J. Appl. Math. & Informatics Vol. **35**(*2017*)*, No. 1 - 2, pp. 165 - 180*
https://doi.org/10.14317/jami.2017.165

A NEW APPROACH FOR NUMERICAL SOLUTION OF LINEAR AND NON-LINEAR SYSTEMS

HALİL ZEYBEK* AND İHSAN TİMUÇİN DOLAPÇI

Abstract. In this study, Taylor matrix algorithm is designed for the approximate solution of linear and non-linear differential equation systems. The algorithm is essentially based on the expansion of the functions in differential equation systems to Taylor series and substituting the matrix forms of these expansions into the given equation systems. Using the Mathematica program, the matrix equations are solved and the unknown Taylor coefficients are found approximately. The presented numerical approach is discussed on samples from various linear and non-linear differential equation systems as well as stiff systems. The computational data are then compared with those of some earlier numerical or exact results. As a result, this comparison demonstrates that the proposed method is accurate and reliable.

AMS Mathematics Subject Classification : 65L04, 35F50, 41A58.
Key words and phrases : Stiff systems, Linear and non-linear differential equation systems, Taylor matrix method.

1. Introduction

A system of differential equations of the first order is considered as

$$A_i\,(x)\,y_i'\,(x) + L_i\,[y_1, y_2, \ldots, y_n] + N_i\,[y_1, y_2, \ldots, y_n] = G_i\,(x)\,, \qquad i = 1, 2, \ldots, n, \quad a \le x \le b \tag{1}$$

with initial conditions

$$\sum_{j=1}^{n} \alpha_{ij} y_j\,(x_0) + \sum_{j=1}^{n}\sum_{k=1}^{n} \beta_{ijk} y_j\,(x_0)\, y_k\,(x_0) = \lambda_i, \quad i = 1, 2, \ldots,\ n, \quad a \le x_0 \le b. \tag{2}$$

Received May 24, 2016. Revised September 10, 2016. Accepted September 19, 2016.
*Corresponding author.

The approximate solution is denoted by

$$y_j\left(x\right)=\sum_{t=0}^{N}\frac{{y_j}^{(t)}\left(c\right)\left(x-c\right)^t}{t!};\quad j=1,2,\ldots,n \tag{3}$$

which is a Taylor polynomial of degree N at $x=c$. ${y_j}^{(t)}\left(c\right)$, $(t=0,1,\ldots,N)$ are the unknown coefficients to be determined. Besides, where

$$L_i\left[y_1,\ y_2,\ldots,\ y_n\right]=\sum_{j=1}^{n}B_{ij}\left(x\right)y_j\left(x\right);\quad i=1,2,\ldots,n \tag{4}$$

$$N_i\left[y_1,\ y_2,\ldots,\ y_n\right]=\sum_{j=1}^{n}\sum_{k=1}^{n}C_{ijk}\left(x\right)y_j\left(x\right)y_k\left(x\right);\quad i=1,2,\ldots,n \tag{5}$$

The systems of differential equation, which are like Equation (1), are often encountered in many branches of physical, chemical and engineering applications. These systems of differential equation were solved with various methods such as He's variational iteration, differential transformation, Adomian's decomposition, power series method in advance. The exact, analytic or numerical solutions of these were obtained [1–4]. The purpose of this paper is to employ Taylor matrix algorithm which is numerical method to Equation (1). Since the beginning of the 1994s, to find the approximate solutions of many different types of equations, Taylor matrix method has been used by Kanwal and Liu, Sezer et al. [5–22]. Recently, stiff systems have been solved by Kome et al., Atay et al. [23–24]. Now, the Taylor matrix algorithm has been applied to the linear and non-linear systems of ordinary differential equations including the stiff systems. Using the computer program, the approximate solutions of these equation systems have been easily obtained.

2. Description of the method

2.1. The transformation of differential equation system to matrix equation. In this section, we consider equation (1) and write the truncated Taylor series expansions of each function in equation (1) at $x=c$ and their matrix forms. We first consider the desired solutions $y_j\left(x\right)$ of equation (1) defined by a truncated Taylor series (3). Then we can write equation (3) in the matrix form

$$\left[y_j\left(x\right)\right]=\mathbf{X}M\mathbf{Y_j}, \tag{6}$$

where

$$\mathbf{X}=\left[\begin{array}{cccc}1 & (x-c) & (x-c)^2\cdots(x-c)^N\end{array}\right]_{1\times(N+1)},$$

$$\mathbf{M}=\begin{bmatrix} \frac{1}{0!} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1!} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{N!} \end{bmatrix}_{(N+1)\times(N+1)}, \quad \mathbf{Y_j}=\begin{bmatrix} {y_j}^{(0)}(c) \\ {y_j}^{(1)}(c) \\ {y_j}^{(2)}(c) \\ \vdots \\ {y_j}^{(N)}(c) \end{bmatrix}_{(N+1)\times 1}.$$

Now we evaluate the function $B_{ij}(x)\,y_j(x)$ of equation (1) and can write it as the truncated Taylor series expansions of degree N at $x=c$ in the form

$$B_{ij}(x)\,y_j(x)=\sum_{t=0}^{N}\frac{\left[B_{ij}(x)\,y_j(x)\right]_{x=c}^{(t)}(x-c)^t}{t!}. \tag{7}$$

By the Leibnitz's rule we appraise

$$\left[B_{ij}(x)\,y_j(x)\right]_{x=c}^{(t)}=\sum_{m=0}^{t}\binom{t}{m}{B_{ij}}^{(t-m)}(c)\,{y_j}^{(m)}(c)$$

and substitute in expression (7). So expression (7) becomes

$$B_{ij}(x)\,y_j(x)=\sum_{t=0}^{N}\sum_{m=0}^{t}\frac{\binom{t}{m}B_{ij}^{(t-m)}(c)\,y_j^{(m)}(c)\,(x-c)^t}{t!}$$

and its matrix form as follows

$$\left[B_{ij}(x)\,y_j(x)\right]=\mathbf{X}\mathbf{B_{i}}_j\mathbf{Y_j}, \tag{8}$$

where

$$\mathbf{B_i}_j=\begin{bmatrix} \frac{{B_{ij}}^{(0)}(c)}{0!0!} & 0 & 0 & \dots & 0 \\ \frac{{B_{ij}}^{(1)}(c)}{1!0!} & \frac{{B_{ij}}^{(0)}(c)}{0!1!} & 0 & \dots & 0 \\ \frac{{B_{ij}}^{(2)}(c)}{2!0!} & \frac{{B_{ij}}^{(1)}(c)}{1!1!} & \frac{{B_{ij}}^{(0)}(c)}{0!2!} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{{B_{ij}}^{(N)}(c)}{N!0!} & \frac{{B_{ij}}^{(N-1)}(c)}{(N-1)!1!} & \frac{{B_{ij}}^{(N-2)}(c)}{(N-2)!2!} & \dots & \frac{{B_{ij}}^{(0)}(c)}{0!N!} \end{bmatrix}_{(N+1)\times(N+1)}.$$

