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ORTHOGONAL TWO-DIRECTION WAVELETS OF ORDER 2 FROM ORTHOGONAL SYMMETRIC/ANTISYMMETRIC MULTIWAVELETS†

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Abstract. A method for recovering Chui-Lian's orthogonal symmetric/antisymmetric multiwavelets of order 2 from orthogonal two-direction wavelets of order 2 was proposed by Yang and Xie. In this paper we pursue the converse, that is, we propose a method for constructing orthogonal two-direction wavelets of order 2 from orthogonal symmetric/antisymmetric multiwavelets of order 2.

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1. Introduction

A standard (one-direction) scaling function of dilation factor 2 is a real-valued function ϕ which satisfies a recursion relation of the form

$$\phi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} p_k \, \phi(2x-k) \tag{1.1}$$

and generates a multiresolution approximation (MRA) of $L^2(\mathbb{R})$. The recursion coefficients p_k are scalars.

Two-direction scaling function ϕ and wavelet function ψ, which are a more general setting than the one-direction scaling function and wavelet, are investigated in [2, 3, 4, 5, 6, 7].

A *two-direction refinable function* of dilation factor 2 is a real-valued function $\phi(x)$ which satisfies a recursion relation

$$\phi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} \left[p_k^+ \, \phi(2x-k) + p_k^- \, \phi(k-2x)\right] \tag{1.2}$$

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and generates a multiresolution approximation of $L^2(\mathbb{R})$.

The *two-direction wavelet function* ψ associated with ϕ satisfy

$$\psi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} \left[q_k^+ \, \phi(2x-k) + q_k^- \, \phi(k-2x)\right]. \tag{1.3}$$

The two-direction scaling function and wavelet function together will be called a *two-direction wavelet.*

One-direction wavelet theory needs to be appropriately modified for the two-direction setting. For example, a basis of the space V_0 of the two-direction MRA is given by

$$\{\phi(x-k),\, \phi(k-x) : k \in \mathbb{Z}\}.$$

The *deduced multiscaling function* $\mathbf{\Phi}$, of multiplicity 2, is a standard (one-direction) multiscaling function which satisfies the *deduced refinement equation*

$$\mathbf{\Phi}(x) = \begin{bmatrix} \phi(x) \\ \phi(-x) \end{bmatrix} = \sqrt{2} \sum_k \begin{bmatrix} p_k^+ & p_k^- \\ p_{-k}^- & p_{-k}^+ \end{bmatrix} \mathbf{\Phi}(2x-k). \tag{1.4}$$

Many properties of $\boldsymbol{\phi}$, such as approximation order, smoothness, and orthogonality, are defined and investigated in terms of corresponding properties of $\mathbf{\Phi}$.

In this paper we only consider real recursion coefficients p_k^+, p_k^-, q_k^+, and q_k^- in $\mathbb{R}$ for $k \in \mathbb{Z}$.

In [7], a method for recovering Chui-Lian's orthogonal symmetric/antisymmetric multiwavelets of order 2 from orthogonal two-direction wavelets of order 2 was proposed. Motivated by [7], we pursue the converse of [7] in this paper, that is, we propose a method for constructing orthogonal two-direction scaling function of order 2 and wavelet function ψ associated with ϕ from orthogonal symmetric/antisymmetric multiscaling function $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ of order 2 and wavelet $\boldsymbol{\psi} = [\psi_1, \psi_2]^T$, respectively.

For an example, we take Chui and Lian's orthogonal symmetric/antisymmetric multiscaling functions $\boldsymbol{\phi}$ of order 2 and multiwavelets $\boldsymbol{\psi}$ in [1]. We obtain two-direction scaling function ϕ of order 2 supported on $[0, 2]$ and wavelet ψ. The constructed two-direction wavelets are the same as [7, Example 4.1].

