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A RESEARCH ON A NEW APPROACH TO EULER POLYNOMIALS AND BERNSTEIN POLYNOMIALS WITH VARIABLE $[x]_q$

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Abstract. In this paper, we consider a modified Euler polynomials $\widetilde{E}_{n,q}(x)$ with variable $[x]_q$ and investigate some interesting properties of the Euler polynomials. We also give some relationships between the modified Euler polynomials and their Hurwitz zeta function. Finally, we derive some identities associated with Bernstein polynomials.

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1. Introduction

The Euler numbers and polynomials have been studied by researchers in many areas of mathematics, mathematical physics and statistical physics(cf. [1, 3, 5, 6, 7]). Furthermore, many generalized and modified theories are investigated by them(see [2, 4, 8, 9, 10, 11, 12]). It is well known that the ordinary Euler numbers and polynomials are defined as below, respectively.

$$F(t) = \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \tag{1.1}$$

$$F(t,x) = \frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.2}$$

Observe that $E_n(0) = E_n$ (cf. [1, 3, 5, 6, 7]).

Let p be an odd prime number. Throughout this paper, the symbol $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively.

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$\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ denote the ring of integers, the field of rational numbers and the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$(cf. [1-12]).
Also, in this paper, we use the definition of q-number :

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [3-7])} . \tag{1.3}$$

Hence, $\lim_{q\to 1} \frac{1-q^x}{1-q} = x$ for all x with $|x|_p \leq 1$ in the present p-adic case.

For $g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function$\}$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-1}} \sum_{x=0}^{p^N-1} g(x)(-1)^x \text{ (cf. [2, 3, 4, 5])}. \tag{1.4}$$

Let $g_n(x)$ be the translation with $g_n(x) = g(x+n)$. Then we have following integral equation:

$$I_{-1}(g_n) + (-1)^{n-1} I_{-1}(g) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} g(l) \text{ (cf. [2, 3, 4, 7])}. \tag{1.5}$$

From (1.5), we get

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \tag{1.6}$$

where $g_1(x) = g(x+1)$. By the definition of p-adic integral and the generating function of Euler polynomials, we can obtain the relationship between Euler numbers and polynomials as belows

$$E_n(x) = (E+x)^n,$$

$$E_0 = 1, (E+1)^n + E_n = 2\delta_{0,n}, \tag{1.7}$$

where $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$

In this paper, we define the modified Euler polynomials with variable $[x]_q$ and investigate some properties. We try to find relation between modified Euler polynomials and ordinary Euler numbers and polynomials. Also, we define the analogue Hurwitz zeta function using modified Euler polynomials. And we study on the relation between the modified Euler polynomials and Bernstein polynomials with variable $[x]_q$.

2. The modified Euler polynomials with variable $[x]_q$

In this section, we introduce a new approach to Euler polynomials using variable $[x]_q$. We define a modified Euler polynomials and find some interesting properties which are related to the Euler polynomials with variable $[x]_q$. From $\lim_{q\to 1}\frac{1-q^x}{1-q} = x$, we consider the relation between the modified Euler polynomials $\widetilde{E}_{n,q}(x)$ and the ordinary Euler polynomials $E_n(x)$.

Definition 2.1. For $q \in \mathbb{C}_p$ with $|1-q|_p \leqslant 1$, we define a modified Euler polynomials $\widetilde{E}_{n,q}(x)$ as below:

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(\frac{1-q^x}{1-q}+y)t} d\mu_{-1}(y).$$

By using p-adic integral on $\mathbb{Z}_p$, we get the generating function of the modified Euler polynomials as follows:

$$F_q(t,x) = \frac{2}{e^t+1}e^{(\frac{1-q^x}{1-q})t} = \frac{2}{e^t+1}e^{[x]_q t} = \sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

The generating function is related to that of the ordinary Euler polynomials from (2.1).

$$\lim_{q\to 1} F_q(t,x) = \lim_{q\to 1}\frac{2}{e^t+1}e^{(\frac{1-q^x}{1-q})t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

By using Definition 2.1, we obtain the modified Euler polynomials as follows:

$$\widetilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (\frac{1-q^x}{1-q}+y)^n d\mu_{-1}(y), \quad \widetilde{E}_{0,q}(x) = 0. \tag{2.2}$$

From (2.2), we get the next Theorem.

