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OPTIMIZATION PROBLEMS WITH DIFFERENCE OF SET-VALUED MAPS UNDER GENERALIZED CONE CONVEXITY

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Abstract. In this paper, we establish the necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions for an optimization problem with difference of set-valued maps under generalized cone convexity assumptions. We also study the duality results of Mond-Weir (MWD), Wolfe (WD) and mixed (Mix D) types for the weak solutions of the problem (P).

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1. Introduction

In the theory of nonsmooth optimization, authors are mainly interested to study the analysis and applications of nonconvex mappings. One type of such nonconvex mappings is the D. C. mappings i.e. the difference of convex mappings. This type of mappings has been studied in optimization theory in [7–10, 12–14]. In 1989, Hiriart-Urruty [12] studied the D. C. optimization problems. He established the sufficient optimality conditions for such type of problems with the difference of convex, proper and lower semicontinuous mappings using the notion of ϵ-subdifferential. In 2009, Lahoussine et al. [13] characterized the difference of locally Lipschitz D.C. mappings in terms of set-valued mapping monotonicity. In the last few years, authors like Flores-Bazán [7], Gadhi et al. [9, 10] and Taa [14] studied the optimization problems with the difference of cone convex vector-valued mappings. In 2005, Taa [14] established the optimality conditions for D.C. vector optimization problems by using the Lagrange-Fritz-John and Lagrange-Karush-Kuhn-Tucker multipliers rules. In [9], Gadhi

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and Metrane established the sufficient optimality conditions for D.C. vector optimization problems in ordered Banach space. Later in 2005, Gadhi [8] also established the necessary optimality conditions for the optimization problems with the difference of cone convex set-valued mappings by using the notion of subdifferential, introduced by Baier and Jahn [1]. In 2012, Guo et al. [11] established the sufficient optimality condition for generalized D.C. multiobjective optimization problems by using the notion of subdifferential, introduced by Borwein [3].

In this paper, we are mainly interested to establish the necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions for the optimization problem (P) with the difference of set-valued maps under generalized cone convexity assumptions. We also study the duality results of Mond-Weir (MWD), Wolfe (WD) and mixed (Mix D) types for the weak solutions of the problem (P).

This paper is organized as follows. In Section 2, we recall some definitions and preliminary concepts of set-valued mappings and set-valued D.C. optimization problems. In Section 3, we establish the necessary and sufficient KKT conditions for the set-valued D.C. optimization problem (P) and prove the duality results of various types under generalized cone convexity assumption on set-valued maps.

2. Definition and preliminaries

Let Y be a real normed space and K be a non-empty subset of Y. Then K is said to be a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Further, K is called pointed if $K \cap (-K) = \{\theta_Y\}$, solid if $\mathrm{int}(K) \neq \emptyset$, closed if $\overline{K} = K$ and convex if $\lambda K + (1-\lambda)K \subseteq K$, for all $\lambda \in [0,1]$, where $\mathrm{int}(K)$ and $\overline{K}$ denote the interior and closure of K, respectively and θ_Y is the zero element of Y.

Let Y^* be the continuous dual of Y. Suppose that $y \in Y$ and $y^* \in Y^*$. Then, by $\langle y^*, y\rangle$, we mean the canonical bilinear form with respect to the duality between Y^* and Y.

Let K be a solid pointed convex cone of Y. We have the following two types of cone-orderings in Y with respect to K.
For any $y, y' \in Y$,

$$y \leq y' \text{ if } y' - y \in K$$

and

$$y < y' \text{ if } y' - y \in \mathrm{int}(K).$$

The following notions of minimality are mainly used in Y with respect to a solid pointed convex cone K.

Definition 2.1. Let B be a non-empty subset of a real normed space Y. Then strongly minimal, minimal and weakly minimal points of B are defined as:

(i) $y' \in B$ is a strongly minimal point of B if $y' \leq y$ for all $y \in B$.
(ii) $y' \in B$ is a minimal point of B if there is no $y \in B \setminus \{y'\}$ such that $y \leq y'$.

(iii) $y' \in B$ is a weakly minimal point of B if there is no $y \in B$ such that $y < y'$.

The sets of strongly minimal, minimal and weakly minimal points of B are denoted by s-min(B), min(B) and w-min(B), respectively and characterized as:

$$\text{s-min}(B) = \Big\{y' \in B : B \subseteq \{y'\} + K\Big\},$$

$$\min(B) = \Big\{y' \in B : (y' - B) \cap B = \{y'\}\Big\}$$

and

$$\text{w-min}(B) = \Big\{y' \in B : (y' - \operatorname{int}(K)) \cap B = \emptyset\Big\}.$$

Similarly, the sets of strongly maximal, maximal and weak maximal points of B can be defined and characterized.

Let X and Y be real normed spaces, 2^Y be the set of all subsets of Y and K be a solid pointed convex cone in Y. Let $F : X \to 2^Y$ be a set-valued map from X to Y i.e., $F(x) \subseteq Y$, for all $x \in X$.

The effective domain, graph and epigraph of F are defined by:

$$\operatorname{dom}(F) = \Big\{x \in X : F(x) \neq \emptyset\Big\},$$

$$F(A) = \bigcup_{x \in A} F(x), \text{ for any } \emptyset \neq A \subseteq X,$$

$$\operatorname{gr}(F) = \Big\{(x, y) \in X \times Y : y \in F(x)\Big\}$$

and

$$\operatorname{epi}(F) = \Big\{(x, y) \in X \times Y : y \in F(x) + K\Big\}.$$

Definition 2.2. ([15]). Let $\emptyset \neq A \subseteq X$ and $F : A \to 2^Y$ be a set-valued map. Let $x' \in A$ and $y' \in F(x')$. A bounded linear operator $T : X \to Y$ is called a weak subgradient for y' of F at x' if

$$y' - T(x') \in \text{w-min} \bigcup_{x \in A} \Big(F(x) - T(x)\Big).$$

The set of all weak subgradients for y' of F at x' is called the weak subdifferential for y' of F at x' and is denoted by $\partial_w F(x'; y')$. Moreover, F is called weak subdifferentiable at x' if $\partial_w F(x'; y) \neq \emptyset$, for all $y \in F(x')$.