This paper is organized as follows. Constructions of two-direction scaling functions of order 2 and wavelets from orthogonal symmetric/antisymmetric multiscaling functions of order 2 and multiwavelets, respectively, are introduced in section 2. An example for illustrating the general theory in sections 1 and 2 is given in section 3.

2. Two-direction wavelets of order 2 from orthogonal symmetric/antisymmetric multiwavelets

In this section we propose a method for constructing orthogonal two-direction scaling function of order 2 and wavelet from orthogonal symmetric/antisymmetric multiscaling function $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ of order 2 and multiwavelet $\boldsymbol{\psi} = [\psi_1, \psi_2]^T$.

2.1. Orthogonal two-direction scaling function of order 2. Orthogonal symmetric/antisymmetric multiscaling function $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ of order 2 supported on $[0, 2]$ is given as

$$\begin{aligned}\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix} &= \begin{bmatrix}a_0 & b_0\\ c_0 & d_0\end{bmatrix}\begin{bmatrix}\phi_1(2x)\\ \phi_2(2x)\end{bmatrix} + \begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}\begin{bmatrix}\phi_1(2x-1)\\ \phi_2(2x-1)\end{bmatrix}\\ &\quad + \begin{bmatrix}a_0 & -b_0\\ -c_0 & d_0\end{bmatrix}\begin{bmatrix}\phi_1(2x-2)\\ \phi_2(2x-2)\end{bmatrix},\end{aligned} \tag{2.1}$$

where $a_0, a_1, b_0, b_1, c_0, c_1, d_0$, and d_1 are constants. (Existence is guaranteed by Chui-Lian in [1].)

Construct a function ϕ by

$$\phi(x) = \frac{\sqrt{2}}{2}\left[\phi_1(x) - \phi_2(x)\right]. \tag{2.2}$$

Since $\phi_1(2-x) = \phi_1(x)$ and $\phi_2(2-x) = -\phi_2(x)$ by symmetric/antisymmetric property, we have

$$\phi(2-x) = \frac{\sqrt{2}}{2}\left[\phi_1(2-x) - \phi_2(2-x)\right] = \frac{\sqrt{2}}{2}\left[\phi_1(x) + \phi_2(x)\right]. \tag{2.3}$$

By solving (2.2) and (2.3) for ϕ_1 and ϕ_2, we have

$$\phi_1(x) = \frac{1}{\sqrt{2}}\left[\phi(x) + \phi(2-x)\right], \qquad \phi_2(x) = \frac{1}{\sqrt{2}}\left[\phi(2-x) - \phi(x)\right]. \tag{2.4}$$

Clearly, ϕ provides approximation order 2, since $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ provides approximation order 2. ϕ is supported on $[0, 2]$, since ϕ_1 and ϕ_2 are supported on $[0, 2]$. ϕ is refinable, since ϕ_1 and ϕ_2 are refinable.

Now we want to prove that ϕ is a two-direction refinable function of the form

$$\phi(x) = \sum_{k=0}^{2} p_k^+ \phi(2x-k) + \sum_{k=2}^{4} p_k^- \phi(k-2x), \tag{2.5}$$

for some p_k^+ and p_k^-.

By applying (2.2), we have

$$\begin{aligned}\sqrt{2}\,\phi(x) &= \phi_1(x) - \phi_2(x)\\ &= (a_0 - c_0)\phi_1(2x) + (a_1 - c_1)\phi_1(2x-1) + (a_0 + c_0)\phi_1(2x-2)\\ &+ (b_0 - d_0)\phi_2(2x) + (b_1 - d_1)\phi_2(2x-1) + (-b_0 - d_0)\phi_2(2x-2).\end{aligned}$$

