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UNIQUENESS AND VALUE SHARING PROBLEMS IN CLASS $\mathcal{A}$ OF MEROMORPHIC FUNCTIONS

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Abstract. *In this paper, we study the uniqueness and value sharing problems in class $\mathcal{A}$ of meromorphic functions. We obtain significant results which improve as well as generalize the result of C.C Yang and Xinhou Hua* [10].

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1. Introduction

In this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane. Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a \in \overline{\mathbb{C}}$. We say that f and g share the value a CM if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities. We shall use the standard notations of value distribution theory, $T(r,f), m(r,f), N(r,f), \overline{N}(r,f)$,... (Hayman[14], Yang[18], Laine[16] and Navanlinna[17]). We denote by $S(r,f)$ any function satisfying $S(r,f) = o\{T(r,f)\}$, as $r \to +\infty$, possibly outside of finite measure.
Let $f(z)$ and $g(z)$ are non-constant meromorphic functions and a be a finite complex number. We denote by $\overline{N}_L(r,f)$ the counting function for the poles of both f and g about which f has larger multiplicity than g, where multiplicity is not counted. Similarly, we have the notation for $\overline{N}_L(r,g)$.
We denote by $\mathcal{A}$ the class of meromorphic functions f in $\mathbb{C}$ which satisfy the condition $\overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) = S(r,f)$. Clearly all functions in $\mathcal{A}$ are transcendental meromorphic functions.
In 1920's R. Nevanlinna[17] proved the following result (the Nevanlinna four value theorem.)

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Theorem A. Let f and g be two nonconstant meromorphic functions. If f and g share four distinct values CM, then f is a Mobius transformation of g.
For instance, $f = e^z$, $g = e^{-z}$ share $0, \pm 1, \infty$, and $f = \frac{1}{g}$.
In 1997, Yang and Hua[10], obtained following result.
Theorem B. Let f and g be two non-constant meromorphic functions, $n \geq 11$ an integer and $a \in C - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)th$ root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

2. Some Lemmas

Lemma 2.1([6]). Let f be a meromorphic function of finite order and P a homogeneous differential polynomial in f of degree n. If $\Theta(0, f) = \Theta(\infty, f) = 1$, then

$$T(r, p) \sim nT(r, f).$$

Lemma 2.2 ([11]). Let $f_j (j = 1, 2, 3)$ be meromorphic functions that satisfy

$$\sum_{j=1}^{3} f_j = 1$$

Assume that f_1 is not a constant, and

$$\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \bar{N}(r, f_j) < (\lambda + 0(1))T(r), r \in I,$$

where $\lambda < 1$, $T(r) = max\{T(r, f_1), T(r, f_2), T(r, f_3)\}$, $N_2(r, \frac{1}{f_j})$ is the counting function of zeros of $f_j (j = 1, 2, 3)$, where a multiple zero is counted two times and a simple zero is counted once. Then $f_2 = 1$ or $f_3 = 1$.
Lemma 2.3([13]). Let f be a non-constant meromorphic function. Then

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\bar{N}(r, f) + S(r, f).$$

where k is a positive integer.
Lemma 2.4([13]). Let F and G be two distinct non-constant meromorphic functions, and let c be a complex number such that $c \neq 0, 1$. If F and G share 1 and c IM, and if $\bar{N}(r, \frac{1}{F}) + \bar{N}(r, F) = S(r, F)$ and $\bar{N}(r, \frac{1}{G}) + \bar{N}(r, G) = S(r, G)$, then F and G share $0, 1, c, \infty$ CM.
Lemma 2.5 ([17]). If f and g are distinct non-constant meromorphic functions that share four values a_1, a_2, a_3, a_4 CM, then f is Mobius transformation of g: two of the shared values, say a_1 and a_2 are picard exceptional values and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.
Lemma 2.6([13]). If $f(z) \in \mathcal{A}$ and k is a positive integer, then

$$T(r, \frac{f^{(k)}}{f}) = S(r, f).$$

Lemma 2.7([14]). Let f be a non-constant meromorphic functions and a_1, a_2, a_3 be three distinct small meromorphic functions of f, then