Similarly, the notion of strong subgradient and subdifferential have been defined for set-valued case.

Definition 2.3. ([3]). Let $\emptyset \neq A \subseteq X$ and $F : A \to 2^Y$ be a set-valued map. Let $x' \in A$ and $y' \in F(x')$. A bounded linear operator $T : X \to Y$ is called a strong subgradient for y' of F at x' if

$$y' - T(x') \in \text{s-min} \bigcup_{x \in A} \Big(F(x) - T(x)\Big).$$

The set of all strong subgradients for y' of F at x' is called the strong subdifferential for y' of F at x' and is denoted by $\partial_s F(x'; y')$. Moreover, F is called strong subdifferentiable at x' if $\partial_s F(x'; y) \neq \emptyset$, for all $y \in F(x')$.

Definition 2.4. ([2]). Let A be a non-empty convex subset of X. A set-valued map $F : X \to 2^Y$, with $A \subseteq \mathrm{dom}(F)$, is called K-convex on A if $\forall x_1, x_2 \in A$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K.$$

It is clear that if the set-valued map $F : X \to 2^Y$ is K-convex on A, then $\mathrm{epi}(F)$ is a convex subset of $X \times Y$.

Let X, Y and Z be real normed spaces and A be a nonempty closed convex subset of X. Let K and L be solid pointed closed convex cones in Y and Z, respectively. Suppose that $F_1 : X \to 2^Y$, $F_2 : X \to 2^Y$, $G_1 : X \to 2^Z$ and $G_2 : X \to 2^Z$ are set-valued maps with

$$A \subseteq \mathrm{dom}(F_1) \cap \mathrm{dom}(F_2) \cap \mathrm{dom}(G_1) \cap \mathrm{dom}(G_2).$$

Consider the optimization problems with difference of set-valued maps:

$$\begin{aligned} &\underset{x \in A}{\text{minimize}} \quad && F_1(x) - F_2(x), \\ &\text{subject to,} && \Big(G_1(x) - G_2(x)\Big) \bigcap (-L) \neq \emptyset. \end{aligned} \tag{P}$$

Here, the feasible set S of the problem (P) is defined by

$$S = \Big\{x \in A : \Big(G_1(x) - G_2(x)\Big) \bigcap (-L) \neq \emptyset\Big\}.$$

Definition 2.5. A point $(x', y_1' - y_2') \in X \times Y$, with $x' \in S$, $y_1' \in F_1(x')$ and $y_2' \in F_2(x')$, is called a minimizer of the problem (P) if there exist no $(x, y_1 - y_2) \in X \times Y$, with $x \in S$, $y_1 \in F_1(x)$ and $y_2 \in F_2(x)$, such that

$$(y_1 - y_2) - (y_1' - y_2') \in -K \setminus \{\theta_Y\}.$$

Definition 2.6. A point $(x', y_1' - y_2') \in X \times Y$, with $x' \in S$, $y_1' \in F_1(x')$ and $y_2' \in F_2(x')$, is called a weak minimizer of the problem (P) if there exist no $(x, y_1 - y_2) \in X \times Y$, with $x \in S$, $y_1 \in F_1(x)$ and $y_2 \in F_2(x)$, such that

$$(y_1 - y_2) - (y_1' - y_2') \in -\mathrm{int}(K).$$

3. Main results

We introduce the notion of ρ-cone convexity of set-valued maps in [6]. For $\rho = 0$, we have the usual notion of cone convexity of set-valued maps.

Definition 3.1. ([6]). Let X, Y be real normed spaces, A be a nonempty convex subset of X, K be a solid pointed convex cone in Y, $e \in \mathrm{int}(K)$ and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \mathrm{dom}(F)$. Then F is said to be ρ-K-convex with respect to e on A if there exists $\rho \in \mathbb{R}$ such that

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \rho\lambda(1 - \lambda)\|x_1 - x_2\|^2 e + K,$$
$$\forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

If $\rho > 0$, then F is said to be strongly ρ-cone convex, for $\rho = 0$ we get the usual notion of cone convexity and if $\rho < 0$, then F is said to be weakly ρ-cone convex. In [6], we give an example of ρ-cone convex set-valued map, which is not cone convex.

In the following lemma, we characterize ρ-cone convexity in terms of strong subdifferential of set-valued maps.

Lemma 3.1. Let A be a convex subset of X, $x' \in A$ and $e \in \operatorname{int}(K)$. Let $F : X \to 2^Y$ be ρ-K-convex with respect to e on A. Then, for any $y' \in F(x')$ and $T' \in \partial_s F(x'; y')$, we have

$$F(x) - y' - T'(x - x') - \rho\|x - x'\|^2 e \subseteq K, \forall x \in A.$$

Proof. Let $x \in A$ and $\lambda \in [0, 1]$.
Since $F : X \to 2^Y$ is a ρ-K-convex with respect to e on A,

$$\lambda F(x) + (1 - \lambda)F(x') \subseteq F(\lambda x + (1 - \lambda)x') + \rho\lambda(1 - \lambda)\|x - x'\|^2 e + K.$$

Let $y \in F(x)$ and $y' \in F(x')$.
Therefore,

$$\lambda y + (1 - \lambda)y' = u + \rho\lambda(1 - \lambda)\|x - x'\|^2 e + c,$$

for some $u \in F(\lambda x + (1 - \lambda)x')$ and $c \in K$.
Since $T' \in \partial_s F(x'; y')$,

$$y' - T'(x') \in \text{s-min} \bigcup_{x \in A} \Big(F(x) - T'(x)\Big).$$

It follows that

$$F\Big(\lambda x + (1 - \lambda)x'\Big) - y' \geq T'\Big(\lambda x + (1 - \lambda)x' - x'\Big) = \lambda T'(x - x').$$