By (2.4), we have

$$\begin{aligned}2\,\phi(x) &= (a_0 - c_0)[\phi(2x) + \phi(2-2x)] + (a_1 - c_1)[\phi(2x-1) + \phi(3-2x)]\\ &+ (a_0 + c_0)[\phi(2x-2) + \phi(4-2x)] + (b_0 - d_0)[\phi(2-2x) - \phi(2x)]\\ &+ (b_1 - d_1)[\phi(3-2x) - \phi(2x-1)] + (-b_0 - d_0)[\phi(4-2x) - \phi(2x-2)]\\ &= (a_0 - b_0 - c_0 + d_0)\phi(2x) + (a_1 - b_1 - c_1 + d_1)\phi(2x-1)\end{aligned}$$

$$
\begin{aligned}
&+ \quad (a_0+b_0+c_0+d_0)\phi(2x-2)+(a_0+b_0-c_0-d_0)\phi(2-2x)\\
&+ \quad (a_1+b_1-c_1-d_1)\phi(3-2x)+(a_0-b_0+c_0-d_0)\phi(4-2x).
\end{aligned}
$$

Hence, we have

$$\phi(x)=\sum_{k=0}^{2}p_k^{+}\phi(2x-k)+\sum_{k=2}^{4}p_k^{-}\phi(k-2x), \tag{2.6}$$

where

$$
\begin{aligned}
p_0^{+} &= \tfrac{1}{2}(a_0-b_0-c_0+d_0), p_1^{+}=\tfrac{1}{2}(a_1-b_1-c_1+d_1),\\
p_2^{+} &= \tfrac{1}{2}(a_0+b_0+c_0+d_0), p_2^{-}=\tfrac{1}{2}(a_0+b_0-c_0-d_0),\\
p_3^{-} &= \tfrac{1}{2}(a_1+b_1-c_1-d_1), p_4^{-}=\tfrac{1}{2}(a_0-b_0+c_0-d_0).
\end{aligned} \tag{2.7}
$$

Hence, ϕ is a two-direction refinable function of order 2 supported on $[0,2]$.

2.2. Orthogonal two-direction wavelet function ψ. Orthogonal symmetric/antisymmetric multiwavelet function $\boldsymbol{\psi}=[\psi_1,\psi_2]^T$ supported on $[0,2]$ is given as

$$
\begin{aligned}
\begin{bmatrix}\psi_1(x)\\ \psi_2(x)\end{bmatrix} &= \begin{bmatrix}a_0' & b_0'\\ c_0' & d_0'\end{bmatrix}\begin{bmatrix}\phi_1(2x)\\ \phi_2(2x)\end{bmatrix}+\begin{bmatrix}a_1' & b_1'\\ c_1' & d_1'\end{bmatrix}\begin{bmatrix}\phi_1(2x-1)\\ \phi_2(2x-1)\end{bmatrix}\\
&+\begin{bmatrix}a_0' & -b_0'\\ -c_0' & d_0'\end{bmatrix}\begin{bmatrix}\phi_1(2x-2)\\ \phi_2(2x-2)\end{bmatrix},
\end{aligned} \tag{2.8}
$$

where $a_0', a_1', b_0', b_1', c_0', c_1', d_0'$, and d_1' are constants. (Existence is guaranteed by Chui-Lian in [1].)

Construct a function ψ by

$$\psi(x)=\frac{\sqrt{2}}{2}\left[-\psi_1(-x)+\psi_2(-x)\right]. \tag{2.9}$$

(By constructing this way, we are able to recover ψ by Yang in [5], see Example 3.1 in section 3. There exist many other ways of constructing ψ, which is orthogonal wavelet corresponding to ϕ.)

Since $\psi_1(2-x)=\psi_1(x)$ and $\psi_2(2-x)=-\psi_2(x)$ by symmetric/antisymmetric property, we have

$$\psi(x-2)=\frac{\sqrt{2}}{2}\left[-\psi_1(2-x)+\psi_2(2-x)\right]=-\frac{\sqrt{2}}{2}\left[\psi_1(x)+\psi_2(x)\right]. \tag{2.10}$$

By solving (2.9) and (2.10) for ψ_1 and ψ_2, we have

$$\psi_1(x)=-\frac{1}{\sqrt{2}}\left[\psi(-x)+\psi(x-2)\right], \qquad \psi_2(x)=-\frac{1}{\sqrt{2}}\left[\psi(x-2)-\psi(-x)\right]. \tag{2.11}$$

Clearly, ψ is supported on $[-2,0]$, since ϕ_1 and ϕ_2 are supported on $[0,2]$. ψ is refinable, since ψ_1 and ψ_2 are refinable.