Theorem 2.2. *Let $n \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|1-q|_p \leqslant 1$. Then we have*

$$\widetilde{E}_{n,q}(x) = (\frac{1-q^x}{1-q}+E)^n.$$

Proof. By using p-adic integral, we have

$$\begin{aligned}\widetilde{E}_{n,q}(x) &= \int_{\mathbb{Z}_p} (\frac{1-q^x}{1-q}+y)^n d\mu_{-1}(y)\\ &= \sum_{l=0}^{n}\binom{n}{l}(\frac{1-q^x}{1-q})^{n-l}E_l.\end{aligned} \tag{2.3}$$

Hence, we get the result as below

$$\widetilde{E}_{n,q}(x) = (\frac{1-q^x}{1-q}+E)^n.$$

□

The result can be expressed $\widetilde{E}_{n,q}(x) = ([x]_q + E)^n$ by the equation (1.6). From Theorem 2.2, we have the following example.

Remark 2.1. We get the following equations when we substitute $n = 1, 2, 3, 4$.

$$\begin{aligned}
\widetilde{E}_{1,q}(x) &= (\frac{1-q^x}{1-q}) - \frac{1}{2},\\
\widetilde{E}_{2,q}(x) &= (\frac{1-q^x}{1-q})^2 - (\frac{1-q^x}{1-q}),\\
\widetilde{E}_{3,q}(x) &= (\frac{1-q^x}{1-q})^3 - \frac{3}{2}(\frac{1-q^x}{1-q})^2 + \frac{1}{4},\\
\widetilde{E}_{4,q}(x) &= (\frac{1-q^x}{1-q})^4 - 2(\frac{1-q^x}{1-q})^3 + (\frac{1-q^x}{1-q}),\\
&\vdots
\end{aligned}$$

From remark 2.1, we note that $\lim_{q\to 1} \widetilde{E}_{n,q}(x) = E_n(x)$. So, we can see that the coefficients of $\widetilde{E}_{n,q}(x)$ are equal to the coefficients of Euler polynomials $E_n(x)$ (cf. [1]).

From Theorem 2.2, we also have the following corollary.

Corollary 2.3. *Let $n \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|1-q|_p \leqslant 1$. Then we get*

$$\widetilde{E}_{n,q}(x+y) = \sum_{l=0}^{n} \binom{n}{l} \widetilde{E}_{l,q}(x) q^{x(n-l)} [y]_q^{n-l}.$$

Proof. By the equation (2.1) and Cauchy product,

$$\begin{aligned}
\sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x+y)\frac{t^n}{n!} &= \frac{2}{e^t+1} e^{[x+y]_q t}\\
&= \sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x)\frac{t^n}{n!} \sum_{m=0}^{\infty} (q^x[y]_q)^m \frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n} \binom{n}{l} \widetilde{E}_{l,q}(x)(q^x[y]_q)^{n-l}\frac{t^n}{n!}.
\end{aligned}$$

Hence, we have

$$\widetilde{E}_{n,q}(x+y) = \sum_{l=0}^{n} \binom{n}{l} \widetilde{E}_{l,q}(x) q^{x(n-l)} [y]_q^{n-l} \frac{t^n}{n!}.$$