In the same way we obtain

$$A_i(x)\,y_i'(x)=\sum_{t=0}^{N}\sum_{m=0}^{t}\frac{\binom{t}{m}A_i^{(t-m)}(c)\,y_i^{(m+1)}(c)\,(x-c)^t}{t!}, \tag{9}$$

$$C_{ijk}(x)\,Y_{jk}(x)=\sum_{t=0}^{N}\sum_{m=0}^{t}\frac{\binom{t}{m}C_{ijk}^{(t-m)}(c)\,Y_{jk}^{(m)}(c)\,(x-c)^t}{t!}, \tag{10}$$

$$G_i(x)=\sum_{t=0}^{N}\frac{G_i^{(t)}(c)\,(x-c)^t}{t!} \tag{11}$$

so that

$$Y_{jk}(x) = y_j(x)\, y_k(x), \quad Y_{jk}^{(m)}(c) = \sum_{s=0}^{m} \binom{m}{s} y_j^{(m-s)}(c)\, y_k^{(s)}(c).$$

Their matrix forms, in order as follows

$$[A_i(x)\, y_i'(x)] = \mathbf{X}\mathbf{A_i}\mathbf{Y_i}, \tag{12}$$

$$[C_{ijk}(x)\, y_j(x)\, y_k(x)] = [C_{ijk}(x)\, Y_{jk}(x)] = \mathbf{X}\mathbf{C_{i}}_{jk}\mathbf{Y_{j}}_{k}, \tag{13}$$

$$[G_i(x)] = \mathbf{X}M\mathbf{G_i}, \tag{14}$$

where

$$\mathbf{A_i} = \begin{bmatrix} 0 & \frac{A_i^{(0)}(c)}{0!0!} & 0 & \dots & 0 \\ 0 & \frac{A_i^{(1)}(c)}{1!0!} & \frac{A_i^{(0)}(c)}{0!1!} & \dots & 0 \\ 0 & \frac{A_i^{(2)}(c)}{2!0!} & \frac{A_i^{(1)}(c)}{1!1!} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{A_i^{(N)}(c)}{N!0!} & \frac{A_i^{(N-1)}(c)}{(N-1)!1!} & \dots & \frac{A_i^{(1)}(c)}{1!(N-1)!} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\mathbf{C_{i}}_{jk} = \begin{bmatrix} \frac{C_{ijk}^{(0)}(c)}{0!0!} & 0 & 0 & \dots & 0 \\ \frac{C_{ijk}^{(1)}(c)}{1!0!} & \frac{C_{ijk}^{(0)}(c)}{0!1!} & 0 & \dots & 0 \\ \frac{C_{ijk}^{(2)}(c)}{2!0!} & \frac{C_{ijk}^{(1)}(c)}{1!1!} & \frac{C_{ijk}^{(0)}(c)}{0!2!} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{C_{ijk}^{(N)}(c)}{N!0!} & \frac{C_{ijk}^{(N-1)}(c)}{(N-1)!1!} & \frac{C_{ijk}^{(N-2)}(c)}{(N-2)!2!} & \dots & \frac{C_{ijk}^{(0)}(c)}{0!N!} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\mathbf{G_i} = \begin{bmatrix} G_i^{(0)}(c) \\ G_i^{(1)}(c) \\ G_i^{(2)}(c) \\ \vdots \\ G_i^{(N)}(c) \end{bmatrix}_{(N+1)\times 1}, \quad \mathbf{Y_{j}}_{k} = \begin{bmatrix} Y_{jk}^{(0)}(c) \\ Y_{jk}^{(1)}(c) \\ Y_{jk}^{(2)}(c) \\ \vdots \\ Y_{jk}^{(N)}(c) \end{bmatrix}_{(N+1)\times 1}.$$

Substituting the matrix forms (8), (12), (13) and (14) corresponding to the functions $B_{ij}(x)\, y_j(x)$, $A_i(x)\, y_i'(x)$, $C_{ijk}(x)\, y_j(x)\, y_k(x)$ and $G_i(x)$, into Eq. (1), and then simplifying the resulting equation system, we get the matrix equation system

$$\begin{array}{l} \mathbf{A_1Y_1}+\mathbf{B}_{11}\mathbf{Y_1}+\dots+\mathbf{B_{1n}Y_n}+\mathbf{C}_{111}\mathbf{Y}_{11}+\mathbf{C}_{112}\mathbf{Y}_{12}+\dots+\mathbf{C_{1n}}_n\mathbf{Y_n}_n=\mathbf{MG_1} \\ \mathbf{A_2Y_2}+\mathbf{B_2}_1\mathbf{Y_1}+\dots+\mathbf{B_{2n}Y_n}+\mathbf{C_2}_{11}\mathbf{Y}_{11}+\mathbf{C_2}_{12}\mathbf{Y_1}_{??}+\dots+\mathbf{C_{2n}}_n\mathbf{Y_n}_n=\mathbf{MG_2} \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ \mathbf{A_nY_n}+\mathbf{B_{n1}Y_1}+\dots+\mathbf{B_n}_n\mathbf{Y_n}+\mathbf{C_{n11}Y}_{11}+\mathbf{C_{n12}Y}_{12}+\dots+\mathbf{C_n}_{nn}\mathbf{Y_n}_n=\mathbf{MG_n} \end{array} \tag{15}$$

