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A NEW FIFTH-ORDER WEIGHTED RUNGE-KUTTA ALGORITHM BASED ON HERONIAN MEAN FOR INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. A new fifth-order weighted Runge-Kutta algorithm based on heronian mean for solving initial value problem in ordinary differential equations is considered in this paper. Comparisons in terms of numerical accuracy and size of the stability region between new proposed Runge-Kutta(5,5) algorithm, Runge-Kutta (5,5) based on Harmonic Mean, Runge-Kutta(5,5) based on Contra Harmonic Mean and Runge-Kutta(5,5) based on Geometric Mean are carried out as well. The problems, methods and comparison criteria are specified very carefully. Numerical experiments show that the new algorithm performs better than other three methods in solving variety of initial value problems. The error analysis is discussed and stability polynomials and regions have also been presented.

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Key words and phrases : Fifth-Order Weighted Runge-Kutta (WRK), Harmonic Mean (HM), Contra Harmonic Mean (CoM), Heronian Mean (HeM), Geometric Mean (GM), Ordinary Differential Equations(ODEs), Initial value problems(IVPs).

1. Introduction

It is well known that most of the Initial Value Problems(IVPs) are solved by Runge-Kutta methods which in turn being applied to compute numerical solutions for variety of problems that are modeled as the differential equations and their systems(Alexander and Coyle[9], Evans[10] , Hung[15] and Shampine and Gorden[3]). Runge-Kutta algorithms are used to solve differential equations efficiently that are equivalent to approximate the exact solutions by matching 'n' terms of the Taylor series expansion. Shampine and Watts [4, 6, 5] have

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developed mathematical codes for the Runge-Kutta fourth order method. Several types of four-stage fourth-order Runge-Kutta methods based on varieties of means are developed to solve industrially applicable problems including digital image processing simulation [16, 18, 24].

Evans and Yaacob[19] have developed new fourth-order accurate composite Runge-Kutta type method using the Heronian Mean and compared with several Runge-Kutta methods of fourth-order based on variety of means. They have found that four stage fourth-order Runge-Kutta method based on harmonic mean has less computational error in numerical solutions of the considered test problems as compared to other four stage fourth-order Runge-Kutta methods based on Centroidal Mean(CeM), Root Means Square(RMS), Arithmetic Mean(AM), Contra Harmonic Mean(CoM) and Geometric Mean(GM) respectively.

Butcher [1] has developed Runge-Kutta formula of fifth-order. The development of fifth-order Runge-Kutta have been done by Evans and Yaakub[12] and it is shown that their method is better than RK4, RH4(5)-Merson and RK5(6)-Nystrom methods. Sanugi and Yaacob have developed a new 5-stage explicit fifth-order nonlinear Runge-Kutta method based on GM. The fifth-order Runge-Kutta(5, 5) method with error control has been introduced by Evans and Yaakub [17]. They computed numerical results and compared with other well known methods RKF (4, 5) and RK (4, 5) Merson. Razali et al. [22] have applied the fifth order Runge-Kutta method to investigate the problem of Lorenz system. A systematic scheme for solving a system of time varying singular ordinary differential equation(ODE) has been discussed by Ponalagusamy [21]. Various physical day to day problems in the field of Robot arm, motions of the planet in a gravity field like kepler problem, electric circuits, chemical Reaction are in the form of system of equations. The dynamics of Robot Arm problem was first proposed by Taha [8]. The robot arm problem was successfully investigated by using the Runge-Kutta methods (Ponalagusamy and Senthilkumar[23]and Senthilkumar et. al. [26]).

Evans and Yaakub [12] computed approximate solutions of several types of differential equations using fifth order Runge-Kutta method based on CoM. Recent developments of fifth-order Runge-Kutta methods based on HM, GM, CoM are performed by ([12], [13], [20]). These existing methods are compared with fifth-order Runge-Kutta method based on heronian mean in view of numerical errors and stability [25]. It is pertinent to mention that no effort, so far, has been made to develop a five stage fifth-order weighted Runge-Kutta method based on HeM. Keeping this in view, a modest effort has been made in the present paper to develop such a new efficient numerical algorithm which is, for the first time added to the literature. It is observed that the presently developed algorithm has also been found to be more suitable one to solve the system of ODEs.