Therefore,

$$u - y' \geq \lambda T'(x - x').$$

Hence,

$$\begin{aligned}\lambda(y - y') - \rho\lambda(1 - \lambda)\|x - x'\|^2 e &\geq \lambda(y - y') - \rho\lambda(1 - \lambda)\|x - x'\|^2 e - c \\ &= u - y' \\ &\geq \lambda T'(x - x').\end{aligned}$$

Consequently,

$$y - y' - \rho\|x - x'\|^2 e + \rho\lambda\|x - x'\|^2 e \geq T'(x - x'),$$

which is true for all $\lambda \in [0, 1]$.
Therefore,

$$y - y' - \rho\|x - x'\|^2 e \geq T'(x - x').$$

Since $y \in F(x)$ be arbitrary, we have

$$F(x) - y' - \rho\|x - x'\|^2 e \geq T'(x - x').$$

Therefore,

$$F(x) - y' - T'(x - x') - \rho\|x - x'\|^2 e \subseteq K, \forall x \in A.$$

□

We establish the necessary KKT conditions for the set-valued D.C. optimization problem (P) under ρ-cone convexity assumption on set-valued maps.

Theorem 3.1. Let A be a convex subset of X, $\rho_1, \rho_2, \rho_1', \rho_2' \in \mathbb{R}$, $e_1, e_2 \in \text{int}(K)$ and $e_1', e_2' \in \text{int}(L)$. Let $(x', y_1' - y_2')$, with $x' \in A$, $y_1' \in F_1(x')$ and $y_2' \in F_2(x')$, be a weak minimizer of the problem (P) and there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with

$$z_1' - z_2' \in -L.$$

Suppose that $F_1 : X \to 2^Y$ is weakly ρ_1-K-convex with respect to e_1 and $G_1 : X \to 2^Z$ is strongly ρ_1'-L-convex with respect to e_1', on A, with

$$\rho_1\langle y^*, e_1\rangle + \rho_1'\langle z^*, e_1'\rangle \geq 0. \tag{3.1}$$

Also, suppose that $F_2 : X \to 2^Y$ is weakly ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is strongly ρ_2'-L-convex with respect to e_2', on A.
Then there exists $(\theta_{Y^*}, \theta_{Z^*}) \neq (y^*, z^*) \in K^+ \times L^+$ such that

$$y^*T_1 + z^*T_2 \in \partial_s(y^*F_1 + z^*G_1)(x'; \langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle),$$
$$\forall T_1 \in \partial_s F_2(x'; y_2') \text{ and } T_2 \in \partial_s G_2(x'; z_2')$$

and

$$\langle z^*, z_1' - z_2'\rangle = 0.$$

Proof. Let $T_1 \in \partial_s F_2(x'; y_2')$ and $T_2 \in \partial_s G_2(x'; z_2')$.
We claim that the system

$$F_1(x) - y_1' - T_1(x - x') < \rho_1\|x - x'\|^2 e_1$$
$$G_1(x) - z_2' - T_2(x - x') < \rho_1'\|x - x'\|^2 e_1'$$

has no solution in A.
Suppose that the system has a solution $x_0 \in A$.
Therefore,

$$F_1(x_0) - y_1' - T_1(x_0 - x') - \rho_1\|x_0 - x'\|^2 e_1 \subseteq -\text{int}(K)$$

and

$$G_1(x_0) - z_2' - T_2(x_0 - x') - \rho_1'\|x_0 - x'\|^2 e_1' \subseteq -\text{int}(L).$$

As $F_2 : X \to 2^Y$ is weakly ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is strongly ρ_2'-L-convex with respect to e_2', on A and $T_1 \in \partial_s F_2(x'; y_2')$ and $T_2 \in \partial_s G_2(x'; z_2')$, we have

$$F_2(x_0) - y_2' - T_1(x_0 - x') - \rho_2\|x_0 - x'\|^2 e_2 \subseteq K$$

and

$$G_2(x_0) - z_2' - T_2(x_0 - x') - \rho_2'\|x_0 - x'\|^2 e_2' \subseteq L.$$

Hence,

$$-F_2(x_0) + y_2' + T_1(x_0 - x') + \rho_2\|x_0 - x'\|^2 e_2 \subseteq -K$$

and

$$-G_2(x_0) + z_2' + T_2(x_0 - x') + \rho_2'\|x_0 - x'\|^2 e_2' \subseteq -L.$$

Therefore,

$$\begin{aligned} F_1(x_0) - y_1' &\subseteq T_1(x_0 - x') + \rho_1\|x_0 - x'\|^2 e_1 - \mathrm{int}(K) \\ &\subseteq F_2(x_0) - y_2' + \|x_0 - x'\|^2(\rho_1 e_1 - \rho_2 e_2) - \mathrm{int}(K) - K \\ &\subseteq F_2(x_0) - y_2' + \|x_0 - x'\|^2(\rho_1 e_1 - \rho_2 e_2) - \mathrm{int}(K). \end{aligned}$$

Since $\rho_1 \leq 0$ and $\rho_2 \geq 0$,

$$\rho_1 e_1 - \rho_2 e_2 \in -K.$$

Hence,

$$F_1(x_0) - y_1' \subseteq F_2(x_0) - y_2' - \mathrm{int}(K).$$

Similarly, we have

$$G_1(x_0) \subseteq G_2(x_0) + (z_2' - z_2') - \mathrm{int}(L) \subseteq G_2(x_0) - \mathrm{int}(L).$$