Now we want to prove that ψ is a two-direction wavelet function associated with ϕ of the form

$$\psi(x)=\sum_{k=-4}^{-2} q_k^+\phi(2x-k)+\sum_{k=-2}^{0} q_k^-\phi(k-2x), \tag{2.12}$$

for some q_k^+ and q_k^-.

By applying (2.9), we have

$$\begin{aligned}-\sqrt{2}\,\psi(-x) &= \phi_1(x)-\phi_2(x)\\ &= (a_0'-c_0')\phi_1(2x)+(a_1'-c_1')\phi_1(2x-1)+(a_0'+c_0')\phi_1(2x-2)\\ &+ (b_0'-d_0')\phi_2(2x)+(b_1'-d_1')\phi_2(2x-1)+(-b_0'-d_0')\phi_2(2x-2).\end{aligned}$$

By (2.11), we have

$$\begin{aligned}-2\,\psi(-x) &= (a_0'-b_0'-c_0'+d_0')\phi(2x)+(a_1'-b_1'-c_1'+d_1')\phi(2x-1)\\ &+ (a_0'+b_0'+c_0'+d_0')\phi(2x-2)+(a_0'+b_0'-c_0'-d_0')\phi(2-2x)\\ &+ (a_1'+b_1'-c_1'-d_1')\phi(3-2x)+(a_0'-b_0'+c_0'-d_0')\phi(4-2x).\end{aligned}$$

That is,

$$\begin{aligned}-2\,\psi(x) &= (a_0'-b_0'-c_0'+d_0')\phi(-2x)+(a_1'-b_1'-c_1'+d_1')\phi(-2x-1)\\ &+ (a_0'+b_0'+c_0'+d_0')\phi(-2x-2)+(a_0'+b_0'-c_0'-d_0')\phi(2x+2)\\ &+ (a_1'+b_1'-c_1'-d_1')\phi(2x+3)+(a_0'-b_0'+c_0'-d_0')\phi(2x+4).\end{aligned}$$

Hence, we have

$$\psi(x)=\sum_{k=-4}^{-2} q_k^+\phi(2x-k)+\sum_{k=-2}^{0} q_k^-\phi(k-2x), \tag{2.13}$$

where

$$\begin{aligned}q_{-4}^+ &= -\tfrac{1}{2}(a_0'-b_0'+c_0'-d_0'), q_{-3}^+ = -\tfrac{1}{2}(a_1'+b_1'-c_1'-d_1'),\\ q_{-2}^+ &= -\tfrac{1}{2}(a_0'+b_0'-c_0'-d_0'), q_{-2}^- = -\tfrac{1}{2}(a_0'+b_0'+c_0'+d_0'),\\ q_{-1}^- &= -\tfrac{1}{2}(a_1'-b_1'-c_1'+d_1'), q_0^- = -\tfrac{1}{2}(a_0'-b_0'-c_0'+d_0').\end{aligned} \tag{2.14}$$

Hence, ψ is a two-direction wavelet function associated with ϕ supported on $[-2,0]$.

2.3. Main Theorem. Before discussing the main Theorem, we need to discuss the normalization of ϕ. Since $\phi_2(x)$ is antisymmetric about $x=1$, we have $\int_{-\infty}^{\infty}\phi_2(x)\,\mathrm{d}x=\int_0^2\phi_2(x)\,\mathrm{d}x=0$. If $\int_{-\infty}^{\infty}\phi_1(x)\,\mathrm{d}x=1$, then

$$\int_{-\infty}^{\infty}\phi(x)\,\mathrm{d}x=\frac{\sqrt{2}}{2}\int_{-\infty}^{\infty}[\phi_1(x)-\phi_2(x)]\,\mathrm{d}x=\frac{\sqrt{2}}{2},$$

which is a correct normalization for the two-direction scaling functions (For normalizing condition for ϕ, see [2]). Hence, our construction of ϕ is correctly normalized.