As we are doing mathematical modeling for investigating problems in daily life, we oftenly use the initial value problems(IVPs) such that

$$y' = f(x, y(x)) \tag{1.1}$$

where x is the independent variable which may indicate the time in a physical problem and the dependent variable *y(x)* is the solution. Moreover, since *y(x)* could be a N-dimensional vector valued function, the domain and range of the differential equation f and the solution y are given by

$$f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \tag{1.2}$$

$$y : \mathbb{R} \longrightarrow \mathbb{R}^N \tag{1.3}$$

The above equation (1.1) where f is a function of both x and y which is called "non-autonomous". However, by simply introducing an extra variable which is always exactly equal to x, it can easily be rewritten in an equivalent "autonomous" form below, where f is a function of y only:

$$y'(x) = f(y(x)) \tag{1.4}$$

Even though several problems are naturally expressed in the non-autonomous form, the autonomous form of differential equation (1.4) is preferred for most of the theoretical investigations. Further, the autonomous form has some advantages in numerical analysis since it gives a greater possibility that numerical methods can solve the differential equation exactly. It is of interest to note that the differential equation by itself is not enough to find a unique solution. Hence, some other additional information is needed. However, if all components of y are given at a certain value of x, that is, "initial conditions", then the differential equation is called an "initial value problem (IVP)" which is closely and naturally involved with physical modeling. An initial value problem with the given initial condition $y(x_0) = y_0$ is of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \tag{1.5}$$

in non-autonomous form and

$$y'(x) = f(y(x)), \quad y(x_0) = y_0 \tag{1.6}$$

in autonomous form.

Before we seek for a numerical solution to an initial value problem it is important to consider whether the solution is unique, or even a solution exists at all. There are several mathematical criteria for determining these two considerations, but the most commonly used approach is the Lipschitz condition.

Definition 1.1. The function $f : [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ is said to satisfy a Lipschitz condition in its second variable if there exists a constant L, known as a Lipschitz constant, such that for any $x \in [a, b]$ and $y, Z \in \mathbb{R}^N$, $\| f(x, Y) - f(x, Z) \| \leq L \| Y - Z \|$.

This definition is used in the following theorem.

Theorem 1.2. *Consider an initial value problem (1.5) or (1.6) where* $f : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ *is continuous in its first variable and satisfies a Lipschitz condition in its second variable. There there exists a unique solution to this problem.*

Proof The proof of theorem can be found in [7]. □

The plan of the paper is as follows. In section 2, the formulation and derivation of the fifth-order weighted Runge-Kutta method based on heronian mean is presented. In section 3 the stability polynomial and stability region of all the considered methods are presented and compared. The first order and system of first order ordinary differential equations are tested on variety of initial value problems in section 4. The paper ends with conclusions.

2. Formulation and derivation of the Fifth-Order Weighted Runge-Kutta Heronian Mean Scheme

The general p-stage Runge-Kutta method for solving an IVP $y' = f(x, y(x))$ with the initial condition $y(x_0) = y_0$ is defined by

$$y_{n+1} = y_n + \sum_{i=1}^{p} b_i k_i \tag{2.1}$$

$$Where,\ k_i = f\left(x_n + c_i h,\ y_n + h\sum_{j=1}^{p} a_{ij} k_j\right)\ and \tag{2.2}$$

$$c_i = \sum_{j=1}^{p} a_{ij}; i = 1, 2, ..., p \tag{2.3}$$

with b and c are p-dimensional vectors and $A(a_{ij})$ be the $p \times p$ matrix. Then the Butcher array is of the form as mentioned in figure 1.

$$\begin{array}{c|ccccccc}
c_1 & a_{11} & & & & & & \\
c_2 & a_{21} & a_{22} & & & & & \\
c_3 & a_{31} & a_{32} & a_{33} & & & & \\
\cdot & \cdot & \cdot & & \cdot & & & \\
\cdot & \cdot & \cdot & & & \cdot & & \\
\cdot & \cdot & \cdot & & & & \cdot & \\
c_{p-1} & a_{p-1,1} & a_{p-1,2} & a_{p-1,3} & \cdot\ \cdot\ \cdot & & a_{p-1,p-1} & \\
c_p & a_{p1} & a_{p2} & a_{p3} & \cdot\ \cdot\ \cdot & & a_{p,p-1} & a_{p,p} \\
\hline
 & b_1 & b_2 & b_3 & \cdot\ \cdot\ \cdot & & b_{p-1} & b_p
\end{array}$$