□

Theorem 2.4. *Let $n \in \mathbb{N}$ and $q \in \mathbb{C}_p$. We get*

$$(-1)^n \widetilde{E}_{n,q}(1-x) = \widetilde{E}_{n,q^{-1}}(x).$$

Proof. If $t=-t$ and $x=1-x$ in (2.1), then

$$F_q(-t,1-x)=\frac{2}{e^{-t}+1}e^{[1-x]_q(-t)}=\frac{2}{e^{t}+1}e^{[x]_{q^{-1}}t}$$
$$=\sum_{n=0}^{\infty}\widetilde{E}_{n,q^{-1}}(x)\frac{t^n}{n!}.$$

Hence we have

$$F_q(-t,1-x)=\sum_{n=0}^{\infty}\widetilde{E}_{n,q^{-1}}(x)\frac{t^n}{n!}.$$

Therefore we can see that

$$(-1)^n\widetilde{E}_{n,q}(1-x)=\widetilde{E}_{n,q^{-1}}(x).$$

□

From the equation (1.5), we get the following theorem.

Theorem 2.5. *Let $n\in\mathbb{Z}_+$ and $q\in\mathbb{C}_p$ with $|1-q|_p\leqslant 1$. Then we have*

$$([x]_q+m+E)^n+(-1)^{m-1}\widetilde{E}_{n,q}(x)=2\sum_{l=0}^{m-1}(-1)^{m-1-l}([x]_q+l)^n.$$

Proof. Let $f(y)=e^{([x]_q+y)t}$. The Equation 1.5 is expressed as below:

$$\int_{\mathbb{Z}_p}e^{([x]_q+y+n)t}d\mu_{-1}(y)+(-1)^{n-1}\int_{\mathbb{Z}_p}e^{([x]_q+y)t}d\mu_{-1}(y)$$
$$=2\sum_{l=0}^{n-1}(-1)^{n-1-l}e^{([x]_q+l)^k}.$$

Then, left-hand side is

$$\int_{\mathbb{Z}_p}e^{([x]_q+y+n)t}d\mu_{-1}(y)+(-1)^{n-1}\int_{\mathbb{Z}_p}e^{([x]_q+y)t}d\mu_{-1}(y)$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\binom{k}{l}n^l\widetilde{E}_{k-l,q}(x)\frac{t^k}{k!}+(-1)^{n-1}\sum_{n=0}^{\infty}\widetilde{E}_{k,q}(x)\frac{t^k}{k!}$$
$$=\sum_{k=0}^{\infty}([x]_q+n+E)^k\frac{t^k}{k!}+(-1)^{n-1}\sum_{k=0}^{\infty}\widetilde{E}_{k,q}(x)\frac{t^k}{k!}.$$

Right-hand side is

$$2\sum_{l=0}^{n-1}(-1)^{n-1-l}e^{([x]_q+l)^t}=\sum_{k=0}^{\infty}\sum_{l=0}^{n-1}2(-1)^{n-1-l}([x]_q+l)^k\frac{t^k}{k!}.$$

By using comparing coefficients of $\frac{t^n}{n!}$, we have

$$([x]_q+n+E)^k+(-1)^{n-1}\widetilde{E}_{k,q}(x)=2\sum_{l=0}^{n-1}(-1)^{n-1-l}([x]_q+l)^k.$$

□

Theorem 2.6. *Let $n \in \mathbb{Z}_+$. If $m \equiv 0 \pmod 2$, then*

$$([x]_q + m + E)^n - \widetilde{E}_{n,q}(x) = 2\sum_{l=0}^{m-1}(-1)^{l+1}([x]_q + l)^n.$$

If $m \equiv 1 \pmod 2$, then

$$([x]_q + m + E)^n + \widetilde{E}_{n,q}(x) = 2\sum_{l=0}^{m-1}(-1)^{l}([x]_q + l)^n.$$

Let $m = 1$ in Theorem 2.6. Then we have the following theorem.