$$T(r,f) \leq \sum_{j=1}^{3} \bar{N}(r, \frac{1}{f-a_j}) + S(r,f).$$

Lemma 2.8([14]). Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If

$$N(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{f^{(k)}}) = S(r,\frac{f'}{f}),$$

then $f = e^{az+b}$, where $a \neq 0, b$ are constants.
Following lemmas play a prominent role in improving our results.
Lemma 2.9. Let $f, g \in \mathcal{A}, n \geq m+k+1$ and k be a positive integer. If $f^n[P(f)]^{(k)}$ and $g^n[P(g)]^{(k)}$ share 1 CM, then

$$T(r,g) \leq \left(\frac{n+m-k}{n-m-k}\right) T(r,f) + S(r,g).$$

Proof. Let $G = g^n[P(g)]^{(k)}$. Then it is a polynomial of degree $(n+m-k)$. By lemma 2.1, we have

$$(n+m-k)T(r,g) \sim T(r,G). \tag{1}$$

Applying Lemma 2.7 to $T(r,G)$, we get

$$\begin{aligned}(n+m-k)T(r,g) &\leq \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1}) + S(r,G)\\ &= \overline{N}(r, g^n[P(g)]^{(k)}) + \overline{N}\left(r, \frac{1}{g^n[P(g)]^{(k)}}\right)\\ &+ \overline{N}\left(r, \frac{1}{g^n[P(g)]^{(k)}-1}\right) + S(r, g^n[P(g)]^{(k)})\end{aligned}$$

Noting that

$$\begin{aligned}\overline{N}(r, g^n[P(g)]^{(k)}) &\leq \overline{N}(r,g^n) + N(r,[P(g)]^{(k)})\\ &\leq \overline{N}(r,g) + mN(r,g) + k\overline{N}(r,g)\\ &= mN(r,g) + (k+1)\overline{N}(r,g)\end{aligned}$$

$$\textit{and } S(r,G) = S(r,g), \quad (by (2.1))$$

So,

$$\begin{aligned}(n+m-k)T(r,g) &\leq mN(r,g) + (k+1)\overline{N}(r,g) + \overline{N}(r,\frac{1}{g})\\ &+ N(r,\frac{1}{[P(g)]^{(k)}}) + \overline{N}(r,\frac{1}{g^n[P(g)]^{(k)}-1}) + S(r,g)\end{aligned}$$

Since $f^n[P(f)]^{(k)}$ and $g^n[P(g)]^{(k)}$ share 1 CM, it implies that $f^n[P(f)]^{(k)} - 1$ and $g^n[P(g)]^{(k)} - 1$ have same zeros with same multiplicities, using this with Lemma 2.3, we obtain that

$$\begin{aligned}(n+m-k)T(r,g) \leq mN(r,g) + (k+1)\overline{N}(r,g) + \overline{N}(r,\frac{1}{g}) + mN(r,\frac{1}{g})\\ + k\overline{N}(r,g) + \overline{N}(r,\frac{1}{f^n[P(f)]^{(k)} - 1}) + S(r,g)\end{aligned} \tag{2}$$

By hypothesis, we have

$$\overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) = S(r,f),$$
$$\overline{N}(r,g) + \overline{N}(r,\frac{1}{g}) = S(r,g).$$

Using Nevanlinna's first fundamental theorem and Lemma 2.1, we have

$$\begin{aligned}\overline{N}(r,\frac{1}{f^n[P(f)]^{(k)} - 1}) &\leq T(r,\frac{1}{f^n[P(f)]^{(k)} - 1})\\ &= T(r,f^n[P(f)]^{(k)}) + O(1).\\ &\sim (n+m-k)T(r,f) + O(1).\end{aligned}$$

So,

$$\overline{N}(r,\frac{1}{f^n[P(f)]^{(k)} - 1}) \leq (n+m-k)T(r,f) + O(1). \tag{3}$$

using (3), (2) becomes

$$\begin{aligned}(n+m-k)T(r,g) &\leq mN(r,g) + mN(r,\frac{1}{g}) + (n+m-k)T(r,f) + S(r,g).\\ &\leq 2mT(r,g) + (n+m-k)T(r,f) + S(r,g)\\ (n-m-k)T(r,g) &\leq (n+m-k)T(r,f) + S(r,g)\\ T(r,g) &\leq (\frac{n+m-k}{n-m-k})T(r,f) + S(r,g).\end{aligned}$$

This completes the proof of Lemma. □

Lemma 2.10. Let $f, g \in \mathcal{A}, n \geq m+1$ and k be a positive integer. If $f^n[P(f)]^k$ and $g^n[P(g)]^k$ share 1 CM, then $S(r,f) = S(r,g)$.