Figure 1. Butcher Array Table

To construct our new formula fifth-order weighted RKHeM, we recall that which is combination of AM and GM. Consider the arbitrary numbers a,b and c are in Arithmetic progression and d,e and f are in Geometric progression, such that C is called AM and f is called the GM. That is,

$$c = \frac{a+b}{2} \ and \ \frac{f}{d} = \frac{e}{f} \Rightarrow f^2 = ed \Rightarrow f = \sqrt{ed}, \tag{2.4}$$

Considering,

$$\alpha = \frac{k_{j-1}+k_j}{2} \ and \ \beta = \sqrt{k_{j-1}k_j} \ for \ j = 2,3,4, \tag{2.5}$$

rearranging k_i for i=1,2,3,4,5 and using (2.4) in equation (2.1)-(2.3), we obtain

$$\begin{aligned} k_1 &= f\left(x_n, y_n\right) \\ k_2 &= f\left(x_n + c_1 h, y_n + h a_{11} k_1\right) \\ k_3 &= f\left(x_n + c_2 h, y_n + h(a_{21}k_1 + a_{22}k_2)\right) \\ k_4 &= f\left(x_n + c_3 h, y_n + h(a_{31}k_1 + a_{32}k_2 + a_{33}k_3)\right) \\ k_5 &= f\left(x_n + c_4 h, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3 + a_{44}k_4)\right) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} y_{n+1} = y_n + h(&b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5 + b_6\sqrt{|k_1k_2|} \\ &+ b_7\sqrt{|k_2k_3|} + b_8\sqrt{|k_3k_4|} + b_9\sqrt{|k_4k_5|}) \end{aligned} \tag{2.7}$$

and the parameters $b_1, b_2, ..., b_9, a_{11}, a_{21}, ..., a_{44}$ are to be determined. It is to be noticed that for simplicity, the algebra function f is considered as a function of y only, without loss of generality. Taylor series expansion of an exact solution $y(x_n + h)$ up to sixth order is given by

$$\begin{aligned} y(x_n+h) &= y_n + hf + \frac{h^2}{2} f f_y + \frac{h^3}{6}(f f_y^2 + f^2 f_{yy}) + \frac{h^4}{24}(f^3 f_{yyy} \\ &+ 4f^2 f_y f_{yy} + f f_y^3) + \frac{h^5}{120}(f f_y^4 + 11 f^2 f_y^2 f_y y + 4 f^3 f_y^2 f_{yy} \\ &+ 7 f^3 f_y f_{yyy} + f^4 f_{yyyy}) + \frac{h^6}{720}(f^5 f_{yyyyy} + 11 f^4 f_y f_{yyyy} \\ &+ 15 f^4 f_{yy} f_{yyy} + 32 f^3 f_y^2 f_{yyy} + 34 f^3 f_y f_{yy}^2 + 26 f^2 f_y^3 f_{yy} \\ &+ f f_y^5) + O\left(h^7\right) \end{aligned} \tag{2.8}$$

Expanding k_1, k_2, k_3, k_4 and k_5 in Taylor series about x_n, substituting in equation (2.7) and comparing the co-efficients of the same in equation (2.8), one can obtain 11 equations with 19 parameters. By taking $b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8+b_9 = 1, b_1 = b_5 = b_6 = b_7 = b_8 = b_9 = \frac{1}{14}, b_2 = \frac{1}{7}$ and solving 11 non-linear equations simultaneously, the required values of the parameters have been computed as given below:

$$\begin{aligned}
b_3 &= \frac{2}{7},\ b_4 = \frac{1}{7},\ a_{11} = \frac{38789865}{163066739},\ a_{21} = -\frac{38926273}{154969080},\\
a_{22} &= \frac{49239911}{69362921},\ a_{31} = -\frac{46861542}{132568883},\ a_{32} = \frac{415527214}{419413821},\\
a_{33} &= \frac{38385758}{106762407},\ a_{41} = \frac{311954026}{192782691},\ a_{42} = -\frac{40359572}{29023473},\\
a_{43} &= -\frac{25308289}{156858746},\ a_{44} = \frac{45864583}{76174812}. \qquad (2.9)
\end{aligned}$$

By substituting the values of above parameters in equation (2.6)-(2.7), we get a new fifth-order weighted Runge-Kutta based on HeM as follows:

$$y_{n+1} = y_n + \frac{h}{14}(k_1 + 2(k_2 + k_3) + 2(k_3 + k_4) + k_5 + \sqrt{|k_1k_2|} + \sqrt{|k_2k_3|} + \sqrt{|k_3k_4|} + \sqrt{|k_4k_5|}) \qquad (2.10)$$