It contradicts that $(x', y_1' - y_2')$ is a weak minimizer of the problem (P).
Therefore, the system has no solution in A.
As $F_1 : X \to 2^Y$ is weakly ρ_1-K-convex with respect to e_1 and $G_1 : X \to 2^Z$ is strongly ρ_1'-L-convex with respect to e_1', on A,
$F_1(.) - y_1' - T_1(. - x') : X \to 2^Y$ is weakly ρ_1-K-convex with respect to e_1 and $G_1(.) - z_2' - T_2(. - x') : X \to 2^Z$ is strongly ρ_1'-L-convex with respect to e_1', on A.
Therefore, there exists $(\theta_{Y^*}, \theta_{Z^*}) \neq (y^*, z^*) \in K^+ \times L^+$ such that

$$\begin{aligned} &\Big\langle y^*, F_1(x) - y_1' - T_1(x - x')\Big\rangle + \Big\langle z^*, G_1(x) - z_2' - T_2(x - x')\Big\rangle \\ &- \Big(\rho_1\langle y^*, e_1\rangle + \rho_1'\langle z^*, e_1'\rangle\Big)\|x - x'\|^2 \geq 0. \end{aligned}$$

From (3.1), we have

$$\Big\langle y^*, F_1(x) - y_1' - T_1(x - x')\Big\rangle + \Big\langle z^*, G_1(x) - z_2' - T_2(x - x')\Big\rangle \geq 0. \tag{3.2}$$

Putting $x = x'$ in (3.2), we have

$$\langle z^*, z_1' - z_2'\rangle \geq 0.$$

Again, since $z_1' - z_2' \in -L$,

$$\langle z^*, z_1' - z_2'\rangle \leq 0.$$

Therefore,

$$\langle z^*, z_1' - z_2'\rangle = 0.$$

Hence,

$$\langle z^*, z_1'\rangle = \langle z^*, z_2'\rangle.$$

Consequently, from (3.2), we have

$$\Big\langle y^*, F_1(x) - y_1' - T_1(x - x')\Big\rangle + \Big\langle z^*, G_1(x) - z_1' - T_2(x - x')\Big\rangle \geq 0. \tag{3.3}$$

It shows that

$$\begin{aligned} &\langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle - (y^*T_1 + z^*T_2)(x') \\ &\leq \langle y^*, F_1(x)\rangle + \langle z^*, G_1(x)\rangle - (y^*T_1 + z^*T_2)(x). \end{aligned}$$

Therefore,

$$\begin{aligned} &\langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle - (y^*T_1 + z^*T_2)(x') \\ &\in \min \bigcup_{x\in A} \Big((y^*F_1 + z^*G_1)(x) - (y^*T_1 + z^*T_2)(x)\Big). \end{aligned}$$

Hence,

$$y^*T_1 + z^*T_2 \in \partial_s(y^*F_1 + z^*G_1)(x'; \langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle).$$

□

3.1. Sufficient optimality conditions. We establish the sufficient KKT conditions for the set-valued D.C. optimization problem (P) under ρ-cone convexity assumption on set-valued maps.

Theorem 3.2. Let A be a convex subset of X, $\rho_1, \rho_2, \rho_1', \rho_2' \in \mathbb{R}$, $e_1, e_2 \in \mathrm{int}(K)$ and $e_1', e_2' \in \mathrm{int}(L)$. Suppose that $x' \in S$, $y_1' \in F_1(x')$, $y_2' \in F_2(x')$ and there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with

$$z_1' - z_2' \in -L.$$

Suppose that $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A, satisfying

$$\rho_2\langle y^*, e_2\rangle + \rho_2'\langle z^*, e_2'\rangle \geq 0. \tag{3.4}$$

Assume that for any $x \in A$, $y_2 \in F_2(x)$ and $z_2 \in G_2(x)$, $\partial_s F_2(x; y_2) \neq \emptyset$ and $\partial_s G_2(x; z_2) \neq \emptyset$. If there exist $y^* \in K^+ \setminus \{\theta_{Y^*}\}$ and $z^* \in L^+$ such that

$$\begin{aligned} &y^*T_1 + z^*T_2 \in \partial_s(y^*F_1 + z^*G_1)(x'; \langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle), \\ &\forall T_1 \in \partial_s F_2(x; y_2), T_2 \in \partial_s G_2(x; z_2), x \in A, y_2 \in F_2(x) \text{ and } z_2 \in G_2(x) \end{aligned} \tag{3.5}$$

and

$$\langle z^*, z_1' - z_2'\rangle = 0, \tag{3.6}$$

then $(x', y_1' - y_2')$ is a weak minimizer of the problem (P).

Proof. Let $x \in S$, $y_1 \in F_1(x)$ and $y_2 \in F_2(x)$.
Hence, there exist $z_1 \in G_1(x)$ and $z_2 \in G_2(x)$ such that

$$z_1 - z_2 \in -L.$$

As $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A and $T_1 \in \partial_s F_2(x; y_2)$ and $T_2 \in \partial_s G_2(x; z_2)$,

$$F_2(x') - y_2 - T_1(x' - x) - \rho_2\|x' - x\|^2 e_2 \subseteq K$$

and

$$G_2(x') - z_2 - T_2(x'-x) - \rho_2'\|x'-x\|^2 e_2' \subseteq L.$$

Therefore,

$$y_2' - y_2 - T_1(x'-x) - \rho_2\|x'-x\|^2 e_2 \in K$$

and

$$z_2' - z_2 - T_2(x'-x) - \rho_2'\|x'-x\|^2 e_2' \in L.$$

Therefore,

$$\langle y^*, y_2' - y_2\rangle - y^*T_1(x'-x) - \rho_2\|x'-x\|^2\langle y^*, e_2\rangle \geq 0$$

and

$$\langle z^*, z_2' - z_2\rangle - z^*T_2(x'-x) - \rho_2'\|x'-x\|^2\langle z^*, e_2'\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle y^*, y_2' - y_2\rangle + \langle z^*, z_2' - z_2\rangle &- (y^*T_1 + z^*T_2)(x'-x)\\ &\geq (\rho_2\langle y^*, e_2\rangle + \rho_2'\langle z^*, e_2'\rangle)\|x'-x\|^2.\end{aligned}$$