We can write Eq. (15) as simpler and in this way we get the matrix equation

$$\mathbf{W}Y+\mathbf{C}\overline{\mathbf{Y}}=\mathbf{G} \tag{16}$$

where

$$\mathbf{W}=\begin{bmatrix} A_1+B_{11} & B_{12} & B_{13} & \dots & B_{1n} \\ B_{21} & A_2+B_{22} & B_{23} & \dots & B_{2n} \\ B_{31} & B_{32} & A_3+B_{33} & \dots & B_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ B_{n1} & B_{n2} & B_{n3} & \dots & A_n+B_{nn} \end{bmatrix}_{n(N+1)\times n(N+1)},$$

$$\mathbf{C}=\begin{bmatrix} C_{111} & \dots & C_{11n} & C_{121} & \dots & C_{12n} & \dots & C_{1n1} & \dots & C_{1nn} \\ C_{211} & \dots & C_{21n} & C_{221} & \dots & C_{22n} & \dots & C_{2n1} & \dots & C_{2nn} \\ C_{311} & \dots & C_{31n} & C_{321} & \dots & C_{32n} & \dots & C_{3n1} & \dots & C_{3nn} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ C_{n11} & \dots & C_{n1n} & C_{n21} & \dots & C_{n2n} & \dots & C_{nn1} & \dots & C_{nnn} \end{bmatrix}_{n(N+1)\times n^2(N+1)},$$

$$\overline{\mathbf{Y}}=\begin{bmatrix} Y_{11} & \cdots & Y_{1n} & Y_{21} & \cdots & Y_{2n} & \cdots & Y_{n1} & \cdots & Y_{nn} \end{bmatrix}^T_{n^2(N+1)\times 1},$$

$$\mathbf{Y}=\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix}_{n(N+1)\times 1}, \quad \mathbf{G}=\begin{bmatrix} MG_1 \\ MG_2 \\ \vdots \\ MG_n \end{bmatrix}_{n(N+1)\times 1}.$$

The matrix Eq. (16) is a fundamental relation for system of differential equations of the first order Eq. (1).

2.2. The transformation of initial conditions to matrix equation. Now, we consider the initial conditions (2) and write matrix forms of each function in equation (2). Using the relation (6), we can write the matrix forms of the functions at the points x_0 as follows

$$[y_j(x_0)]=\mathbf{X_0MY_j} \tag{17}$$

where

$$\mathbf{X_0}=\begin{bmatrix} 1 & (x_0-c) & (x_0-c)^2\cdots(x_0-c)^N \end{bmatrix}_{1\times(N+1)}.$$

By taking $C_{ijk}(x)=1$ and $x=x_0$ in Eq. (13), we obtain

$$[y_j(x_0)\,y_k(x_0)]=[Y_{jk}(x_0)]=\mathbf{X_0MY_{j}}_k. \tag{18}$$

Substituting the matrix forms (17) and (18) into Eq. (2), we get the matrix equation system

$$\begin{array}{l} \alpha_{11}\mathbf{X_0MY_1}+\cdots+\alpha_{1n}\mathbf{X_0MY_n}+\beta_{111}\mathbf{X_0MY_{11}}+\cdots+\beta_{1nn}\mathbf{X_0MY_n}_n=\lambda_1 \\ \alpha_{21}\mathbf{X_0MY_1}+\cdots+\alpha_{2n}\mathbf{X_0MY_n}+\beta_{211}\mathbf{X_0MY_{11}}+\cdots+\beta_{2nn}\mathbf{X_0MY_n}_n=\lambda_2 \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \alpha_{n1}\mathbf{X_0MY_1}+\cdots+\alpha_{nn}\mathbf{X_0MY_n}+\beta_{n11}\mathbf{X_0MY_{11}}+\cdots+\beta_{nnn}\mathbf{X_0MY_n}_n=\lambda_n \end{array} \tag{19}$$

We can write Eq. (19) as simpler and in this way we obtain the matrix equation

$$\mathbf{U}Y+\mathbf{V}\overline{\mathbf{Y}}=\lambda \tag{20}$$

where

$$\mathbf{U}=\begin{bmatrix} \alpha_{11}\mathbf{X_0M} & \alpha_{12}\mathbf{X_0M} & \alpha_{13}\mathbf{X_0M} & \dots & \alpha_{1n}\mathbf{X_0M} \\ \alpha_{21}\mathbf{X_0M} & \alpha_{22}\mathbf{X_0M} & \alpha_{23}\mathbf{X_0M} & \dots & \alpha_{2n}\mathbf{X_0M} \\ \alpha_{31}\mathbf{X_0M} & \alpha_{32}\mathbf{X_0M} & \alpha_{33}\mathbf{X_0M} & \dots & \alpha_{3n}\mathbf{X_0M} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1}\mathbf{X_0M} & \alpha_{n2}\mathbf{X_0M} & \alpha_{n3}\mathbf{X_0M} & \dots & \alpha_{nn}\mathbf{X_0M} \end{bmatrix}_{n\times n(N+1)}, \quad \lambda=\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}_{n\times 1},$$

$$\mathbf{V}=\begin{bmatrix} \beta_{111}\mathbf{X_0M} & \dots & \beta_{11n}\mathbf{X_0M} & \dots & \beta_{1n1}\mathbf{X_0M} & \dots & \beta_{1nn}\mathbf{X_0M} \\ \beta_{211}\mathbf{X_0M} & \dots & \beta_{21n}\mathbf{X_0M} & \dots & \beta_{2n1}\mathbf{X_0M} & \dots & \beta_{2nn}\mathbf{X_0M} \\ \beta_{311}\mathbf{X_0M} & \dots & \beta_{31n}\mathbf{X_0M} & \dots & \beta_{3n1}\mathbf{X_0M} & \dots & \beta_{3nn}\mathbf{X_0M} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \beta_{n11}\mathbf{X_0M} & \dots & \beta_{n1n}\mathbf{X_0M} & \dots & \beta_{nn1}\mathbf{X_0M} & \dots & \beta_{nnn}\mathbf{X_0M} \end{bmatrix}_{n\times n^2(N+1)}.$$