Where

$$\begin{aligned}
k_1 &= f(x_n, y_n)\\
k_2 &= f(x_n + \frac{38789865}{163066739}h, y_n + \frac{38789865}{163066739}hk_1)\\
k_3 &= f(x_n + \frac{41063576}{89521497}h, y_n + h(-\frac{38926273}{154969080}k_1\\
&+ \frac{49239911}{69362921}k_2))\\
k_4 &= f(x_n + \frac{142104621}{142562423}h, y_n + h(-\frac{46861542}{132568883}k_1\\
&+ \frac{415527214}{419413821}k_2 + \frac{38385758}{106762407}k_3))\\
k_5 &= f(x_n + \frac{73755049}{110356862}h, y_n + h(\frac{311954026}{192782691}k_1\\
&- \frac{40359572}{29023473}k_2 - \frac{25308289}{156858746}k_3 + \frac{45864583}{76174812}k_4)) \qquad (2.11)
\end{aligned}$$

The Local Truncation Error for the Heronian Mean method is given by

$$\begin{aligned}
LTE^{HeM} &= h^6[-0.0012557831ff_y^5 - 0.0072638436f^2f_y^3f_{yy}\\
&-0.0039318538f^3f_yf_{yy}^2 + 0.0010892514f^3f_y^2f_{yyy} - 0.0002046344\\
&f^4f_{yy}f_{yyy} + 0.0021303658f^4f_yf_{yyyy} + 0.0004306801f^5f_{yyyyy}] + ...
\end{aligned} \qquad (2.12)$$

For a detailed scheme of mathematical derivations to derive the formulas since by equations (2.13)-(2.15), one may refer [23].

From that, the error control and step size selection can be shown as below:

$$0.004512P^5Qh^6 < TOL \qquad (2.13)$$

$$h < \left[\frac{TOL}{0.004512P^5Q}\right]^{\frac{1}{6}} \tag{2.14}$$

The Global Truncation Error(GTE) is given below:

$$|\varepsilon| \leq \left(\frac{h^5}{414720LD}\right) M \left(e^{DL(x_n - x_0)} - 1\right) \tag{2.15}$$

we can conclude that if the LTE of a numerical method is $O(h^{p+1})$ then the GTE is $O(h^6)$. The estimate of the GTE can not be used for practical error estimation or error control because the value from the GTE is less accurate than the LTE.

3. Stability Regions

In this section we discuss the stability regions for the new fifth-order weighted Runge-Kutta method based on heronian mean with the existing fifth order Runge-Kutta formulas based on Contra Harmonic Mean, Geometric Mean and Harmonic Mean. To evaluate stability polynomial, we use simple test equation $y' = \lambda y$, where λ is a complex constant. The stability polynomials for the methods considered in the present investigation are as follows:

The stability polynomial for fifth-order Runge-Kutta based on HeM:

$$Q = 1 + z + 0.5z^2 + 0.162594z^3 + 0.034799z^4 + 0.004883z^5 + O(z)^6 \tag{3.1}$$

The stability polynomial for fifth-order Runge-Kutta based on HM:

$$Q = 1+z+0.499999z^2+0.166667z^3+0.04166667z^4+0.00833333z^5+O(z)^6 \tag{3.2}$$

The stability polynomial for fifth-order Runge-Kutta based on CoM:

$$Q = 1 + z + 0.5z^2 + 0.166667z^3 + 0.04166667z^4 + 0.00833333z^5 + O(z)^6 \tag{3.3}$$

The stability polynomial for fifth-order Runge-Kutta based on GM:

$$Q = 1.00000064z - 0.02245424z^2 - 0.13963200z^3 - 0.03860570z^4 \\ -0.00790011z^5 + O(z)^6 \tag{3.4}$$

The condition $|\frac{y_{n+1}}{y_n}| = Q < 1$ must be satisfied in order to determine the stability region of the fifth-order Runge-Kutta formula in the complex plane. With the help of stability polynomials, the stability regions for the fifth-order Runge-Kutta formula based on Harmonic Mean (shown as star format), the fifth-order Runge-Kutta method based on Contra harmonic Mean (shown as square format), the fifth-order Runge-Kutta method based on Geometric Mean (shown as small circle format) and the proposed fifth-order Runge-Kutta method based on Heronian Mean (shown as big circle format) are depicted in Figure 2. It is observed that the present fifth-order weighted Runge-Kutta method based on heronian mean has a wider stability region in comparison with other three fifth-order methods. It is concluded from Table 1 that our new fifth order method (HeM) has got the better stability region in the negative real axis and both in the positive and negative imaginary axis as compared to the existing fifth order methods(HM, CoM and GM).