By (3.4), we have

$$\langle y^*, y_2' - y_2\rangle + \langle z^*, z_2' - z_2\rangle - (y^*T_1 + z^*T_2)(x'-x) \geq 0. \tag{3.7}$$

By assumption, there exist $y^* \in K^+ \setminus \{\theta_{Y^*}\}$ and $z^* \in L^+$ such that

$$y^*T_1 + z^*T_2 \in \partial_s(y^*F_1 + z^*G_1)(x'; \langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle).$$

Therefore,

$$\begin{aligned}&\langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle - (y^*T_1 + z^*T_2)(x')\\ &\in \min \bigcup_{x\in A}\Big((y^*F_1 + z^*G_1)(x) - (y^*T_1 + z^*T_2)(x)\Big).\end{aligned}$$

Hence,

$$\begin{aligned}&\langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle - (y^*T_1 + z^*T_2)(x')\\ &\leq (y^*F_1 + z^*G_1)(x) - (y^*T_1 + z^*T_2)(x).\end{aligned}$$

Since $y_1 \in F_1(x)$ and $z_1 \in G_1(x)$,

$$\langle y^*, y_1 - y_1'\rangle + \langle z^*, z_1 - z_1'\rangle - (y^*T_1 + z^*T_2)(x - x') \geq 0. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\langle y^*, y_1 - y_1'\rangle + \langle z^*, z_1 - z_1'\rangle - \Big(\langle y^*, y_2 - y_2'\rangle + \langle z^*, z_2 - z_2'\rangle\Big) \geq 0.$$

As $\langle z^*, z_1' - z_2'\rangle = 0$, we have

$$\langle y^*, y_1 - y_1'\rangle - \langle y^*, y_2 - y_2'\rangle + \langle z^*, z_1 - z_2\rangle \geq 0.$$

As $z_1 - z_2 \in -L$ and $z^* \in L^+$, we have

$$\langle z^*, z_1 - z_2\rangle \leq 0.$$

It shows that

$$\langle y^*, y_1 - y_1'\rangle - \langle y^*, y_2 - y_2'\rangle \geq 0.$$

It implies that

$$\langle y^*, y_1 - y_2 - (y_1' - y_2')\rangle \geq 0.$$

Hence,

$$y_1 - y_2 - (y_1' - y_2') \in Y \setminus (-\mathrm{int}(K)).$$

Therefore, $(x', y_1' - y_2')$ is a weak minimizer of the problem (P). □

3.2. Mond-Weir type dual. We prove the duality results of Mond-Weir type for the set-valued D.C. optimization problems (P).

Consider Mond-Weir type dual (MWD) corresponding to the problem (P). Assume that for any $x \in A$, $y_2 \in F_2(x)$ and $z_2 \in G_2(x)$, $\partial_s F_2(x; y_2) \neq \emptyset$ and $\partial_s G_2(x; z_2) \neq \emptyset$.

$$\underset{x \in A}{\text{minimize}} \quad y_1' - y_2', \tag{MWD}$$

subject to,

$$\begin{aligned}
&y^* T_1 + z^* T_2 \in \partial_s (y^* F_1 + z^* G_1)(x'; \langle y^*, y_1' \rangle + \langle z^*, z_1' \rangle),\\
&\forall T_1 \in \partial_s F_2(x; y_2), T_2 \in \partial_s G_2(x; z_2), x \in A, y_2 \in F_2(x) \text{ and } z_2 \in G_2(x),\\
&\langle z^*, z_1' - z_2' \rangle \geq 0,\\
&x' \in A, y_1' \in F_1(x'), y_2' \in F_2(x'), z_1' \in G_1(x'), z_2' \in G_2(x'),\\
&y^* \in K^+ \setminus \{\theta_{Y^*}\} \text{ and } z^* \in L^+.
\end{aligned}$$

A point $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ satisfying all the constraints of (MWD) is called a feasible point of the problem (MWD).

Definition 3.2. A feasible point $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ of the problem (MWD) is said to be a weak maximizer of (MWD) if there exists no feasible point $(x, y_1, y_2, z_1, z_2, y_1^*, z_1^*)$ of (MWD) such that

$$(y_1 - y_2) - (y_1' - y_2') \in \mathrm{int}(K).$$

Theorem 3.3. (Weak duality) Let A be a convex subset of X, $\rho_2, \rho_2' \in \mathbb{R}$, $e_2 \in \mathrm{int}(K)$ and $e_2' \in \mathrm{int}(L)$. Let $x_0 \in S$ and $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ be a feasible point of the problem (MWD). Suppose that $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A, satisfying (3.4).

Then,

$$F_1(x_0) - F_2(x_0) - (y_1' - y_2') \subseteq Y \setminus -\mathrm{int}(K).$$

Proof. We prove the theorem by the method of contradiction.

Suppose that for some $y_1 \in F_1(x_0)$ and $y_2 \in F_2(x_0)$,

$$y_1 - y_2 - (y_1' - y_2') \in -\mathrm{int}(K).$$

As $y^* \in K^+ \setminus \{\theta_{Y^*}\}$,

$$\Big\langle y^*, y_1 - y_2 - (y_1' - y_2') \Big\rangle < 0. \tag{3.9}$$

Since $x_0 \in S$, there exist $z_1 \in G_1(x_0)$ and $z_2 \in G_2(x_0)$ such that

$$z_1 - z_2 \in -L.$$

As $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A and $T_1 \in \partial_s F_2(x_0; y_2)$ and $T_2 \in \partial_s G_2(x_0; z_2)$,