Proof. Proceeding as in the proof of Lemma 2.9, we have

$$T(r,g) \leq (\frac{n+m-k}{n-m-k})T(r,f) + S(r,g).$$

Similarly, we have

$$T(r,f) \leq (\frac{n+m-k}{n-m-k})T(r,g) + S(r,f)$$

using above two inequalities we easily obtain

$$S(r,f) = S(r,g).$$

This completes the proof of Lemma. □

Lemma 2.11. Let $f, g \in \mathcal{A}, n \geq m+1$ and k be a positive integer. If $f^n[P(f)]^{(k)}g^n[P(g)]^{(k)} = 1$, then $f = c_3e^{pz}$ and $g = c_4e^{-pz}$ where c_3, c_4 and p are constants such that $(-1)^k(c_3c_4)^{n+1}p^{2k} = 1$.

Proof. Let

$$F = f^n[P(f)]^{(k)} \, and \, G = g^n[P(g)]^{(k)} \tag{4}$$

By Lemma 2.1, we have

$$T(r,F) \sim (n+m-k)T(r,f),$$
$$T(r,G) \sim (n+m-k)T(r,g)$$

clearly $S(r,F) = S(r,f) and S(r,G) = S(r,g)$. By Lemma 2.10, we have

$$S(r,f) = S(r,g).$$

Thus

$$S(r,F) = S(r,f) = S(r,g) = S(r,G). \tag{5}$$

By hypothesis, we have

$$f^n[P(f)]^{(k)}g^n[P(g)]^{(k)} = 1 \; or \; FG = 1. \tag{6}$$

From 6 and f and g are transcendental functions, it follows that

$$N(r, \frac{1}{f}) = 0 \; and \; N(r, \frac{1}{g}) = 0 \tag{7}$$

By hypothesis, we have

$$\begin{aligned} \bar{N}(r,f) + \bar{N}(r, \frac{1}{f}) &= S(r,f) \\ \bar{N}(r,g) + \bar{N}(r, \frac{1}{g}) &= S(r,g) \end{aligned} \tag{8}$$

(6) can be expressed as

$$f^n[P(f)]^{(k)} = \frac{1}{g^n[P(g)]^{(k)}}$$

So we deduce that

$$N(r, f^n[P(f)]^{(k)}) = N\left(r, \frac{1}{g^n[P(g)]^{(k)}}\right) \tag{9}$$

Using (8), we get

$$\begin{aligned} N(r, f^n[P(f)]^{(k)}) &= N(r, f^n) + N(r, [P(f)]^{(k)}) \\ &= nN(r,f) + mN(r,f) + k\bar{N}(r,f) \\ &= (n+m)N(r,f) + k\bar{N}(r,f) \\ &= (n+m)N(r,f) + S(r,f) \end{aligned}$$

Using this with Lemma 2.3 with (5), (7) and (8), (9) can be written as

$$\begin{aligned}(n+m)N(r,f)+S(r,f) &\le N(r,\frac{1}{g^n})+N\left(r,\frac{1}{[P(g)]^{(k)}}\right)\\ &\le (n+m)N(r,\frac{1}{g})+k\bar{N}(r,g)+S(r,g)\\ &= S(r,g).\end{aligned}$$

which implies that

$$N(r,f)=S(r,f). \tag{10}$$

Similarly

$$N(r,g)=S(r,g). \tag{11}$$

By (7), (8) and Lemma 2.3, we have

$$\begin{aligned}\bar{N}(r,\frac{1}{F}) &= \bar{N}\left(r,\frac{1}{f^n[P(f)]^{(k)}}\right)\\ &\le \bar{N}(r,\frac{1}{f})+N\left(r,\frac{1}{[P(f)]^{(k)}}\right)\\ &\le \bar{N}(r,\frac{1}{f})+mN(r,\frac{1}{f})+k\bar{N}(r,\frac{1}{f})+S(r,f)\\ &= S(r,f)\end{aligned}$$