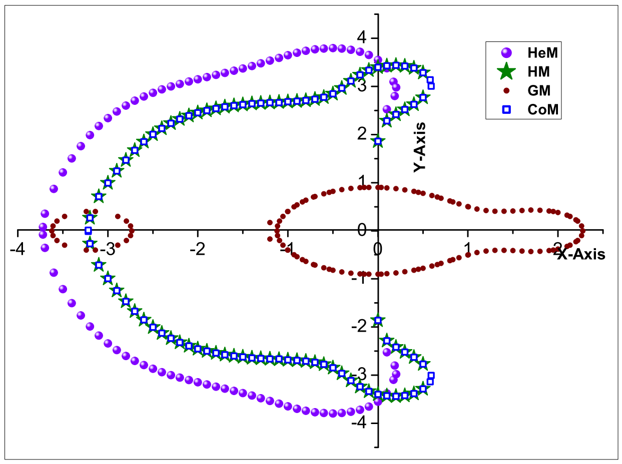

FIGURE 2. Comparison of Stability regions for the fifth-order Runge-Kutta based on HeM, HM, GM and CoM.

TABLE 1. *Range of the Stability Regions*

5^{th}-Order	Real-Axis		Imaginary-Axis	
	Negative	Positive	Negative	Positive
HM	-3.200	0.5	-3.39575000	3.39575000
C_oM	-3.127	0.5	-3.39575159	3.39575159
H_eM	-3.72	0.1	-3.55298172	3.55298172
GM	-1.12174	2.27479	-0.902166	0.902166
	-2.73261	-3.62235	-0.4	0.4

4. Numerical Results

In this section, five first order and one system of first order IVPs are considered to illustrate efficiency and suitability of the computational methods discussed in this paper. The problems can be evaluated with the step size h=0.01. The results are presented in the tables (Table 2 - Table7).

Example 4.1. *Consider a single equation Oscillatory problem of the form* [2]*:*

$$\begin{aligned} y' &= y\,cos(x),\ y(0)=1 \\ \textit{with exact solution } y &= e^{sin(x)}. \end{aligned}$$

Example 4.2. *Consider a Single of a logistic curve of the form* [2]*:*

$$\begin{aligned} y' &= \frac{y}{4}\left(1-\frac{y}{20}\right),\ y(0)=1 \\ \textit{with exact solution } y &= \frac{20}{1+19e^{-x/4}}. \end{aligned}$$

Example 4.3. *Consider the first order initial value problem of the form:*

$$\begin{aligned} y' &= y,\ y(0)=1 \\ \textit{with exact solution } y &= exp(x). \end{aligned}$$

Example 4.4. *Consider the first order initial value problem of the form:*

$$\begin{aligned} y' &= -y+x+1,\ y(0)=1 \\ \textit{with exact solution } y &= x+exp(-x). \end{aligned}$$

Example 4.5. *Consider the first order initial value problem of the form:*

$$\begin{aligned} y' &= 1/y,\ y(0)=1 \\ \textit{with exact solution } y &= \sqrt{2x+1}. \end{aligned}$$

Example 4.6. *The reduced robot arm model to the system of linear equation of the form:*[23]

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= 0.2140x_2 - 0.1730x_1 + 0.0265, \\ x_3' &= x_4, \\ x_4' &= -0.130321x_4 - 0.00191844x_3 + 0.00935089, \quad \textit{with,} \\ x_1(0) &= -1, x_2(0)=0, x_3(0)=-1 \textit{ and } x_4(0)=0. \quad \textit{and} \end{aligned}$$

the corresponding exact solutions are given by

$$\begin{aligned} x_1(t) &= e^{0.107t}[-1.15317919cos(0.401934074t) + 0.306991074 \\ &\quad sin(0.401934074t)] + 0.15317919 \\ x_2(t) &= e^{0.107t}[0.463502009sin(0.401934074t) + 0.123390173 \\ &\quad cos(0.401934074t)] + e^{0.107t}[-1.15317919cos(0.401934074t) \\ &\quad +0.306991074sin(0.401934074t)] \\ x_3(t) &= 1.029908976e^{-0.113404416t} - 6.904124484e^{-0.016916839t} \\ &\quad +4.874215508 \\ x_4(t) &= -0.116795962e^{-0.113404416t} + 0.116795962e^{-0.016916839t} \end{aligned}$$

The considered robot arm problem can be solved by new fifth-order Runge-Kutta method based on heronian mean. The error values for four system of first order equations are presented in the table 7, along with other Runge-Kutta (5,5) methods.