$$F_2(x') - y_2 - T_1(x' - x_0) - \rho_2\|x' - x_0\|^2 e_2 \subseteq K$$

and

$$G_2(x') - z_2 - T_2(x' - x_0) - \rho_2'\|x' - x_0\|^2 e_2' \subseteq L.$$

Therefore,

$$y_2' - y_2 - T_1(x' - x_0) - \rho_2\|x' - x_0\|^2 e_2 \in K$$

and

$$z_2' - z_2 - T_2(x' - x_0) - \rho_2'\|x' - x_0\|^2 e_2' \in L.$$

Therefore,

$$\langle y^*, y_2' - y_2\rangle - y^* T_1(x' - x_0) - \rho_2\|x' - x_0\|^2 \langle y^*, e_2\rangle \geq 0$$

and

$$\langle z^*, z_2' - z_2\rangle - z^* T_2(x' - x_0) - \rho_2'\|x' - x_0\|^2 \langle z^*, e_2'\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle y^*, y_2' - y_2\rangle + \langle z^*, z_2' - z_2\rangle - (y^*T_1 + z^*T_2)(x' - x_0)\\ \geq (\rho_2\langle y^*, e_2\rangle + \rho_2'\langle z^*, e_2'\rangle)\|x' - x_0\|^2.\end{aligned}$$

From (3.4), we have

$$\langle y^*, y_2' - y_2\rangle + \langle z^*, z_2' - z_2\rangle - (y^*T_1 + z^*T_2)(x' - x_0) \geq 0. \tag{3.10}$$

From the constraints of (MWD), we have

$$y^*T_1 + z^*T_2 \in \partial_s(y^*F_1 + z^*G_1)(x'; \langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle).$$

Therefore,

$$\begin{aligned}&\langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle - (y^*T_1 + z^*T_2)(x')\\ &\in \min \bigcup_{x \in A} \Big((y^*F_1 + z^*G_1)(x) - (y^*T_1 + z^*T_2)(x)\Big).\end{aligned}$$

Hence,

$$\begin{aligned}&\langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle - (y^*T_1 + z^*T_2)(x')\\ &\leq (y^*F_1 + z^*G_1)(x_0) - (y^*T_1 + z^*T_2)(x_0).\end{aligned}$$

Since $y_1 \in F_1(x_0)$ and $z_1 \in G_1(x_0)$,

$$\langle y^*, y_1 - y_1'\rangle + \langle z^*, z_1 - z_1'\rangle - (y^*T_1 + z^*T_2)(x_0 - x') \geq 0. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\langle y^*, y_1 - y_1'\rangle + \langle z^*, z_1 - z_1'\rangle - \Big(\langle y^*, y_2 - y_2'\rangle + \langle z^*, z_2 - z_2'\rangle\Big) \geq 0.$$

From the constraints of (MWD), we have

$$\langle z^*, z_1' - z_2'\rangle \geq 0.$$

So,

$$\langle y^*, y_1 - y_1'\rangle - \langle y^*, y_2 - y_2'\rangle + \langle z^*, z_1 - z_2\rangle \geq 0.$$

As $z_1 - z_2 \in -L$ and $z^* \in L^+$, we have

$$\langle z^*, z_1 - z_2 \rangle \leq 0.$$

It shows that

$$\langle y^*, y_1 - y_1' \rangle - \langle y^*, y_2 - y_2' \rangle \geq 0.$$

It implies that

$$\langle y^*, y_1 - y_2 - (y_1' - y_2') \rangle \geq 0,$$

which contradicts (3.9).
Therefore,

$$F_1(x_0) - F_2(x_0) - (y_1' - y_2') \subseteq Y \setminus -\text{int}(K).$$

□

Theorem 3.4. (Strong duality) Let A be a convex subset of X, $x' \in S$, $y_1' \in F_1(x')$ and $y_2' \in F_2(x')$. Suppose that there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with $z_1' - z_2' \in -L$. Assume that for some $(y^*, z^*) \in K^+ \times L^+$, with $\langle y^*, e \rangle = 1$, Eqs. (3.5) and (3.6) are satisfied at $(x', y_1', y_2', z_1', z_2', y^*, z^*)$. Then $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is a feasible solution for (MWD). If the weak duality Theorem 3.3 between (P) and (MWD) holds, then $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is a weak maximizer of (MWD).

Proof. As the Eqs. (3.5) and (3.6) are satisfied at $(x', y_1', y_2', z_1', z_2', y^*, z^*)$,

$$\begin{aligned}
&y^*T_1 + z^*T_2 \in \partial_s(y^*F_1 + z^*G_1)(x'; \langle y^*, y_1' \rangle + \langle z^*, z_1' \rangle), \\
&\forall T_1 \in \partial_s F_2(x; y_2), T_2 \in \partial_s G_2(x; z_2), x \in A, y_2 \in F_2(x) \text{ and } z_2 \in G_2(x)
\end{aligned}$$

and

$$\langle z^*, z_1' - z_2' \rangle = 0.$$

Hence, $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is a feasible solution for (MWD).
Suppose that the weak duality Theorem 3.3 holds between (P) and (MWD) and $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is not a weak maximizer of (MWD).
Then, there exists a feasible point $(x, y_1, y_2, z_1, z_2, y_1^*, z_1^*)$ of (MWD), such that

$$(y_1' - y_2') - (y_1 - y_2) \in -\text{int}(K).$$

It contradicts the weak duality Theorem 3.3 between (P) and (MWD).
Consequently, $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is a weak maximizer for (MWD). □

Theorem 3.5. (Converse duality) Let A be a convex subset of X, $\rho_2, \rho_2' \in \mathbb{R}$, $e_2 \in \text{int}(K)$, $e_2' \in \text{int}(L)$ and $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ be a feasible point of the problem (MWD). Suppose that $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A, satisfying (3.4). If $x' \in S$, then $(x', y_1' - y_2')$ is a weak minimizer of (P).