3. Method of solution

Firstly, we take the fundamental matrix relation Eq. (16) for system of differential equations of the first order Eq. (1). The augmented matrix form of Eq. (16) can be written as follows

$$[\mathbf{W};\mathbf{C};\mathbf{G}] \tag{21}$$

where

$$\mathbf{W}=[w_{i,j}]=\begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{(N+1)-1,1} & w_{(N+1)-1,2} & \cdots & w_{(N+1)-1,n(N+1)} \\ w_{(N+1)+1,1} & w_{(N+1)+1,2} & \cdots & w_{(N+1)+1,n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{2(N+1)-1,1} & w_{2(N+1)-1,2} & \cdots & w_{2(N+1)-1,n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{(n-1)(N+1)+1,1} & w_{(n-1)(N+1)+1,2} & \cdots & w_{(n-1)(N+1)+1,n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{n(N+1)-1,1} & w_{n(N+1)-1,2} & \cdots & w_{n(N+1)-1,n(N+1)} \\ w_{(N+1),1} & w_{(N+1),2} & \cdots & w_{(N+1),n(N+1)} \\ w_{2(N+1),1} & w_{2(N+1),2} & \cdots & w_{2(N+1),n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{n(N+1),1} & w_{n(N+1),2} & \cdots & w_{n(N+1),n(N+1)} \end{bmatrix}_{n(N+1)\times n(N+1)},$$

$$\mathbf{C}=[c_{i,j}]=\begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{(N+1)-1,1} & c_{(N+1)-1,2} & \cdots & c_{(N+1)-1,n^2(N+1)} \\ c_{(N+1)+1,1} & c_{(N+1)+1,2} & \cdots & c_{(N+1)+1,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{2(N+1)-1,1} & c_{2(N+1)-1,2} & \cdots & c_{2(N+1)-1,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{(n-1)(N+1)+1,1} & c_{(n-1)(N+1)+1,2} & \cdots & c_{(n-1)(N+1)+1,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{n(N+1)-1,1} & c_{n(N+1)-1,2} & \cdots & c_{n(N+1)-1,n^2(N+1)} \\ c_{(N+1),1} & c_{(N+1),2} & \cdots & c_{(N+1),n^2(N+1)} \\ c_{2(N+1),1} & c_{2(N+1),2} & \cdots & c_{2(N+1),n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{n(N+1),1} & c_{n(N+1),2} & \cdots & c_{n(N+1),n^2(N+1)} \end{bmatrix}_{n(N+1)\times n^2(N+1)},$$

$$\mathbf{G}=\left[\frac{G_1^{(0)}(c)}{0!}\cdots\frac{G_1^{(N-1)}(c)}{(N-1)!}\cdots\frac{G_n^{(0)}(c)}{0!}\cdots\frac{G_n^{(N-1)}(c)}{(N-1)!}\,\frac{G_1^{(N)}(c)}{N!}\cdots\frac{G_n^{(N)}(c)}{N!}\right]^T_{n(N+1)\times 1}.$$

In the similar way, we consider the matrix equation (20) of the initial conditions (2). The augmented matrix form of Eq. (20) can be written as follows

$$[\mathbf{U};\mathbf{V};\lambda] \tag{22}$$

where

$$\mathbf{U}=[u_{i,j}]=\begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n(N+1)} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n(N+1)} \\ \vdots & \vdots & & \vdots \\ u_{n,1} & u_{n,2} & \cdots & u_{n,n(N+1)} \end{bmatrix}_{n\times n(N+1)}, \quad \lambda=\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}_{n\times 1},$$

$$\mathbf{V}=[v_{i,j}]=\begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,n^2(N+1)} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ v_{n,1} & v_{n,2} & & v_{n,n^2(N+1)} \end{bmatrix}_{n\times n^2(N+1)}.$$

Then, to find the unknown Taylor coefficients for the approximate solution of system of differential equations of the first order Eq. (1) under the initial conditions (2), we replace the last n row matrix (21) with the augmented matrix (22). Thus we obtain new augmented matrix

$$[\mathbf{W}^*;\mathbf{C}^*;\mathbf{G}^*] \tag{23}$$

so that

$$\mathbf{W}^* = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{(N+1)-1,1} & w_{(N+1)-1,2} & \cdots & w_{(N+1)-1,n(N+1)} \\ w_{(N+1)+1,1} & w_{(N+1)+1,2} & \cdots & w_{(N+1)+1,n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{2(N+1)-1,1} & w_{2(N+1)-1,2} & \cdots & w_{2(N+1)-1,n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{(n-1)(N+1)+1,1} & w_{(n-1)(N+1)+1,2} & \cdots & w_{(n-1)(N+1)+1,n(N+1)} \\ \vdots & \vdots & & \vdots \\ w_{n(N+1)-1,1} & w_{n(N+1)-1,2} & \cdots & w_{n(N+1)-1,n(N+1)} \\ u_{1,1} & u_{1,2} & \cdots & u_{1,n(N+1)} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n(N+1)} \\ \vdots & \vdots & & \vdots \\ u_{n,1} & u_{n,2} & \cdots & u_{n,n(N+1)} \end{bmatrix}_{n(N+1)\times n(N+1)},$$

$$\mathbf{C}^* = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{(N+1)-1,1} & c_{(N+1)-1,2} & \cdots & c_{(N+1)-1,n^2(N+1)} \\ c_{(N+1)+1,1} & c_{(N+1)+1,2} & \cdots & c_{(N+1)+1,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{2(N+1)-1,1} & c_{2(N+1)-1,2} & \cdots & c_{2(N+1)-1,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{(n-1)(N+1)+1,1} & c_{(n-1)(N+1)+1,2} & \cdots & c_{(n-1)(N+1)+1,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ c_{n(N+1)-1,1} & c_{n(N+1)-1,2} & \cdots & c_{n(N+1)-1,n^2(N+1)} \\ v_{1,1} & v_{1,2} & \cdots & v_{1,n^2(N+1)} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,n^2(N+1)} \\ \vdots & \vdots & & \vdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,n^2(N+1)} \end{bmatrix}_{n(N+1)\times n^2(N+1)},$$

$$\mathbf{G}^* = \begin{bmatrix} \frac{G_1^{(0)}(c)}{0!} \cdots \frac{G_1^{(N-1)}(c)}{(N-1)!} \cdots \frac{G_n^{(0)}(c)}{0!} \cdots \frac{G_n^{(N-1)}(c)}{(N-1)!} \lambda_1 \cdots \lambda_n \end{bmatrix}^T_{n(N+1)\times 1}$$

or the corresponding matrix equation

$$\mathbf{W}^*\mathbf{Y}+\mathbf{C}^*\overline{\mathbf{Y}}=\mathbf{G}^*. \tag{24}$$

From the nonlinear system, the unknown Taylor coefficients ${y_j}^{(t)}(c)$, $(t = 0, 1, \ldots, N)$ are determined and substituting in (3), and thus we obtain the Taylor polynomial solutions

$$y_j(x) = \sum_{t=0}^{N} \frac{{y_j}^{(t)}(c)(x-c)^t}{t!}; \quad j = 1, 2, \ldots, n.$$

We can easily check the accuracy of this solution obtained as follows. Because the truncated Taylor series (3) is an approximate solution of Eq. (1), when the solutions $y_j(x)$ $(j = 1, 2, \ldots, n)$ and their derivatives are substituted in Eq.

