of $C'D'$. Let $C' =$

(c'_{ij}). If $p^2 \mid c_{jj}$ for some j, $1 \leq j \leq n$, then $p^{\alpha-1} \nmid c'_{jj}$. Hence p divides all entries of row j of D'. Thus $p \mid \det(D')$, which contradicts the fact that D is invertible. Since $p^\alpha \mid \det(C)$, p divides at least α entries of the diagonal of C, therefore we conclude that p divides all the entries of at least one column of every $(n-\alpha+1)\times(n-\alpha+1)$ submatrix of DC. Thus p divides the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrix of DC and by the Binet-Cauchy formula it is easy to see that p divides the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrices of $A = (DC)E^{-1}$. □

Theorem 2.9. *Suppose R is a valuation domain with maximal ideal m and $F = R^{(n)}$ $(n \geq 2)$. Let N be a finitely generated submodule of F with at least n generators. Then N is a prime submodule of F if and only if m is a principal ideal of R and N is the row space of a prime matrix.*

Proof. Let N be a prime submodule of F. Then, by Corollary 2.3 and Theorem 2.4, $(N : F) = m$ is a principal ideal of R and there exist a matrix $A \in M_{n\times n}(R)$ and a positive integer $\alpha \leq n$ such that $N = \langle A\rangle$, $\langle \det A\rangle = m^\alpha$ and the ideal J' of R generated by entries of A' is $m^{\alpha-1}$. Let $m = \langle p\rangle$ for some $p \in R$. So by Proposition 2.8 and Theorem 2.7, $N \subseteq \langle B\rangle$ for some prime matrix B with $\det(B) = p^\alpha$. Thus $A = CB$ for some $C \in M_{n\times n}(R)$ and therefore $up^\alpha = \det(A) = \det(C)\det(B) = \det(C)p^\alpha$. Thus $\det(C) = u$ and so C is invertible. Hence $C^{-1}B = A$. It follows that $\langle B\rangle \subseteq N = \langle A\rangle$. Therefore $N = \langle B\rangle$. Conversely, by Theorem 2.4, the row space of every prime matrix is a prime submodule. □

3. Prime submodules of $F = R^{(n)}$ with at most n-generators over a Prüfer domain R

In this section we characterize the prime submodules of $F = R^{(n)}$ $(n \geq 2)$ with at most n-generators over a Prüfer domain.

Theorem 3.1. *Suppose R is a Prüfer domain and $F = R^{(n)}$ $(n \geq 2)$. Let $B = [X_1 \cdots X_m]$ for some $X_i \in F$ $(1 \leq i \leq m, m < n)$ and rank $B = m$. Then $N = \langle B\rangle$ is a prime submodule of F if and only if the ideal J generated by the determinants of all $m \times m$ submatrices of B is R.*

Proof. Let N be a prime submodule of F. Then by [9, Proposition 1.2], $(N : F) = \langle 0\rangle$. Suppose that $J \neq R$ and P is a prime ideal of R with $J \subset P$. Then by [5, Lemma 2.2], $P \neq 0$ and N_P is a prime submodule of F_P with $(N_P : F_P) = \langle 0\rangle$. Since R is a Prüfer domain, R_P is a valuation domain [4, Theorem 4.22.1]. Therefore by Theorem 2.1, $R_P = J_P$. It follows that $1 = \frac{r}{s}\frac{\det B(j_1,\ldots,j_m)}{1}$ for some $1 \leq j_1 < \cdots < j_m < n$, $r \in R$ and $s \in R\backslash P$. So $s = r\det B(j_1,\ldots,j_m) \in J \subset P$, which is a contradiction. Therefore $J = R$. The converse follows from Proposition 1.2. □

Theorem 3.2. *Suppose R is a Prüfer domain and $F = R^{(n)}$. Let $B \in M_{n\times n}(R)$ and rank $B = n$. Then $N = \langle B\rangle$ is prime in F if and only if* *there exist a maximal ideal P of R and a positive integer $\alpha \leq n$ such that $\langle \det B \rangle = P^{\alpha}$ and the ideal J' of R generated by entries of B' is $P^{\alpha-1}$, where B' is the adjoint matrix of B.*

Proof. Let N be a prime submodule of F. By Proposition 2.2, $P = (N : F) = \sqrt{\langle \det B \rangle}$ is a finitely generated ideal of R and so by [4, Theorem 4.23.3], is maximal. Since R is a Prüfer domain, R_P is a valuation domain. Since N_P is a P_P-prime submodule of F_P, by Theorem 2.4, $\langle \frac{\det B}{1} \rangle_P = P_P^{\alpha}$ and $J'_P = P_P^{\alpha-1}$ for some positive integer $\alpha \leq n$.

Let $\phi : R \to R_P$ be the natural homomorphism. Since $\langle \det B \rangle$ is P-primary, $\varphi^{-1}(\langle \frac{\det B}{1} \rangle_P) = \langle \det B \rangle$. So $\langle \det B \rangle = P^{\alpha}$. Now let $r \in \varphi^{-1}(J'_P)$. Then $\frac{r}{1} \in J'_P$ and hence $sr \in J'$ for some $s \in R - P$. Since P is a maximal ideal of R, $1 = sx + y^{\alpha}$ for some $x \in R$ and $y \in P$. So $r = sxr + y^{\alpha} r \in J'$. Therefore $\varphi^{-1}(J'_P) = J'$. Thus $J' = P^{\alpha-1}$. □

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Fatemeh Mirzaei
Department of Pure Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
E-mail address: mirzaee0269@yahoo.com

Reza Nekooei
Department of Pure Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
E-mail address: rnekooei@uk.ac.ir