Proof. We prove the theorem by the method of contradiction.
Suppose that $(x', y' - y_2')$ is not a weak minimzer of the problem (P).
Then there exist $x \in S$ and $y \in F(x)$ such that

$$y_1 - y_2 - (y_1' - y_2') \in -\mathrm{int}(K).$$

Therefore,

$$\Big\langle y^*, y_1 - y_2 - (y_1' - y_2')\Big\rangle < 0, \text{ as } \theta_{Y^*} \neq y^* \in K^+.$$

Again, since $x \in S$,

$$\Big(G_1(x) - G_2(x)\Big) \bigcap (-L) \neq \emptyset.$$

Let there exist $z_1 \in G_1(x)$ and $z_2 \in G_2(x)$ such that

$$z_1 - z_2 \in \Big(G_1(x) - G_2(x)\Big) \bigcap (-L).$$

So,

$$\langle z^*, z_1 - z_2\rangle \leq 0.$$

From the constraints of (MWD), we have

$$\langle z^*, z_1' - z_2'\rangle \geq 0.$$

Therefore,

$$\Big\langle z^*, z_1 - z_2 - (z_1' - z_2')\Big\rangle = \langle z^*, z_1 - z_2\rangle - \langle z^*, z_1' - z_2'\rangle \leq 0.$$

Hence,

$$\Big\langle y^*, y_1 - y_2 - (y_1' - y_2')\Big\rangle + \Big\langle z^*, z_1 - z_2 - (z_1' - z_2')\Big\rangle < 0. \tag{3.12}$$

As $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A and $T_1 \in \partial_s F_2(x; y_2)$ and $T_2 \in \partial_s G_2(x; z_2)$,

$$F_2(x') - y_2 - T_1(x' - x) - \rho_2\|x' - x\|^2 e_2 \subseteq K$$

and

$$G_2(x') - z_2 - T_2(x' - x) - \rho_2'\|x' - x\|^2 e_2' \subseteq L.$$

Therefore,

$$y_2' - y_2 - T_1(x' - x) - \rho_2\|x' - x\|^2 e_2 \in K$$

and

$$z_2' - z_2 - T_2(x' - x) - \rho_2'\|x' - x\|^2 e_2' \in L.$$

Therefore,

$$\langle y^*, y_2' - y_2\rangle - y^* T_1(x' - x) - \rho_2\|x' - x\|^2\langle y^*, e_2\rangle \geq 0$$

and

$$\langle z^*, z_2' - z_2\rangle - z^* T_2(x' - x) - \rho_2'\|x' - x\|^2\langle z^*, e_2'\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle y^*, y_2' - y_2\rangle + \langle z^*, z_2' - z_2\rangle - (y^* T_1 + z^* T_2)(x' - x)\\ \geq (\rho_2\langle y^*, e_2\rangle + \rho_2'\langle z^*, e_2'\rangle)\|x' - x\|^2.\end{aligned}$$

From (3.4), we have

$$\langle y^*, y_2' - y_2\rangle + \langle z^*, z_2' - z_2\rangle - (y^*T_1 + z^*T_2)(x' - x) \geq 0. \tag{3.13}$$

From the constraints of (MWD), we have

$$y^*T_1 + z^*T_2 \in \partial_s(y^*F_1 + z^*G_1)(x'; \langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle).$$

Therefore,

$$\begin{aligned}&\langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle - (y^*T_1 + z^*T_2)(x')\\ &\in \min \bigcup_{x\in A} \Big((y^*F_1 + z^*G_1)(x) - (y^*T_1 + z^*T_2)(x)\Big).\end{aligned}$$

Hence,

$$\begin{aligned}&\langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle - (y^*T_1 + z^*T_2)(x')\\ &\leq (y^*F_1 + z^*G_1)(x) - (y^*T_1 + z^*T_2)(x).\end{aligned}$$

Since $y_1 \in F_1(x)$ and $z_1 \in G_1(x)$,

$$\langle y^*, y_1 - y_1'\rangle + \langle z^*, z_1 - z_1'\rangle - (y^*T_1 + z^*T_2)(x - x') \geq 0. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\langle y^*, y_1 - y_1'\rangle + \langle z^*, z_1 - z_1'\rangle - \Big(\langle y^*, y_2 - y_2'\rangle + \langle z^*, z_2 - z_2'\rangle\Big) \geq 0,$$

which contradicts (3.12). □

We also prove the duality results of Wolfe and mixed types for the set-valued D.C. optimization problems (P). The proofs are almost same as the above, hence omitted.

3.3. Wolfe type dual. Consider Wolfe type dual (WD) corresponding to the problem (P). Assume that for any $x \in A$, $y_2 \in F_2(x)$ and $z_2 \in G_2(x)$, $\partial_s F_2(x; y_2) \neq \emptyset$ and $\partial_s G_2(x; z_2) \neq \emptyset$.

$$\underset{x\in A}{\text{minimize}} \quad y_1' - y_2' + \langle z^*, z_1' - z_2'\rangle e, \tag{WD}$$

subject to,

$$\begin{aligned}&y^*T_1 + z^*T_2 \in \partial_s(y^*F_1 + z^*G_1)(x'; \langle y^*, y_1'\rangle + \langle z^*, z_1'\rangle),\\ &\forall T_1 \in \partial_s F_2(x; y_2), T_2 \in \partial_s G_2(x; z_2), x \in A, y_2 \in F_2(x) \text{ and } z_2 \in G_2(x),\\ &x' \in A, y_1' \in F_1(x'), y_2' \in F_2(x'), z_1' \in G_1(x'), z_2' \in G_2(x'),\\ &y^* \in K^+ \setminus \{\theta_{Y^*}\}, z^* \in L^+ \text{ and } \langle y^*, e\rangle = 1.\end{aligned}$$

A point $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ satisfying all the constraints of (WD) is called a feasible point of the problem (WD).