Therefore

$$\bar{N}(r,\frac{1}{F})=S(r,F) \tag{12}$$

Similarly

$$\bar{N}(r,\frac{1}{G})=S(r,G) \tag{13}$$

Moreover by using (8) and (10), we have

$$\begin{aligned}\bar{N}(r,F) =&\bar{N}(r,f^n[P(f)]^{(k)})\\ &\le \bar{N}(r,f)+N(r,[P(f)]^{(k)})\\ &\le \bar{N}(r,f)+mN(r,f)+k\bar{N}(r,f)\\ &= S(r,f).\end{aligned}$$

Therefore

$$\bar{N}(r,F)=S(r,F) \tag{14}$$

Similarly

$$\bar{N}(r,G)=S(r,G) \tag{15}$$

It follows from (12)-(15) that

$$\begin{aligned}\bar{N}(r,\frac{1}{F})+\bar{N}(r,F)=S(r,F),\\ \bar{N}(r,\frac{1}{G})+\bar{N}(r,G)=S(r,G).\end{aligned} \tag{16}$$

In view of (6), we know that F and G share 1 and -1 IM, together this with (16) and Lemma 2.4 implies that F and G share $1, -1, 0, \infty$ CM, thus by Lemma 2.5, we get that 0 and ∞ are picard values of F and G. Thus we deduce from (4) that both f and g are transcendental entire functions. By (7) we have

$$\begin{aligned} f(z) &= e^{\alpha(z)}, \\ g(z) &= e^{\beta(z)} \end{aligned} \tag{17}$$

where $\alpha(z)$ and $\beta(z)$ are non constant entire functions.
Then $T(r, \frac{f'}{f}) = T(r, \frac{e^{\alpha}\alpha'}{e^{\alpha}}) = T(r, \alpha')$. We claim that $\alpha(z) + \beta(z) = c$, c is a constant.
From (17), we know that either α and β are transcendental functions or both α and β are polynomials.
From (6), we have

$$\begin{aligned} N(r, \frac{1}{[P(f)]^{(k)}}) &= N(r, g^n [P(g)]^{(k)} f^n) \\ &\leq nN(r, g) + N(r, [P(g)]^{(k)}) + nN(r, f) \\ &= 0. \end{aligned}$$

From this and (6), we get

$$N(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}}) = 0.$$

If $k \geq 2$, suppose that α is a transcendental entire function. From Lemma 2.7, we have $f = e^{\alpha(z)} = e^{az+b}$, it implies that $\alpha(z) = az + b$, a polynomial, which is a contradiction.
Thus α and β polynomials.
We deduce from (17) that

$$[P(f)]^{(k)} = [(\alpha')^k + P_{(k-1)(\alpha')}]p(e^{\alpha}).$$

$$[P(g)]^{(k)} = [(\beta')^k + Q_{(k-1)(\beta')}]p(e^{\beta}).$$

where $P_{(k-1)(\alpha')}$ and $Q_{(k-1)(\beta')}$ are differential polynomials in α' and β' of degree at most $(k-1)$ respectively. Thus by (6) we obtain that

$$[(\alpha')^k + P_{(k-1)(\alpha')}][(\beta')^k + Q_{(k-1)(\beta')}]p(e^{(n+m-k)(\alpha+\beta)}) = 1 \tag{18}$$

we deduce from (18) that $\alpha(z) + \beta(z) = c$, c is a constant.
If $k = 1$, from (17) we get,

$$(\alpha')(\beta')p(e^{(n+m-k)(\alpha+\beta)}) = 1. \tag{19}$$

Let $\alpha+\beta=\gamma$. If α and β are transcendental entire functions, then γ is not a constant and (19) implies that

$$(\alpha')(\gamma'-\alpha')p(e^{(n+m-k)\gamma})=1. \tag{20}$$

Since

$$\begin{aligned}T(r,\gamma') &= m(r,\gamma')\\ &= m(r,\frac{p(e^{(n+m-k)\gamma'})}{p(e^{(n+m-k)\gamma}}\gamma')\\ &= m(r,\frac{(p(e^{(n+m-k)\gamma}))'}{p(e^{(n+m-k)\gamma})}) = S(r,p(e^{(n+m-k)\gamma}))\end{aligned}$$