(1), the resulting equation must be satisfied approximately; which is, for every $i = 1, 2, \ldots, n$ and $x = x_r \in [a, b]$

$$E_i(x_r) = |A_i(x_r)\, y_i'(x_r) + L_i[y_1,\ y_2, \ldots,\ y_n] + N_i[y_1,\ y_2, \ldots,\ y_n] - G_i(x_r)| \cong 0$$

or

$E_i(x_r) \le 10^{-k_{i,r}}$ ($k_{i,r}$ is any positive integer).

If $\max\left(10^{-k_{i,r}}\right) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E_i(x_r)$ at each of the points becomes smaller than the presciribed 10^{-k}.

4. Numerical examples

In this section, we will solve numerical examples from various linear and non-linear differential equation systems and thus the method will be demonstrated.

Example 1. In the first case, we choose the following stiff system of differential equations

$$\begin{aligned} &\frac{dy_1(x)}{dx} + 20y_1(x) + 0.25y_2(x) + 19.75y_3(x) = 0 \\ &\frac{dy_2(x)}{dx} - 20y_1(x) + 20.25y_2(x) - 0.25y_3(x) = 0 \\ &\frac{dy_3(x)}{dx} - 20y_1(x) + 19.75y_2(x) + 0.25y_3(x) = 0 \end{aligned} \tag{25}$$

with initial conditions

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = -1. \tag{26}$$

The analytic solution of the problem is

$$\begin{aligned} y_1(x) &= \frac{1}{2}\left[e^{-\frac{1}{2}x} + e^{-20x}\left(\cos(20x) + \sin(20x)\ \right)\right], \\ y_2(x) &= \frac{1}{2}\left[e^{-\frac{1}{2}x} - e^{-20x}\left(\cos(20x) - \sin(20x)\ \right)\right], \\ y_3(x) &= -\frac{1}{2}\left[e^{-\frac{1}{2}x} + e^{-20x}\left(\cos(20x) - \sin(20x)\ \right)\right]. \end{aligned} \tag{27}$$

Taking by $N = 3$, the approximate solution $y_j(x)$ is written by

$$y_j(x) = \sum_{t=0}^{3} \frac{y_j{}^{(t)}(0)\, x^t}{t!}; \quad j = 1, 2, 3 \tag{28}$$

When the presented algorithm is applied to system, the unknown parameters are found as

$$\begin{aligned} &y_1{}^{(0)}(0) = 1, \quad y_1{}^{(1)}(0) = -\frac{1}{4}, \quad y_1{}^{(2)}(0) = -\frac{3199}{8}, \quad y_1{}^{(3)}(0) = \frac{255999}{16}, \\ &y_2{}^{(0)}(0) = 0, \quad y_2{}^{(1)}(0) = \frac{79}{4}, \quad y_2{}^{(2)}(0) = -\frac{3199}{8}, \quad y_2{}^{(3)}(0) = -\frac{1}{16}, \\ &y_3{}^{(0)}(0) = -1, \quad y_3{}^{(1)}(0) = \frac{81}{4}, \quad y_3{}^{(2)}(0) = -\frac{3201}{8}, \quad y_3{}^{(3)}(0) = \frac{1}{16}. \end{aligned}$$

Substituting these coefficients in (28), the approximate solutions of this problem

$$y_1(x) \cong 1-\frac{x}{4}-\frac{3199x^2}{16}+\frac{85333x^3}{32},$$
$$y_2(x) \cong \frac{79x}{4}-\frac{3199x^2}{16}-\frac{x^3}{96},$$
$$y_3(x) \cong -1+\frac{81x}{4}-\frac{3201x^2}{16}+\frac{x^3}{96}$$

The numerical results are given in Table 1. It is observed from the Table 1 that the obtained results are consistent with the earlier numerical and exact solutions. Table 2 lists the comparison of our absolute error with the ones obtained by the power series and differential transform method. According to the this table, our absolute errors are less than 0.6×10^{-9} in all computer run.

Table 1. Numeric results of Example 1

	x ↓	Ours-TMM $N=7$	PSM [10] $N=6$	DTM [1] $N=16$	Exact
y_1	0.000	1.000000000000	1.000000000000	1.000000000000	1.000000000000
	0.002	0.998721369939	0.998721369939	0.998721369939	0.998721369939
	0.004	0.995968254090	0.995968254123	0.995968254123	0.995968254090
	0.006	0.991860983771	0.991860984340	0.991860984349	0.991860983780
	0.008	0.986514803617	0.986514807878	0.986514807963	0.986514803703
	0.010	0.980039908167	0.980039928485	0.980039928992	0.980039908675
y_2	0.000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
	0.002	0.038700459859	0.038700458192	0.038700459859	0.038700459859
	0.004	0.075804304896	0.075804278230	0.075804304896	0.075804304896
	0.006	0.111318714900	0.111318579905	0.111318714892	0.111318714892
	0.008	0.145255205958	0.145254779313	0.145255205876	0.145255205876
	0.010	0.177629261818	0.177628220215	0.177629261332	0.177629261332
y_3	0.000	-1.000000000000	-1.000000000000	-1.000000000000	-1.000000000000
	0.002	-0.960300039973	-0.960300039973	-0.960300039973	-0.960300039973
	0.004	-0.922197693770	-0.922197693770	-0.922197693770	-0.922197693770
	0.006	-0.885685780602	-0.885685780602	-0.885685780611	-0.885685780611
	0.008	-0.850752783385	-0.850752783385	-0.850752783467	-0.850752783467
	0.010	-0.817383217374	-0.817383217376	-0.817383217859	-0.817383217859