Theorem 2.7. *Let $m = 1, n \in \mathbb{Z}_+$. Then we have*

$$([x]_q + 1 + E)^n + \widetilde{E}_{n,q}(x) = \begin{cases} 2 & \text{if } n = 0, \\ 2[x]_q^n & \text{if } n \neq 0. \end{cases}$$

3. The analogue of the modified Euler zeta function

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. We define the modified Euler-Hurwitz zeta function with variable $[x]_q$. In other words, the modified Euler-Hurwitz zeta function interpolates the polynomials. Differentiating the equation (2.1) gives the following result.

$$\frac{d^k}{dt^k}F_q(t,x)\bigg|_{t=0} = 2\sum_{m=0}^{\infty}(-1)^m([x]_q + m)^k. \tag{3.1}$$

We define the modified Euler-Hurwitz zeta function as following definition.

Definition 3.1. For $s \in \mathbb{C}$ with $Re(s) > 0$, we define $\zeta_q(s,x)$ by

$$\zeta_q(s,x) = 2\sum_{m=1}^{\infty}\frac{(-1)^m}{(\frac{1-q^x}{1-q} + m)^s} = 2\sum_{m=1}^{\infty}\frac{(-1)^m}{([x]_q + m)^s}.$$

Note that $\zeta_q(s,x)$ is a meromorphic function on $\mathbb{C}$.

Remark 3.1. If $q \to 1$, then we can observe that $\lim_{q\to 1}\zeta_q(s,x) = \zeta(s,x)$.

That is,

$$\lim_{q\to 1}\zeta_q(s,x) = \lim_{q\to 1} 2\sum_{m=1}^{\infty}\frac{(-1)^m}{(\frac{1-q^x}{1-q} + m)^s} = 2\sum_{m=0}^{\infty}\frac{(-1)^m}{(x+m)^s}.$$

Relation between $\zeta_q(s,x)$ and $\widetilde{E}_{k,q}(x)$ is given by the following theorem.

Theorem 3.2. *For $k \in \mathbb{N}$, we have*

$$\zeta_q(-k,x) = \widetilde{E}_{k,q}(x).$$

Observe that $\zeta_q(-k,x)$ function interpolates $\widetilde{E}_{k,q}(x)$ polynomials at non-negative integers.

4. The relations between the modified Euler polynomials and the Bernstein polynomials

In this section, we investigate the relations between the modified Euler polynimials and the Bernstein polynomials. From Theorem 2.4, we can see that $\widetilde{E}_{n,q}(1) = (-1)^n \widetilde{E}_{n,q^{-1}} = (-1)^n E_n$. Then we have the following theorem.

Theorem 4.1. *Let $n \in \mathbb{N}$. Then we have*

$$\widetilde{E}_{n,q}(2) = (q - E)^n.$$

Proof. By using p-adic integral on $\mathbb{Z}_p$, we have

$$\begin{aligned}\widetilde{E}_{n,q}(2) &= \int_{\mathbb{Z}_p} ([2]_q + y)^n d\mu_{-1}(y) = \sum_{l=0}^{n} \binom{n}{l} q^{n-l} \widetilde{E}_{l,q}(1) \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{n-l} (-1)^l \widetilde{E}_{l,q^{-1}} = (q - \widetilde{E}_{q^{-1}})^n.\end{aligned}$$

Thus we can see that

$$\widetilde{E}_{n,q}(2) = (q - E)^n.$$

□

For $x \in \mathbb{Z}_p$, the Bernstein polynomials of degree n are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{where} \quad x \in [0,1], n, k \in \mathbb{Z}_+. \text{ (cf. [3, 5, 8])} \tag{4.1}$$

By the equation (4.1), we get the symmetry of the Bernstein polynomials as below:

$$B_{k,n}(x) = B_{n-k,n}(1-x). \tag{4.2}$$

Using p-aidc integral on $\mathbb{Z}_p$, we get the next theorem.