Thus (20) implies that Since

$$\begin{aligned}T(r,p(e^{(n+m-k)\gamma})) &= T(r,\frac{1}{(\alpha')(\gamma'-\alpha')})\\ &\leq T(r,(\alpha')(\gamma'-\alpha'))+O(1)\\ &\leq 2T(r,\alpha')+S(r,p(e^{(n+m-k)\gamma})).\end{aligned}$$

Which implies that

$$T(r,p(e^{(n+m-k)\gamma}))=O(T(r,\alpha')).$$

Thus $T(r,\gamma')=S(r,\alpha')$. In view of (20) and by Lemma 2.7, we get

$$T(r,\alpha')\leq \bar{N}(r,\alpha')+\bar{N}(r,\frac{1}{\alpha'})+\bar{N}(r,\frac{1}{\alpha'-\gamma')})+S(r,\alpha').$$

Since α and β are transcendental entire function and in view of (20), we obtain $T(r,\alpha')\leq S(r,\alpha')$ and this implies that α' is a constant, which is a contradiction. Thus α and β are both polynomials and $\alpha(z)+\beta(z)=c$, for a constant c. Hence from (18), we get

$$(\alpha')^{2k}=1+P_{(2k-1)}(\alpha') \tag{21}$$

where $P_{(2k-1)}(\alpha')$ is differential polynomial in α' From (21), we have

$$\begin{aligned}2kT(r,\alpha') &= T(r,(\alpha')^{2k})=m(r,(\alpha')^{2k})\\ &\leq m(r,P_{(2k-1)}(\alpha'))+O(1)\\ &= m(r,\frac{P_{(2k-1)}(\alpha')}{(\alpha')^{2k-1}}(\alpha')^{2k-1})+O(1)\\ &\leq m(r,\frac{P_{(2k-1)}(\alpha')}{(\alpha')^{2k-1}})+m(r,(\alpha')^{2k-1})+O(1)\\ &\leq (2k-1)T(r,\alpha')+S(r,\alpha')\end{aligned}$$

Therefore $T(r,\alpha')\leq S(r,\alpha')$, which implies that α' is a constant. Thus $\alpha=pz+c_1$, $\beta=-pz+c_2$. By (17), we represent f and g as
$f=c_3e^{pz}\quad g=c_4e^{-pz}$.

Where c_3, c_4 and p are constants such that $(-1)^k(c_3c_4)^{n+1}p^{2k} = 1$.
This completes the proof of Lemma. □

3. Main Results

The Theorem B motivate us to think that, whether there exists a similar result, if $f^n f'$ is replaced in Theorem B by $f^n[P(f)]^{(k)}$. In this paper, we prove significant result which improves as well as generalize Theorem B in class $\mathcal{A}$.
Theorem 1. If $f, g \in \mathcal{A}, n \geq m + k + 1$ and k be a positive integer. Then $f^n[P(f)]^{(k)} = 1$ has infinitely many zeros.

Proof. Let $F = f^n[P(f)]^{(k)}$. By Lemma 2.1 and 2.6, we have

$$\begin{aligned}(n+m-k)T(r,f) &\sim T(r, f^n[P(f)]^{(k)})\\ &\leq \overline{N}(r, f^n[P(f)]^{(k)}) + \overline{N}(r, \frac{1}{f^n[P(f)]^{(k)}})\\ &+ \overline{N}(r, \frac{1}{f^n[P(f)]^{(k)} - 1}) + S(r, f^n[P(f)]^{(k)})\end{aligned} \tag{22}$$