Example 2. As a second example, we consider the following non-linear system:

$$\begin{aligned} \frac{dy_1(x)}{dx} + y_1(x) &= 0 \\ \frac{dy_2(x)}{dx} - y_1(x) + y_2^2(x) &= 0 \\ \frac{dy_3(x)}{dx} - y_2^2(x) &= 0 \end{aligned} \tag{29}$$

with initial conditions

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0. \tag{30}$$

TABLE 2. Comprasion of the absolute error for the Example 1

x ↓	y_1 TMM Ours	y_1 RKM [1]	y_1 DTM [1]	y_2 TMM Ours	y_2 RKM [1]	y_2 DTM [1]	y_3 TMM Ours	y_3 RKM [1]	y_3 DTM [1]
0.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.002	0.0	0.1E-10	0.2E-12	0.0	0.1E-10	0.3E-16	0.0	0.1E-10	0.0
0.004	0.0	0.6E-10	0.3E-10	0.0	0.2E-10	0.5E-16	0.0	0.2E-10	0.2E-15
0.006	0.9E-11	0.6E-9	0.5E-9	0.8E-11	0.4E-10	0.0	0.9E-11	0.4E-10	0.0
0.008	0.8E-10	0.4E-8	0.4E-8	0.8E-10	0.5E-10	0.0	0.8E-10	0.5E-10	0.0
0.010	0.5E-9	0.2E-7	0.2E-07	0.4E-9	0.7E-10	0.1E-15	0.4E-9	0.7E-10	0.2E-15

For $N = 3$, by applying Taylor matrix algorithm, we obtain the unknown Taylor coefficients as

$$y_1{}^{(0)}(0) = 1, \quad y_1{}^{(1)}(0) = -1, \quad y_1{}^{(2)}(0) = 1, \quad y_1{}^{(3)}(0) = -1,$$
$$y_2{}^{(0)}(0) = 0, \quad y_2{}^{(1)}(0) = 1, \quad y_2{}^{(2)}(0) = -1, \quad y_2{}^{(3)}(0) = -1,$$
$$y_3{}^{(0)}(0) = 0, \quad y_3{}^{(1)}(0) = 0, \quad y_3{}^{(2)}(0) = 0, \quad y_3{}^{(3)}(0) = 2.$$

And thus, for $N = 3$, the approximate solutions of this problem

$$y_1(x) \cong 1 - x + \frac{x^2}{2} - \frac{x^3}{6}, \quad y_2(x) \cong x - \frac{x^2}{2} - \frac{x^3}{6}, \quad y_3(x) \cong \frac{x^3}{3}.$$

The calculated values are presented in Table 3. In this table, the obtained numerical results are consistent with the earlier study.

TABLE 3. Numeric results of Example 2

	x ↓	Ours-TMM $N = 7$	PSM [10] $N = 5$	ADM [12] $N = 4$	DTM [1] $N = 7$
y_1	0.00	1.000000000000	1.000000000000	1.000000000000	1.000000000000
	0.02	0.980198673306	0.980198673306	1.019798673333	0.980198673306
	0.04	0.960789439152	0.960789439146	1.039189440000	0.960789439152
	0.06	0.941764533584	0.941764533520	1.058164540000	0.941764533584
	0.08	0.923116346386	0.923116346026	1.076716373333	0.923116346386
	0.10	0.904837418035	0.904837416666	1.094837500000	0.904837418035
y_2	0.00	0.000000000000	0.000000000000	0.000000000000	0.000000000000
	0.02	0.019798700073	0.019798700106	0.019798700000	0.019798700073
	0.04	0.039189868825	0.039189870080	0.039189866666	0.039189868825
	0.06	0.058166714891	0.058166725919	0.058166699999	0.058166714891
	0.08	0.076723256464	0.076723309226	0.076723200000	0.076723256464
	0.10	0.094854319940	0.094854500000	0.094854166666	0.094854319940
y_3	0.00	0.000000000000	0.000000000000	0.000000000000	0.000000000000
	0.02	0.000002626619	0.000002626613	0.000002626666	0.000002626619
	0.04	0.000020692021	0.000020691626	0.000020693333	0.000020692021
	0.06	0.000068751523	0.000068747039	0.000068759999	0.000068751523
	0.08	0.000160397148	0.000160372053	0.000160426666	0.000160397148
	0.10	0.000308262023	0.000308166666	0.000308333333	0.000308262023

Example 3. Thirdly, we discuss the non-linear stiff system:

$$\begin{aligned} \frac{dy_1(x)}{dx} + \kappa_1 y_1(x) - \kappa_2 y_2(x) y_3(x) = 0 \\ \frac{dy_2(x)}{dx} - \kappa_3 y_1(x) - \kappa_4 y_2(x) y_3(x) + \kappa_5 y_2^2(x) = 0 \\ \frac{dy_3(x)}{dx} - \kappa_6 y_2^2(x) = 0. \end{aligned} \tag{31}$$

The initial conditions are chosen by

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0. \tag{32}$$

Also, the constant parameters are given by

$$\kappa_1 = 0.04, \quad \kappa_2 = 0.01, \quad \kappa_3 = 400, \quad \kappa_4 = 100, \quad \kappa_5 = 3000, \quad \kappa_6 = 30.$$

For $N = 3$, following the solution procedure, the unknown Taylor coefficients are determined as

$$y_1{}^{(0)}(0) = 1, \quad y_1{}^{(1)}(0) = -\kappa_1, \quad y_1{}^{(2)}(0) = \kappa_1^2, \quad y_1{}^{(3)}(0) = -\kappa_1^3,$$

$$y_2{}^{(0)}(0) = 0, \quad y_2{}^{(1)}(0) = \kappa_3, \quad y_2{}^{(2)}(0) = -\kappa_1\kappa_3, \quad y_2{}^{(3)}(0) = \kappa_3(\kappa_1^2 - 2\kappa_3\kappa_5),$$

$$y_3{}^{(0)}(0) = 0, \quad y_3{}^{(1)}(0) = 0, \quad y_3{}^{(2)}(0) = 0, \quad y_3{}^{(3)}(0) = 2\kappa_3^2\kappa_6.$$

The numerical solution is obtained as

$$y_1(x) \cong 1 - \kappa_1 x + \kappa_1^2 \frac{x^2}{2} - \kappa_1^3 \frac{x^3}{6}, \quad y_2(x) \cong \kappa_3 x - \kappa_1\kappa_3 \frac{x^2}{2} + \kappa_1^2\kappa_3 \frac{x^3}{6} - \kappa_3^2\kappa_5 \frac{x^3}{3},$$

$$y_3(x) \cong \kappa_3^2\kappa_6 \frac{x^3}{3}.$$

The numerical results are reported in Table 4. It is observed from the table that the calculated data are close to results in Adomian's decomposition and differential transformation method.