Definition 3.3. A feasible point $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ of the problem (WD) is said to be a weak maximizer of the problem (WD) if there exists no feasible point $(x, y_1, y_2, z_1, z_2, y_1^*, z_1^*)$ of (WD) such that

$$\Big(y_1 - y_2 + \langle z_1^*, z_1 - z_2\rangle e\Big) - \Big(y_1' - y_2' + \langle z^*, z_1' - z_2'\rangle e\Big) \in \text{int}(K).$$

Theorem 3.6. (Weak duality) Let A be a convex subset of X, $\rho_2, \rho_2' \in \mathbb{R}$, $e_2 \in \text{int}(K)$ and $e_2' \in \text{int}(L)$. Let $x_0 \in S$ and $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ be a feasible point for the problem (WD). Suppose that $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A, satisfying (3.4).
Then,

$$F_1(x_0) - F_2(x_0) - \Big(y_1' - y_2' + \langle z^*, z_1' - z_2' \rangle e\Big) \subseteq Y \setminus -\text{int}(K).$$

Theorem 3.7. (Strong duality) Let A be a convex subset of X, $x' \in S$, $y_1' \in F_1(x')$ and $y_2' \in F_2(x')$. Let there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with

$$z_1' - z_2' \in -L.$$

Assume that for some $(y^*, z^*) \in K^+ \times L^+$, with $\langle y^*, e \rangle = 1$, Eqs. (3.5) and (3.6) are satisfied at $(x', y_1', y_2', z_1', z_2', y^*, z^*)$. Then $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is a feasible solution for (WD). If the weak duality Theorem 3.6 between (P) and (WD) holds, then $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is a weak maximizer of (WD).

Theorem 3.8. (Converse duality) Let A be a convex subset of X, $\rho_2, \rho_2' \in \mathbb{R}$, $e_2 \in \text{int}(K)$, $e_2' \in \text{int}(L)$ and $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ be a feasible point of the problem (WD), with

$$\langle z^*, z_1' - z_2' \rangle \geq 0.$$

Suppose that $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A, satisfying (3.4). If $x' \in S$, then $(x', y_1' - y_2')$ is a weak minimizer of (P).

3.4. Mixed type dual. Consider the mixed type dual (Mix D) corresponding to the problem (P). Assume that for any $x \in A$, $y_2 \in F_2(x)$ and $z_2 \in G_2(x)$, $\partial_s F_2(x; y_2) \neq \emptyset$ and $\partial_s G_2(x; z_2) \neq \emptyset$.

$$\underset{x \in A}{\text{minimize}} \qquad y_1' - y_2' + \langle z^*, z_1' - z_2' \rangle e, \tag{Mix D}$$

subject to,

$$\begin{aligned}
&y^* T_1 + z^* T_2 \in \partial_s (y^* F_1 + z^* G_1)(x'; \langle y^*, y_1' \rangle + \langle z^*, z_1' \rangle), \\
&\forall T_1 \in \partial_s F_2(x; y_2), T_2 \in \partial_s G_2(x; z_2), x \in A, y_2 \in F_2(x) \text{ and } z_2 \in G_2(x), \\
&\langle z^*, z_1' - z_2' \rangle \geq 0, \\
&x' \in A, y_1' \in F_1(x'), y_2' \in F_2(x'), z_1' \in G_1(x'), z_2' \in G_2(x'), \\
&y^* \in K^+ \setminus \{\theta_{Y^*}\}, z^* \in L^+ \text{ and } \langle y^*, e \rangle = 1.
\end{aligned}$$

A point $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ satisfying all the constraints of (Mix D) is called a feasible point of the problem (Mix D).

Definition 3.4. A feasible point $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ of the problem (Mix D) is said to be a weak maximizer of the problem (Mix D) if there exists no feasible point $(x, y_1, y_2, z_1, z_2, y_1^*, z_1^*)$ of (Mix D) such that

$$\Big(y_1 - y_2 + \langle z_1^*, z_1 - z_2 \rangle e\Big) - \Big(y_1' - y_2' + \langle z^*, z_1' - z_2' \rangle e\Big) \in \text{int}(K).$$

Theorem 3.9. (Weak duality) Let A be a convex subset of X, $\rho_2, \rho_2' \in \mathbb{R}$, $e_2 \in \text{int}(K)$ and $e_2' \in \text{int}(L)$. Let $x_0 \in S$ and $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ be a feasible point for the problem (Mix D). Suppose that $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A, satisfying (3.4).
Then,

$$F_1(x_0) - F_2(x_0) - (y_1' - y_2' + \langle z^*, z_1' - z_2' \rangle e) \subseteq Y \setminus -\text{int}(K).$$

Theorem 3.10. (Strong duality) Let A be a convex subset of X, $x' \in S$, $y_1' \in F_1(x')$ and $y_2' \in F_2(x')$. Let there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with $z_1' - z_2' \in -L$. Assume that for some $(y^*, z^*) \in K^+ \times L^+$, with $\langle y^*, e \rangle = 1$, Eqs. (3.5) and (3.6) are satisfied at $(x', y_1', y_2', z_1', z_2', y^*, z^*)$. Then $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is a feasible solution for (Mix D). If the weak duality Theorem 3.9 between (P) and (Mix D) holds, then $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ is a weak maximizer of (Mix D).

Theorem 3.11. (Converse duality) Let A be a convex subset of X, $\rho_2, \rho_2' \in \mathbb{R}$, $e_2 \in \text{int}(K)$, $e_2' \in \text{int}(L)$ and $(x', y_1', y_2', z_1', z_2', y^*, z^*)$ be a feasible point of the problem (Mix D). Suppose that $F_2 : X \to 2^Y$ is ρ_2-K-convex with respect to e_2 and $G_2 : X \to 2^Z$ is ρ_2'-L-convex with respect to e_2', on A, satisfying (3.4). If $x' \in S$, then $(x', y_1' - y_2')$ is a weak minimizer of (P).

4. Conclusions

In this paper, we establish the necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions for the set-valued D.C. optimization problem (P) under ρ-cone convexity assumptions. We also prove the weak, strong and converse duality results of various types for the problem (P). Our future plans include to study the symmetric duals of the set-valued D. C. optimization problems, equilibrium problems and variational inequality problems with respect to set-valued D.C. maps under ρ-cone convexity assumptions.

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