Noting that

$$\begin{aligned}\overline{N}(r, f^n[P(f)]^{(k)}) &\leq \overline{N}(r, f^n) + N(r, [P(f)]^{(k)})\\ &\leq \overline{N}(r, f) + mN(r, f) + k\overline{N}(r, f)\\ &\leq mN(r, f) + (k+1)\overline{N}(r, f)\\ \overline{N}(r, \frac{1}{f^n[P(f)]^{(k)}}) &\leq \overline{N}(r, \frac{1}{f^n}) + N(r, \frac{1}{[P(f)]^{(k)}})\\ &\leq \overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^n[P(f)]^{(k)}})\end{aligned}$$

and $(n+m-k)T(r,f) \sim T(r, f^n[P(f)]^{(k)})$. So $S(r, f^n[P(f)]^{(k)}) = S(r, f)$, substituting above inequalities in (22), we obtain,

$$\begin{aligned}(n+m-k)T(r,f) \leq mN(r,f) + (k+1)\overline{N}(r,f) + \overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{[P(f)]^{(k)}})\\ + \overline{N}(r, \frac{1}{f^n[P(f)]^{(k)} - 1}) + S(r, f)\end{aligned}$$

using Lemma 2.3, we get,

$$\begin{aligned}(n+m-k)T(r,f) \leq mN(r,f) + (k+1)\overline{N}(r,f) + \overline{N}(r, \frac{1}{f}) + mN(r, \frac{1}{f})\\ + k\overline{N}(r,f) + \overline{N}(r, \frac{1}{f^n[P(f)]^{(k)} - 1}) + S(r,f).\end{aligned} \tag{23}$$

By hypothesis, we have $\overline{N}(r,f)=S(r,f)$, $\overline{N}(r,\frac{1}{f})=S(r,f)$
Therefore (23) becomes,

$$\begin{aligned}(n+m-k)T(r,f) &\leq mN(r,f)+mN(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f^n[P(f)]^{(k)}-1})+S(r,f)\\ &\leq 2mT(r,f)+\overline{N}(r,\frac{1}{f^n[P(f)]^{(k)}-1})+S(r,f)\\ (n-m-k)T(r,f) &\leq \overline{N}(r,\frac{1}{f^n[P(f)]^{(k)}-1})+S(r,f)\end{aligned}$$

which implies that $f^n[P(f)]^{(k)}-1$ has infinitely many zeros for $n\geq m+k+1$. This completes the proof of Theorem 1. □

Theorem 2. Let $f,g\in\mathcal{A}, n\geq m+k+4$ and k be a positive integer. If $f^n[P(f)]^{(k)}$ and $g^n[P(g)]^{(k)}$ share 1 CM, then either $f\equiv tg$ for a constant t such that $t^{n+1}=1$ or $f(z)=c_3e^{pz}$, $g(z)=c_4e^{-pz}$ where c_3,c_4 and p are constants such that $(-1)^{(k)}(c_3c_4)^{n+1}p^{2k}=1$.

Proof. By hypothesis, $f^n[P(f)]^{(k)}$ and $g^n[P(g)]^{(k)}$ share 1 CM. Let

$$H(z)=\frac{f^n[P(f)]^{(k)}-1}{g^n[P(g)]^{(k)}-1} \tag{24}$$

Then $H(z)$ is a meromorphic function satisfying $T(r,H)=O(T(r,f)+T(r,g))$, by the first fundamental theorem and Lemma 2.1.
From (24), we see that the zeros and poles of $H(z)$ are multiple and satisfy

$$\begin{aligned}\bar{N}(r,H) &\leq \bar{N}_L(r,f)\\ \bar{N}(r,\frac{1}{H}) &\leq \bar{N}_L(r,g)\end{aligned} \tag{25}$$

Let

$$\begin{aligned}f_1 &= f^n[P(f)]^{(k)}\\ f_2 = -Hg^n[P(g)^{(k)}], &\ \ f_3=H\end{aligned} \tag{26}$$

then by using (24), we easily see that

$$\begin{aligned}f_1+f_2+f_3 &= f^n[P(f)]^{(k)}-Hg^n[P(g)]^{(k)}+H\\ &= f^n[P(f)]^{(k)}-H[g^n[P(g)]^{(k)}-1]\\ &= f^n[P(f)]^{(k)}-\left(\frac{f^n[P(f)]^{(k)}-1}{g^n[P(g)]^{(k)}-1}\right)[g^n[P(g)]^{(k)}-1]\\ &= 1.\end{aligned}$$