Example 4. Let us the following non-linear system of ordinary differential equations:

$$\begin{aligned} \frac{dy_1(x)}{dx} - 2e^{4x} y_4^2(x) = 0 \\ \frac{dy_2(x)}{dx} - y_1(x) + y_3(x) = \cos x - e^{2x} \\ \frac{dy_3(x)}{dx} - y_2(x) + y_4(x) = e^{-x} - \sin x \\ \frac{dy_4(x)}{dx} + e^{-5x} y_1^2(x) = 0 \end{aligned} \tag{33}$$

with initial conditions

$$y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 0, \quad y_4(0) = 1 \tag{34}$$

The analytic solution of the problem is

$$y_1(x) = e^{2x}, \quad y_2(x) = \sin x + \cos x, \quad y_3(x) = \sin x, \quad y_4(x) = e^{-x} \tag{35}$$

Table 4. Numeric results of Example 3

	x	Ours-TMM		ADM [12]	DTM [1]
	↓	$N=5$	$N=7$	$N=3$	$N=6$
y_1	0.0000	1.000000000000	1.000000000000	1.000000000000	1.000000000000
	0.0002	0.999992000032	0.999992000032	0.999992000032	0.999992000032
	0.0004	0.999984000141	0.999984000139	0.999984000127	0.999984000141
	0.0006	0.999976000387	0.999976000365	0.999976000287	0.999976000387
	0.0008	0.999968000931	0.999968000762	0.999968000511	0.999968000931
	0.0010	0.999960002079	0.999960001275	0.999960000799	0.999960002079
y_2	0.0000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
	0.0002	0.078744267776	0.078743789936	0.079359680000	0.079999687680
	0.0004	0.150545405957	0.150484255475	0.154878720006	0.159998842880
	0.0006	0.211410705424	0.210365966764	0.222717120023	0.239997742080
	0.0008	0.263246864409	0.255420441642	0.279034880054	0.319996846080
	0.0010	0.316809600017	0.279491108218	0.319992000106	0.399996800000
y_3	0.0000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
	0.0002	0.000012554163	0.000012558940	0.000012799999	0.000012799920
	0.0004	0.000095145824	0.000095145824	0.000102399999	0.000102398674
	0.0006	0.000285874099	0.000296319233	0.000345599999	0.000345593042
	0.0008	0.000567522099	0.000645769452	0.000819199999	0.000819177240
	0.0010	0.000831952000	0.001205056457	0.001599999999	0.001599942559

For $N = 3$, by applying the presented method, we can get the unknown Taylor coefficients

$$y_1{}^{(0)}(0) = 1,\quad y_1{}^{(1)}(0) = 2,\quad y_1{}^{(2)}(0) = 4,\quad y_1{}^{(3)}(0) = 8,$$
$$y_2{}^{(0)}(0) = 1,\quad y_2{}^{(1)}(0) = 1,\quad y_2{}^{(2)}(0) = -1,\quad y_2{}^{(3)}(0) = -1,$$
$$y_3{}^{(0)}(0) = 0,\quad y_3{}^{(1)}(0) = 1,\quad y_3{}^{(2)}(0) = 0,\quad y_3{}^{(3)}(0) = -1,$$
$$y_4{}^{(0)}(0) = 1,\quad y_4{}^{(1)}(0) = -1,\quad y_4{}^{(2)}(0) = 1,\quad y_4{}^{(3)}(0) = -1.$$

And thus, For $N = 3$, the approximate solutions of this problem

$$y_1(x) \cong 1+2x+2x^2+\frac{4x^3}{3},\quad y_2(x) \cong 1+x-\frac{x^2}{2}-\frac{x^3}{6},\quad y_3(x) \cong x-\frac{x^3}{6},$$
$$y_4(x) \cong 1-x+\frac{x^2}{2}-\frac{x^3}{6}$$

Numeric results of Example 4 are shown in Table 5. As can be seen in the Table 5 that our numerical values are close to exact values and the absolute errors are adequately small, as expected.

5. Conclusions

In this study, we have constructed the numerical scheme based on Taylor matrix method to solve the systems of differential equations of the first order. The presented method has been discussed on the four test problems including the stiff systems. Using the computer program, the approximate solutions of systems of differential equations have been easily obtained. The obtained results have

Table 5. Numeric results of Example 4

	x	Ours-TMM		Exact	Absolute
	↓	$N=5$	$N=7$		error
y_1	0.0	1.000000000000	1.000000000000	1.000000000000	0.0
	0.2	1.491818666666	1.491824680634	1.491824697641	0.1E-7
	0.4	2.225130666666	2.225536365714	2.225540928492	0.4E-5
	0.6	3.315136000000	3.319994148571	3.320116922736	0.1E-3
	0.8	4.923114666666	4.951742455873	4.953032424395	0.1E-2
	1.0	7.266666666666	7.380952380952	7.389056098930	0.8E-2
y_2	0.0	1.000000000000	1.000000000000	1.000000000000	0.0
	0.2	1.178736000000	1.178735908571	1.178735908636	0.6E-10
	0.4	1.310485333333	1.310479319365	1.310479336311	0.1E-7
	0.6	1.390048000000	1.389977645714	1.389978088304	0.4E-6
	0.8	1.414464000000	1.414058300952	1.414062800246	0.4E-5
	1.0	1.383333333333	1.381746031746	1.381773290676	0.2E-4
y_3	0.0	0.000000000000	0.000000000000	0.000000000000	0.0
	0.2	0.198669333333	0.198669330793	0.198669330795	0.2E-11
	0.4	0.389418666666	0.389418341587	0.389418342308	0.7E-9
	0.6	0.564647999999	0.564642445714	0.564642473395	0.2E-7
	0.8	0.717397333333	0.717355723174	0.717356090899	0.3E-6
	1.0	0.841666666666	0.841468253968	0.841470984807	0.2E-5
y_4	0.0	1.000000000000	1.000000000000	1.000000000000	0.0
	0.2	0.818730666666	0.818730753015	0.818730753077	0.6E-10
	0.4	0.670314666666	0.670320030476	0.670320046035	0.1E-7
	0.6	0.548752000000	0.548811245714	0.548811636094	0.3E-6
	0.8	0.449002666666	0.449325145396	0.449328964117	0.3E-5
	1.0	0.366666666666	0.367857142857	0.367879441171	0.2E-4

been then compared to numerical or exact results derived from other solution methods of equation systems in tables. As a result, our findings demonstrate that our numerical scheme is very efficient, simple and it is preferable to some recent numerical schemes.