We have the following main Theorem of this paper from subsections 2.1 and 2.2.

Theorem 2.1. Let $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ be an orthogonal symmetric/antisymmetric multiscaling function of order 2 supported on $[0, 2]$ with nonzero 2×2 recursion coefficient matrices h_0, h_1, h_2. Let $\boldsymbol{\psi} = [\psi_1, \psi_2]^T$ be an orthogonal symmetric/antisymmetric multiwavelet function associated with $\boldsymbol{\phi}$ supported on $[0, 2]$ with nonzero 2×2 recursion coefficient matrices g_0, g_1, g_2. Construct functions ϕ and ψ by

$$\begin{aligned} \phi(x) &= \frac{\sqrt{2}}{2}\left[\phi_1(x) - \phi_2(x)\right], \\ \psi(x) &= \frac{\sqrt{2}}{2}\left[-\psi_1(-x) + \psi_2(-x)\right]. \end{aligned} \tag{2.15}$$

Then (i) ϕ is an orthogonal two-direction scaling function of order 2 supported on $[0, 2]$ such that

$$\phi(x) = \sum_{k=0}^{2} p_k^+ \phi(2x - k) + \sum_{k=2}^{4} p_k^- \phi(k - 2x) \tag{2.16}$$

for some p_k^+ and p_k^-;

(ii) ψ is an orthogonal two-direction wavelet function associated with ϕ supported on $[-2, 0]$ such that

$$\psi(x) = \sum_{k=-4}^{-2} q_k^+ \phi(2x - k) + \sum_{k=-2}^{0} q_k^- \phi(k - 2x) \tag{2.17}$$

for some q_k^+ and q_k^-.

3. Example

In this section we provide an example to illustrate the general theory in sections 1, and 2.

Example 3.1. Chui-Lian's orthogonal symmetric/antisymmetric multiscaling function $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ of order 2 supported on $[0, 2]$ is given in [1] as

$$\begin{aligned} \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} &= \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \begin{bmatrix} \phi_1(2x) \\ \phi_2(2x) \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \\ &+ \begin{bmatrix} a_0 & -b_0 \\ -c_0 & d_0 \end{bmatrix} \begin{bmatrix} \phi_1(2x-2) \\ \phi_2(2x-2) \end{bmatrix}, \end{aligned} \tag{3.1}$$

where

$$a_0 = \frac{1}{2}, a_1 = 1, b_0 = \frac{1}{2}, b_1 = 0 = c_1, c_0 = -\frac{\sqrt{7}}{4}, d_0 = -\frac{\sqrt{7}}{4}, d_1 = \frac{1}{2}. \tag{3.2}$$

Construct a function ϕ by

$$\phi(x) = \frac{\sqrt{2}}{2}\left[\phi_1(x) - \phi_2(x)\right]. \tag{3.3}$$

By applying (2.6) and (2.7), we have

$$\phi(x) = \sum_{k=0}^{2} p_k^+ \phi(2x-k) + \sum_{k=2}^{4} p_k^- \phi(k-2x), \tag{3.4}$$

where

$$p_0^+ = 0, p_1^+ = \frac{3}{4}, p_2^+ = \frac{2-\sqrt{7}}{4}, p_2^- = \frac{2+\sqrt{7}}{4}, p_3^- = \frac{1}{4}, p_4^- = 0. \tag{3.5}$$

It turns out that ϕ is the two-direction scaling function

$$\phi(x) = \frac{3}{4}\phi(2x-1) + \frac{2-\sqrt{7}}{4}\phi(2x-1) + \frac{2+\sqrt{7}}{4}\phi(2-2x) + \frac{1}{4}\phi(3-2x) \tag{3.6}$$

in [5, Example 2].