Assuming that f_1 is non-constant and by Lemma 2.2, we have

$$\begin{aligned}
&\sum_{j=1}^{3} N_2(r,\frac{1}{f_j})+\sum_{j=1}^{3}\bar{N}(r,f_j)\\
&=N_2(r,\frac{1}{f_1})+N_2(r,\frac{1}{f_2})+N_2(r,\frac{1}{f_3})+\bar{N}(r,f_1)\\
&+\bar{N}(r,f_2)+\bar{N}(r,f_3)\\
&\le N_2(r,\frac{1}{f^n[P(f)]^{(k)}})+N_2(r,\frac{1}{g^n[P(g)]^{(k)}})+N_2(r,\frac{1}{H})\\
&+\bar{N}(r,f^n[P(f)]^{(k)})+\bar{N}(r,g^n[P(g)]^{(k)})+\bar{N}(r,H).
\end{aligned} \tag{27}$$

Noting that

$$\bar{N}(r,f^n[P(f)]^{(k)})\le mN(r,f)+(k+1)\bar{N}(r,f)$$

$$\bar{N}(r,g^n[P(f)]^{(k)})\le mN(r,g)+(k+1)\bar{N}(r,g)$$

using this with (25) and Lemma 2.3, (27) becomes

$$\begin{aligned}
&\sum_{j=1}^{3} N_2(r,\frac{1}{f_j})+\sum_{j=1}^{3}\bar{N}(r,f_j)\\
&=2\bar{N}(r,\frac{1}{f})+N(r,\frac{1}{[P(f)]^{(k)}})\\
&+2\bar{N}(r,\frac{1}{g})+N(r,\frac{1}{[P(g)]^{(k)}}+(k+1)\bar{N}(r,f)+mN(r,g)\\
&+(k+1)\bar{N}(r,g)+\bar{N}(r,H)\\
&\le 2\bar{N}(r,\frac{1}{f})+mN(r,\frac{1}{f})+k\bar{N}(r,f)\\
&+2\bar{N}(r,\frac{1}{g})+mN(r,\frac{1}{g})+k\bar{N}(r,g)+2\bar{N}_L(r,g)\\
&+mN(r,f)+(k+1)\bar{N}(r,f)+mN(r,g)+(k+1)\bar{N}(r,g)\\
&+\bar{N}_L(r,f)+S(r,f)+S(r,g)\\
&=2(\bar{N}(r,\frac{1}{f})+\bar{N}(r,\frac{1}{g}))+m(N(r,\frac{1}{f})+N(r,\frac{1}{g}))\\
&+(2k+1)(\bar{N}(r,f)+\bar{N}(r,g))+m(N(r,f)+N(r,g))\\
&+2\bar{N}_L(r,g)+\bar{N}_L(r,f)+S(r,f)+S(r,g).
\end{aligned} \tag{28}$$

Since $f,g\in\mathcal{A}$, we have $\bar{N}(r,f)+\bar{N}(r,\frac{1}{f})=S(r,f)$
$\bar{N}(r,g)+\bar{N}(r,\frac{1}{g})=S(r,g)$

Therefore

$$\begin{aligned} &\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \bar{N}(r, f_j) \\ &\leq m(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) + m(N(r, f) + N(r, g)) + 2\bar{N}_L(r, g) \\ &+ \bar{N}_L(r, f) + S(r, f) + S(r, g) \end{aligned} \tag{29}$$