References

1. I.H. Abdel-Halim Hassan, *Application to differential transformation method for solving systems of differential equations*, Appl. Math. Model. **32** (2008), 2552-2559.
2. J. Biazar and H. Ghazvini, *He's variational iteration method for solving linear and non-linear systems of ordinary differential equations*, Appl. Math. Comput. **191** (2007), 287-297.
3. B. Bülbül and M. Sezer, *A Taylor matrix method for the solution of a two-dimensional linear hyperbolic equation*, Appl. Math. Lett. **24** (2011), 1716-1720.
4. M. Gülsu and M. Sezer, *The approximate solution of high-order linear difference equation with variable coefficients in terms of Taylor polynomials*, Appl. Math. Comput. **168** (2005), 76-88.
5. M. Gülsu and M. Sezer, *A method for the approximate solution of the high-order linear difference equations in terms of Taylor polynomials*, Int. J. Comput. Math. **82** (2005), 629-642.
6. M. Gülsu and M. Sezer, *A Taylor polynomial approach for solving differential-difference equations*, J. Comput. and Appl. Math. **186** (2006), 349-364.

7. M. Gülsu and M. Sezer, *On the solution of Riccati equation by the Taylor matrix method*, Appl. Math. Comput. **176** (2006), 414-421.
8. M. Gülsu, M. Sezer and B. Tanay, *A matrix method for solving high-order linear difference equations with mixed argument using hybrid legendre and taylor polynomials*, J. Franklin Inst. **343** (2006), 647-659.
9. M. Gülsu and M. Sezer, *Approximations to the solution of linear Fredholm integro differential–difference equation of high order*, J. Franklin Inst. **343** (2006), 720-737.
10. N. Güzel and M. Bayram, *On the numerical solution of stiff systems*, Appl. Math. Comput. **170** (2005), 230-236.
11. R.P. Kanwal and K.C. Liu, *A Taylor expansion approach for solving integral equations*, Int. J. Math. Educ. Sci. Technol. **20** (1989), 411-414.
12. D. Kaya, *A reliable method for the numerical solution of the kinetics problems*, Appl. Math. Comput. **156** (2004), 261-270.
13. C. Keşan, *Taylor polynomial solutions of linear differential equations*, Appl. Math. Comput. **142** (2003), 155-165.
14. N. Kurt and M. çevik, *Polynomial solution of the single degree of freedom system by Taylor matrix method*, Mechanics Research Communications **35** (2008), 530-536.
15. S. Nas, S. Yalçınbaş and M. Sezer, *A Taylor polynomial approach for solving high-order linear Fredholm integro-differential equations*, Int. J. Math. Educ. Sci. Technol. **31** (2000), 213-225.
16. M. Sezer, *Taylor polynomial solutions of Volterra integral equations*, Int. J. Math. Educ. Sci. Technol. **25** (1994), 625-633.
17. M. Sezer, *A method for approximate solution of the second order linear differential equations in terms of Taylor polynomials*, Int. J. Math. Educ. Sci. Technol. **27** (1996), 821-834.
18. M. Sezer, A. Karamete and M. Gülsu, *Taylor polynomial solutions of systems of linear differential equations with variable coefficients*, Int. J. Comput. Math. **82** (2005), 755-764.
19. M. Sezer and M. Gülsu, *A new polynomial approach for solving difference and Fredholm integro-difference equations with mixed argument*, Appl. Math. Comput. **171** (2005), 332-344.
20. M. Sezer and M. Gülsu, *Polynomial solution of the most general linear Fredholm integro differential-difference equation by means of Taylor matrix method*, Int. J. Complex Variables **50** (2005), 367-382.
21. M. Sezer, B. Tanay and M. Gülsu, *A polynomial approach for solving high-order linear complex differentıal equations with variable coefficients in disc*, Erciyes Üniversitesi Fen Bilimleri Enstitüsü Dergisi **25** (2009), 374-389.
22. S. Yalçınbaş and M. Sezer, *The approximate solution of high-order linear Volterra-Fredholm integro-differential equations in terms of Taylor polynomials*, Appl. Math. Comput. **112** (2000), 291-308.
23. S. Kome, M.T. Atay, A. Eryilmaz, C. Kome and S. Piipponen, *Comparative numerical solutions of stiff ordinary differential equations using Magnus series expansion method*, New Trends in Mathematical Sciences **3** (2015), 35-45.
24. M.T. Atay, A. Eryilmaz and S. Kome, *Magnus series expansion method for solving nonhomogeneous stiff systems of ordinary differential equations*, Kuwait J. Sci. **43** (2016), 25-38.

H. Zeybek received his M.Sc. and Ph.D from Nevşehir Hacı Bektaş Veli University in 2011 and 2016, respectively. Now he is a research assistant. His research interests include numerical analysis, linear and non-linear differential equations, fluid mechanics, water waves, tsunami waves, non-linear wave phenomena and finite element method.

Department of Applied Mathematics, Faculty of Computer Science, Abdullah Gül University, Kayseri 38080, Turkey.
e-mail: zeybek.halil45@gmail.com

İ.T. Dolapçı received his M.Sc. and Ph.D from Dumlupınar University and Eskişehir Osmangazi University in 1996 and 2003, respectively. He is currently an assistant professor at Celal Bayar University. His research interests are applied mathematics, differential equations, iterative methods and numerical methods.

Department of Mechanical Engineering, Faculty of Engineering, Celal Bayar University, Manisa 45140, Turkey.
e-mail: ihsan.dolapci@cbu.edu.tr