$\phi(2-x)$, flipping of $\phi(x)$ about $x = 1$, is also a two-direction scaling function of order 2 supported on $[0, 2]$.

Chui-Lian's [1] orthogonal symmetric/antisymmetric multiwavelet function $\boldsymbol{\psi} = [\psi_1, \psi_2]^T$ supported on $[0, 2]$ is given as

$$\begin{aligned}\begin{bmatrix}\psi_1(x)\\ \psi_2(x)\end{bmatrix} &= \begin{bmatrix}a_0' & b_0'\\ c_0' & d_0'\end{bmatrix}\begin{bmatrix}\phi_1(2x)\\ \phi_2(2x)\end{bmatrix} + \begin{bmatrix}a_1' & b_1'\\ c_1' & d_1'\end{bmatrix}\begin{bmatrix}\phi_1(2x-1)\\ \phi_2(2x-1)\end{bmatrix}\\ &\quad + \begin{bmatrix}a_0' & -b_0'\\ -c_0' & d_0'\end{bmatrix}\begin{bmatrix}\phi_1(2x-2)\\ \phi_2(2x-2)\end{bmatrix},\end{aligned} \tag{3.7}$$

where

$$a_0' = -\frac{1}{2}, a_1' = 1, b_0' = -\frac{1}{2}, b_1' = 0 = c_1', c_0' = \frac{1}{4}, d_0' = \frac{1}{4}, d_1' = \frac{\sqrt{7}}{2}. \tag{3.8}$$

Construct a function ψ by

$$\psi(x) = \frac{\sqrt{2}}{2}\left[-\psi_1(-x) + \phi_2(-x)\right]. \tag{3.9}$$

By applying (2.13) and (2.14), we have

$$\psi(x) = \sum_{k=-4}^{-2} q_k^+ \phi(2x-k) + \sum_{k=-2}^{0} q_k^- \phi(k-2x), \tag{3.10}$$

associated with ϕ, where

$$q_{-4}^+ = 0 = q_0^-, q_{-3}^+ = -\frac{2-\sqrt{7}}{4}, q_{-2}^+ = \frac{3}{4}, q_{-2}^- = \frac{1}{4}, q_{-1}^- = -\frac{2+\sqrt{7}}{4}. \tag{3.11}$$

It turns out that ψ is the two-direction wavelet function

$$\psi(x) = \frac{3}{4}\phi(2x+2) - \frac{2-\sqrt{7}}{4}\phi(2x+3) - \frac{2+\sqrt{7}}{4}\phi(-1-2x) + \frac{1}{4}\phi(-2-2x) \tag{3.12}$$

associated with ϕ in [5, Example 2].

$\psi(-2-x)$, flipping of $\psi(x)$ about $x=-1$, is also a two-direction wavelet function associated with $\phi(2-x)$ supported on $[-2,0]$.

For the graphs of ϕ_1, ϕ_2, ψ_1 and ψ_2, see Fig. 3.1. For the graphs of $\phi(x)$, $\phi(2-x)$, $\psi(x)$ and $\psi(-2-x)$, see Fig. 3.2.