TABLE 2. *Exact solution and Error for the Example 4.1.*

x	$Exact$	$ErrorHM$	$ErrorCoM$	$ErrorHeM$	$ErrorGM$
0.0	1.000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	1.104987	5.36727E-2	1.04992E-1	1.46958E-6	4.38845E-2
0.2	1.219779	1.15072E-1	2.19788E-1	3.06199E-6	9.46901E-2
0.3	1.343826	1.84172E-1	3.43839E-1	4.72663E-6	1.52537E-1
0.4	1.476122	2.60595E-1	4.76140E-1	6.39238E-6	2.17241E-1
0.5	1.615147	3.43542E-1	6.15168E-1	7.96868E-6	2.88246E-1
0.6	1.758819	4.31747E-1	7.58843E-1	9.34937E-6	3.64567E-1
0.7	1.904497	5.23456E-1	9.04524E-1	1.04197E-5	4.44760E-1
0.8	2.049009	6.16441E-1	1.04903E-0	1.10662E-5	5.26919E-1
0.9	2.188742	7.08060E-1	1.18877E-0	1.11896E-5	6.08720E-1
1.0	2.319777	7.95362E-1	1.31981E-0	1.07185E-5	6.87500E-1

TABLE 3. *Exact solution and Error for the Example 4.2.*

x	$Exact$	$ErrorHM$	$ErrorCoM$	$ErrorHeM$	$ErrorGM$
0.0	1.000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	1.024019	1.20702E-2	2.40190E-2	7.12980E-8	1.06322E-2
0.2	1.048583	2.45504E-2	4.85831E-2	1.45632E-7	2.16404E-2
0.3	1.073703	3.74501E-2	7.37031E-2	2.23080E-7	3.30334E-2
0.4	1.099390	5.07788E-2	9.93899E-2	3.03721E-7	4.48205E-2
0.5	1.125655	6.45465E-2	1.25655E-1	3.87635E-7	5.70108E-2
0.6	1.152509	7.87629E-2	1.52509E-1	4.74900E-7	6.96136E-2
0.7	1.179963	9.34379E-2	1.79966E-1	5.65598E-7	8.26384E-2
0.8	1.208030	1.08582E-1	2.08030E-1	6.59808E-7	9.60948E-2
0.9	1.236720	1.24205E-1	2.36721E-1	7.57612E-7	1.09992E-2
1.0	1.266046	1.40317E-2	2.66047E-1	8.59090E-7	1.24341E-2

5. Conclusion

In the present paper, we have developed a new fifth-order weighted Runge-Kutta technique based on heronian mean and obtained the stability polynomial and stability region. Several practically applicable problems have been considered to test the suitability, adoptability and accuracy of the proposed method.

TABLE 4. *Exact solution and Error for the Example 4.3.*

x	$Exact$	$ErrorHM$	$ErrorCoM$	$ErrorHeM$	$ErrorGM$
0.0	1.00000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	1.105171	5.37684E-2	1.05176E-1	1.46839E-6	4.38940E-2
0.2	1.221403	1.15956E-1	2.21413E-1	3.24565E-6	9.50941E-2
0.3	1.349859	1.87589E-1	3.49873E-1	5.38049E-6	1.54533E-1
0.4	1.491825	2.69811E-1	4.91844E-1	7.92849E-6	2.23254E-1
0.5	1.64872	3.63893E-1	6.48746E-1	1.09529E-5	3.02416E-1
0.6	1.822119	4.71247E-1	8.22148E-1	1.45258E-5	3.93316E-1
0.7	2.013753	5.93443E-1	1.01379E-0	1.87291E-5	4.97397E-1
0.8	2.225541	7.32224E-4	1.22558E-0	2.36558E-5	6.16268E-1
0.9	2.459603	8.89526E-1	1.45965E-0	2.94116E-5	7.51719E-1
1.0	2.718282	1.06749E-0	1.71833E-0	3.61165E-5	9.05744E-1