Noting that

$$2\bar{N}_L(r, g) + \bar{N}_L(r, f) \leq 2\bar{N}(r, f) = S(r, f)$$

or

$$2\bar{N}_L(r, g) + \bar{N}_L(r, f) \leq 2\bar{N}(r, g) = S(r, g).$$

Thus (29) becomes

$$\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \bar{N}(r, f_j) \leq 2m(T(r, f) + T(r, g)) + S(r, f) + S(r, g).$$

for $m = 1$, using Lemma 2.9 and 2.10, we get

$$\begin{aligned} \sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \bar{N}(r, f_j) &\leq 2mT(r, f) + 2m\frac{(n+m-k)}{(n-m-k)}T(r, f) + S(r, f) \\ &= \left[2m + 2m(\frac{n+m-k}{n-m-k})\right] T(r, f) + S(r, f) \\ &= 2m \left[1 + \frac{n+m-k}{n-m-k}\right] T(r, f) + S(r, f) \\ &= 2m \left[\frac{n-m-k+n+m-k}{n-m-k}\right] T(r, f) + S(r, f) \\ &= \frac{4m(n-k)}{n-m-k}T(r, f) + S(r, f) \\ &\leq \frac{4m(n-k)}{(n-m-k)(n+m-k)}T(r) + S(r, f) \\ &\leq \left(\frac{4m(n-k)}{(n-m-k)(n+m-k)} + O(1)\right) T(r). \end{aligned}$$

Since $n \geq m + k + 4$, $\frac{4m(n-k)}{(n-m-1)(n+m-1)} < 1$, using Lemma 2.2, we get $f_2 = 1$ or $f_3 = 1$. Next we consider two cases:

Case 1. $f_2 = 1$ i.e., $-Hg^n[P(g)]^{(k)} = 1$ using (24) we have

$$\frac{f^n[P(f)]^{(k)} - 1}{g^n[P(g)]^{(k)} - 1} g^n[P(g)]^{(k)} = 1$$

by simple computing, we get

$$f^n[P(g)]^{(k)} g^n[P(g)]^{(k)} = 1.$$

By Lemma 2.11, we get the conclusion of Theorem 2.
Case 2. $f_3 = 1$ i.e., $H = 1$ using (24), we have

$$\frac{f^n[P(f)]^{(k)} - 1}{g^n[P(g)]^{(k)} - 1} = 1$$

i.e.,

$$f^n[P(f)]^{(k)} = g^n[P(g)]^{(k)}. \tag{30}$$

By Lemma 2.1, we have

$$T(r, f^n[P(f)]^{(k)}) = T(r, g^n[P(g)]^{(k)})$$

$$(n+m)T(r, f) = (n+m)T(r, g)$$

$$T(r, f) = T(r, g) \tag{31}$$

and also

$$S(r, f) = S(r, g). \tag{32}$$

Let $h = \frac{g}{f}$. Then by (30), we have

$$h^n = \frac{[P(f)]^{(k)}}{[P(g)]^{(k)}},$$

$$h^{(n+1)} = \frac{g[P(f)]^{(k)}}{f[P(g)]^{(k)}}.$$

Suppose that h is not a constant.
By (31), we have

$$\begin{aligned} T(r, h) &= T(r, \frac{g}{f}) \\ &\leq T(r, g) + T(r, f) + O(1) \\ &\leq 2T(r, f) + O(1). \end{aligned}$$

Which implies that

$$S(r, h) = S(r, f).$$

Similarly

$$S(r, h) = S(r, g).$$

Thus, by (32)

$$S(r, h) = S(r, f) = S(r, g).$$

By the first fundamental theorem and Lemma 2.6, we have

$$\begin{aligned} T(r,h^{(n+1)}) &= T\left(r,\frac{g[P(f)]^{(k)}}{f[P(g)]^{(k)}}\right) \\ (n+1)T(r,h) &\leq T(r,\frac{[P(f)]^{(k)}}{f}) + T(r,\frac{g}{[P(g)]^{(k)}}) + O(1) \\ &= T(r,\frac{[P(f)]^{(k)}}{f}) + T(r,\frac{[P(g)]^{(k)}}{g}) + O(1) \\ &= S(r,f) + S(r,g) \\ &= S(r,h). \end{aligned}$$

Which is a contradiction since $n \geq m+k+4$. Therefore h is a constant. Since f and g are transcendental meromorphic functions, we have $h \neq 0$.
Let $t = \frac{1}{h}$, which implies that $f = tg$, From (30), we obtain $t^{n+1} = 1$. This completes the proof of the Theorem 2. □

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