Theorem 4.2. *Let $n \in \mathbb{N}$. Then we obtain*

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l (1 + [2]_q)^{n-l} \widetilde{E}_{l,q}(2).$$

Proof. For $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} (-1)^n \sum_{l=0}^{n} \binom{n}{l} (-1 - [2]_q)^{n-l} (x + [2]_q)^l d\mu_{-1}(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} (-1)^{2n-l} (1 + [2]_q)^{n-l} \int_{\mathbb{Z}_p} (x + [2]_q)^l d\mu_{-1}(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} (-1)^l (1 + [2]_q)^{n-l} \widetilde{E}_{l,q}(2).\end{aligned}$$

□

From p-dic integral for the Bernstein polynomials, we get the following theorem.

Theorem 4.3. *For $n,k\in\mathbb{Z}_+$ with $n>k+1$, we have*

$$\sum_{l=0}^{k}\binom{k}{l}(-1)^l((1+[2]_q)-\widetilde{E}_q(2))^{n-k+l}=\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^lE_{k+l}.$$

Proof. For $n>k+1$, we get

$$\int_{\mathbb{Z}_p}B_{k,n}(x)d\mu_{-1}(x)=\int_{\mathbb{Z}_p}\binom{n}{k}x^k(1-x)^{n-k}d\mu_{-1}(x)$$
$$=\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\sum_{s=0}^{n-k+l}\binom{n-k+l}{s}(-1)^s(1+[2]_q)^{n-k+l-s}\widetilde{E}_{s,q}(2)$$
$$=\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^l((1+[2]_q)-\widetilde{E}_q(2))^{n-k+l}.$$

We have that

$$\int_{\mathbb{Z}_p}B_{k,n}(x)d\mu_{-1}(x)=\int_{\mathbb{Z}_p}\binom{n}{k}x^k(1-x)^{n-k}d\mu_{-1}(x)$$
$$=\int_{\mathbb{Z}_p}\binom{n}{k}x^k\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^lx^ld\mu_{-1}(x)$$
$$=\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^lE_{k+l}.$$

Since $\widetilde{E}_{n,q}=E_n$, we can see that

$$\sum_{l=0}^{k}\binom{k}{l}(-1)^l((1+[2]_q)-\widetilde{E}_q(2))^{n-k+l}=\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^lE_{k+l}.$$

□

From Theorem 4.1 and Theorem 4.3, we get Corollary 4.4.

Corollary 4.4. *Let $n,k\in\mathbb{Z}_+$ with $n>k+1$. Then we derive*

$$\binom{n}{k}((1+[2]_q)-(q-E)^s)^{n-k}(-[2]_q+\widetilde{E}_q(2)))^k=\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^lE_{k+l}.$$

Theorem 4.5. *For $n_1,n_2,k\in\mathbb{Z}_+$ with $n_1+n_2>2k+1$, we get*

$$\sum_{l=0}^{2k}\binom{2k}{l}(-1)^l((1+[2]_q)-\widetilde{E}_q(2))^{n_l+n_2-2k+l}$$
$$=\sum_{l=0}^{n_1+n_2-2k}\binom{n_1+n_2-2k}{l}(-1)^lE_{2k+l}.$$

Proof. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-1}(x) \\
&= \int_{\mathbb{Z}_p} \binom{n_1}{k}\binom{n_2}{k} x^{2k}(1-x)^{n_1+n_2-2k} d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l \int_{\mathbb{Z}_p} (1-x)^{n_1+n_2-2k+l} d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l ((1+[2]_q) - \widetilde{E}_q(2))^{n_l+n_2-2k+l}.
\end{aligned}$$

We can see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-1}(x) \\
&= \int_{\mathbb{Z}_p} \binom{n_1}{k}\binom{n_2}{k} x^{2k}(1-x)^{n_1+n_2-2k} d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \int_{\mathbb{Z}_p} x^{2k+l} d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l E_{2k+l}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\sum_{l=0}^{2k} \binom{2k}{l} (-1)^l ((1+[2]_q) - \widetilde{E}_q(2))^{n_l+n_2-2k+l} \\
&= \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l E_{2k+l}.
\end{aligned}$$

□

From Theorem 4.1 and Theorem 4.5, we have Corollary 4.6 that is analogous to Corollary 4.4.