TABLE 5. *Exact solution and Error for the Example 4.4.*

x	$Exact$	$ErrorHM$	$ErrorCoM$	$ErrorHeM$	$ErrorGM$
0.0	1.000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	1.004837	1.94594E-3	3.93119E-3	3.23979E-6	1.22023E-3
0.2	1.018731	8.23366E-3	1.69592E-2	4.26872E-6	6.80205E-3
0.3	1.040818	1.82597E-2	3.82637E-2	4.96717E-6	1.61309E-2
0.4	1.070320	3.14578E-2	6.70572E-2	5.45673E-6	2.86441E-2
0.5	1.106531	4.73255E-2	1.02627E-1	5.79005E-6	4.38418E-2
0.6	1.148812	6.54178E-2	1.44329E-1	6.00032E-6	6.12809E-2
0.7	1.196585	8.53413E-2	1.91579E-1	6.11189E-6	8.05693E-2
0.8	1.249329	1.06749E-1	2.43849E-1	6.14380E-6	1.01361E-1
0.9	1.306570	1.29335E-1	3.00663E-1	6.11150E-6	1.23351E-1
1.0	1.367879	1.52831E-1	3.61588E-1	6.02777E-6	1.46272E-1

It is noticed from the numerical results that the new fifth-order Runge-Kutta method based on heronian mean is more efficient than the well known fifth-order Runge-Kutta method based Harmonic mean(HM), fifth order weighted Runge-Kutta method based on contra harmonic mean(CoM) and fifth-order Runge-Kutta method based Geometric mean(GM). In particularly, the robot arm real time problem is evaluated by the present method which gives more efficient numerical results as compared to other existing fifth-order Runge-Kutta methods. A remarkable result is that the new fifth-order weighted Runge-Kutta method based on heronian mean guarantees the most efficient numerical technique for investigating first order initial value problems and the system of first order initial value problem in ODEs.

TABLE 6. *Exact solution and Error for the Example 4.5.*

x	$Exact$	$ErrorHM$	$ErrorCoM$	$ErrorHeM$	$ErrorGM$
0.0	1.000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	1.095445	4.67493E-2	9.54499E-2	1.15882E-6	4.56243E-2
0.2	1.183216	8.79780E-2	1.83226E-1	1.97034E-6	8.55639E-2
0.3	1.264911	1.25022E-1	2.64926E-1	2.56461E-6	1.21204E-1
0.4	1.341641	1.58779E-1	3.41660E-1	3.01436E-6	1.53475E-1
0.5	1.414214	1.89881E-1	4.14238E-1	3.36341E-6	1.83036E-1
0.6	1.483240	2.18791E-1	4.83269E-1	3.63968E-6	2.10367E-1
0.7	1.549193	2.45859E-1	5.49227E-1	3.86180E-6	2.35832E-1
0.8	1.612452	2.71357E-1	6.12491E-1	4.04267E-6	2.59711E-1
0.9	1.673320	2.95496E-1	6.73364E-1	4.19146E-6	2.82226E-1
1.0	1.732051	3.18448E-1	7.32091E-1	4.31490E-6	3.03554E-1

TABLE 7. *Error in R-K(5,5) based on HeM for the Example 4.6(Robot Arm Problem*[23]*).*

t	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
0.0	0.0000000	0.0000000	0.0000000	0.0000000
0.1	7.47021E-7	4.59391E-8	1.54702E-8	1.02271E-9
0.2	1.08259E-6	8.23844E-8	7.36929E-9	2.02120E-9
0.3	1.40138E-6	1.10219E-7	1.92867E-9	2.99695E-9
0.4	1.71670E-6	1.29260E-7	1.16024E-8	3.95053E-9
0.5	12.0319E-6	1.39193E-7	2.13947E-8	4.88238E-9
0.6	2.34769E-6	1.39663E-7	3.11926E-8	5.79294E-9
0.7	2.66379E-6	1.30312E-7	4.09371E-8	6.68260E-9
0.8	2.97943E-6	1.10785E-7	5.05942E-8	7.55174E-9
0.9	3.29354E-6	8.07423E-8	6.01429E-8	8.40073E-9
1.0	3.60489E-6	3.98600E-8	6.95702E-8	9.22992E-9

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