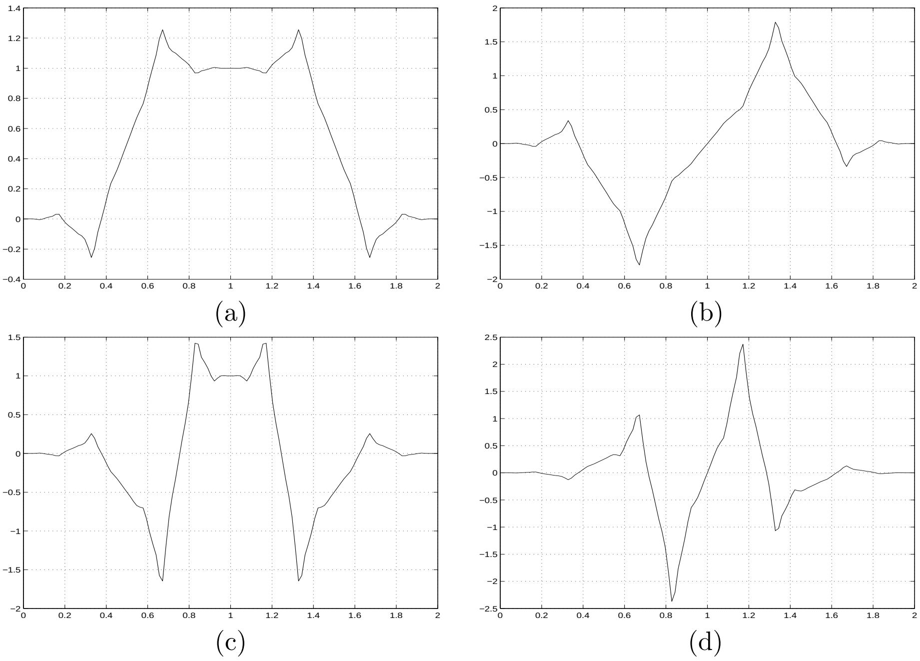

FIGURE 3.1. Chui-Lian's orthogonal symmetric/antisymmetric multiscaling function of order 2 and multiwavelet: (a) ϕ_1. (b) ϕ_2. (c) ψ_1. (d) ψ_2.

□

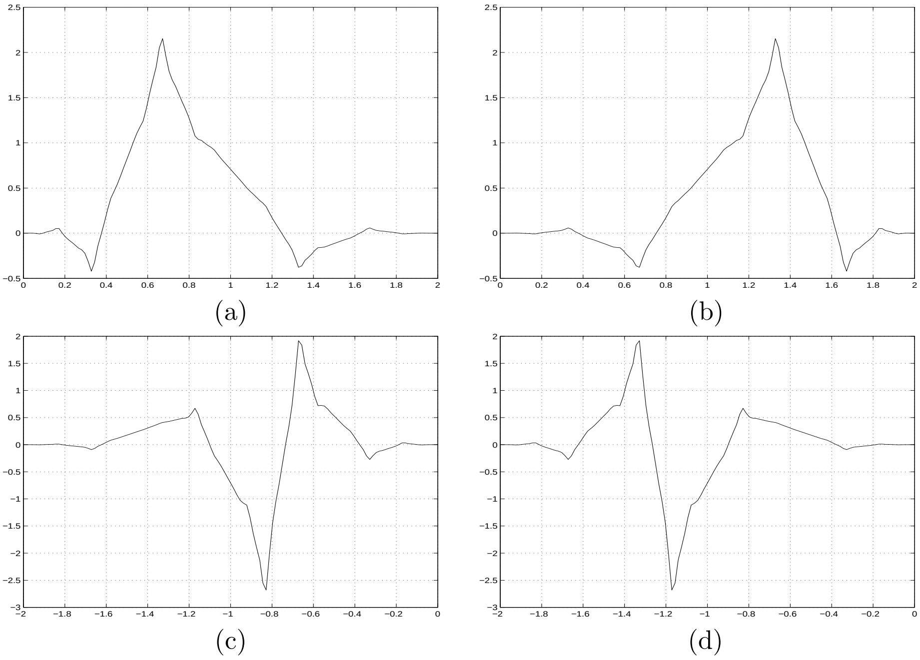

FIGURE 3.2. Orthogonal two-direction scaling functions of order 2 and wavelet functions from CL2: (a) $\phi(x)$. (b) $\phi(2-x)$. (c) $\psi(x)$. (d) $\psi(-2-x)$.

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