Corollary 4.6. *For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$, then we have*

$$\begin{aligned}
&((1+[2]_q) - (q-E))^{n_1+n_2-2k}(-[2]_q + \widetilde{E}_q(2))^{2k} \\
&= \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l E_{2k+l}.
\end{aligned}$$

Using the above Theorem 4.3, Theorem 4.5 and mathematical induction, we have the following theorem.

Theorem 4.7. *Let $n_1, n_2, n_3, \cdots n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > tk + 1$. We get*

$$\sum_{l=0}^{tk} \binom{tk}{l} (-1)^l ((1 + [2]_q) - \widetilde{E}_q(2))^{n_1+n_2+\cdots+n_t-tk+l}$$
$$= \sum_{l=0}^{n_1+n_2+\cdots+n_t-tk} \binom{n_1 + n_2 + \cdots + n_t - tk}{l} (-1)^l E_{tk+l}.$$

Proof. Let $s \in \mathbb{N}$. For $n_1, n_2, n_3, \cdots n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > tk + 1$. Then we can see that

$$\int_{\mathbb{Z}_p} \Big(\prod_{i=1}^{t} B_{k,n_i}(x) \Big) d\mu_{-1}(x)$$
$$= \int_{\mathbb{Z}_p} \Big(\prod_{i=1}^{t} \binom{n_i}{k} \Big) x^{tk} (1 - x)^{n_1+n_2+\cdots+n_t-tk} d\mu_{-1}(x)$$
$$= \Big(\prod_{i=1}^{t} \binom{n_i}{k} \Big) \sum_{l=0}^{tk} \binom{tk}{l} (-1)^l ((1 + [2]_q) - \widetilde{E}_q(2))^{n_1+n_2+\cdots+n_t-tk}.$$

Also, we have

$$\int_{\mathbb{Z}_p} \Big(\prod_{i=1}^{t} B_{k,n_i}(x) \Big) d\mu_{-1}(x)$$
$$= \int_{\mathbb{Z}_p} \Big(\prod_{i=1}^{t} \binom{n_i}{k} \Big) x^{tk} \sum_{l=0}^{n_1+n_2+\cdots+n_t-tk} \binom{n_1 + n_2 + \cdots + n_t - tk}{l} (-1)^l x^l d\mu_{-1}(x)$$
$$= \Big(\prod_{i=1}^{t} \binom{n_i}{k} \Big) \sum_{l=0}^{n_1+n_2+\cdots+n_t-tk} \binom{n_1 + n_2 + \cdots + n_t - tk}{l} (-1)^l E_{tk+l}.$$

Therefore we can see that

$$\sum_{l=0}^{tk} \binom{tk}{l} (-1)^l ((1 + [2]_q) - \widetilde{E}_q(2))^{n_1+n_2+\cdots+n_t-tk+l}$$
$$= \sum_{l=0}^{n_1+n_2+\cdots+n_t-tk} \binom{n_1 + n_2 + \cdots + n_t - tk}{l} (-1)^l E_{tk+l}.$$

□

From Theorem 4.1 and Theorem 4.7, we have the following corollary.

Corollary 4.8. *For $n_1, n_2, n_3, \cdots n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > tk + 1$, we have*

$$((1 + [2]_q) - (q - E))^{n_1+n_2+\cdots+n_t-tk}(-[2]_q + \widetilde{E}_q(2))^{tk}$$
$$= \sum_{l=0}^{n_1+n_2+\cdots+n_t-tk} \binom{n_1 + n_2 + \cdots + n_t - tk}{l} (-1)^l E_{tk+l}$$

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