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POSITIVE SOLUTION FOR A CLASS OF NONLOCAL ELLIPTIC SYSTEM WITH MULTIPLE PARAMETERS AND SINGULAR WEIGHTS

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Abstract. This study is concerned with the existence of positive solution for the following nonlinear elliptic system

$$\begin{cases} -M_1\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right) div(|x|^{-ap}|\nabla u|^{p-2}\nabla u) \\ \qquad\qquad = |x|^{-(a+1)p+c_1}\Big(\alpha_1 A_1(x)f(v)+\beta_1 B_1(x)h(u)\Big),\ x\in\Omega, \\ -M_2\left(\int_\Omega |x|^{-bq}|\nabla v|^q\,dx\right) div(|x|^{-bq}|\nabla v|^{q-2}\nabla v) \\ \qquad\qquad = |x|^{-(b+1)q+c_2}\Big(\alpha_2 A_2(x)g(u)+\beta_2 B_2(x)k(v)\Big),\ x\in\Omega, \\ u=v=0,\ x\in\partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $0\in\Omega,\ 1<p,\ q<N,\ 0\le a<\frac{N-p}{p},\ 0\le b<\frac{N-q}{q}$ and α_i,β_i,c_i are positive parameters. Here $M_i, A_i, B_i, f, g, h, k$ are continuous functions and we discuss the existence of positive solution when they satisfy certain additional conditions. Our approach is based on the sub and super solutions method.

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1. Introduction

In this paper we study the existence of positive solution for the nonlocal and nonlinear elliptic system

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$$
\begin{cases}
-M_1\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right) div(|x|^{-ap}|\nabla u|^{p-2}\nabla u) \\
\qquad\qquad = |x|^{-(a+1)p+c_1}\Big(\alpha_1 A_1(x)f(v)+\beta_1 B_1(x)h(u)\Big),\ x\in\Omega, \\
-M_2\left(\int_\Omega |x|^{-bq}|\nabla v|^q\,dx\right) div(|x|^{-bq}|\nabla v|^{q-2}\nabla v) \\
\qquad\qquad = |x|^{-(b+1)q+c_2}\Big(\alpha_2 A_2(x)g(u)+\beta_2 B_2(x)k(v)\Big),\ x\in\Omega, \\
u=v=0,\ x\in\partial\Omega,
\end{cases} \tag{1}
$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $0\in\Omega, 1<p,\ q<N,\ 0\le a<\frac{N-p}{p},\ 0\le b<\frac{N-q}{q}$ and $c_1,c_2,\alpha_1,\alpha_2,\beta_1,\beta_2$ are positive parameters. Moreover $A_i,B_i:\overline{\Omega}\to\mathbb{R}$ and $M_1,M_2,f,g,h,k:[0,\infty)\to\mathbb{R}$ satisfy the following conditions:

(H1) M_1 and M_2 are two continuous and increasing functions such that

$$0<m_i\le M_i(t)\le m_{i,\infty}$$

for $i=1,2$ and all $t\in[0,\infty)$.

(H2) $A_i,B_i\in C(\overline{\Omega})$ and $A_i(x)\ge a_i>0, B_i(x)\ge b_i>0$, for $i=1,2$ and all $x\in\overline{\Omega}$

(H3) f,g,h and k are C^1 nondecreasing functions such that

$$\lim_{s\to\infty}f(s)=\lim_{s\to\infty}g(s)=\lim_{s\to\infty}h(s)=\lim_{s\to\infty}k(s)=\infty.$$

(H4) For all $N>0$,

$$\lim_{s\to\infty}\frac{f(Ng(s)^{\frac{1}{q-1}})}{s^{p-1}}=0.$$

(H5)

$$\lim_{s\to\infty}\frac{h(s)}{s^{p-1}}=\lim_{s\to\infty}\frac{k(s)}{s^{q-1}}=0$$

System (1.1) is related to the stationary problem of a model introduced by Kirchhoff [19]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{2}$$

where ρ,P_0,h,E are all constants. This equation extends the classical d'Alemberts wave equation by considering the effects of the changes in the length of the string during the vibration. A distinguishing feature of equation (2) is that the equation has a nonlocal coefficient $\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx$ which depends on the average $\frac{1}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx$; hence the equation is no longer a pointwise identity. Nonlocal problems can be used for modeling, for example, physical and biological systems for which u describes a process which depends on the average of itself, such as the population density. Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$, were motivated by the following Caffarelli, Kohn and Nirenberg's inequality (see

[9, 27]). The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [7, 14]). On the other hand, quasilinear elliptic systems has an extensive practical background. It can be used to describe the multiplicate chemical reaction catalyzed by the catalyst grains under constant or variant temperature, it can be used in the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [16, 25]) and can be a simple model of tubular chemical reaction, more naturally, it can be a correspondence of the stable station of dynamical system determined by the reaction-diffusion system, see Ladde and Lakshmikantham et al. [20]. More naturally, it can be the populations of two competing species [15]. So, the study of positive solutions of elliptic systems has more practical meanings. We refer to [1, 2, 3, 4, 5, 6, 8, 17] for additional results on elliptic problems. We are inspired by the ideas introduced in many papers such as [2, 4, 13, 24], to establish the existence of a positive weak solution for (1.1) by using sub- and supersolutions method. Our result in this paper extends the main results of [24] and [13]. In [24] the author considered (1.1) in the case $M_i(t) = A_i(x) = B_i(x) \equiv 1, i = 1, 2$ and in [13], $a = b = 0, c_1 = p, c_2 = q$. The concepts of sub and super solution were introduced by Nagumo [22] in 1937 who proved, using also the shooting method, the existence of at least one solution for a class of nonlinear Sturm-Liouville problems. In fact, the premises of the sub and super solution method can be traced back to Picard. He applied, in the early 1880s, the method of successive approximations to argue the existence of solutions for nonlinear elliptic equations that are suitable perturbations of uniquely solvable linear problems. This is the starting point of the use of sub and super solutions in connection with monotone methods. Picard's techniques were applied later by Poincaré [23] in connection with problems arising in astrophysics.

2. Preliminaries

In this paper, we denote $W_0^{1,r}(\Omega, |x|^{-ar})$, the completion of $C_0^\infty(\Omega)$, with respect to the norm $\|u\| = (\int_\Omega |x|^{-ar}|\nabla u|^r dx)^{\frac{1}{r}}$ with $r = p, q$. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -div(|x|^{-sr}|\nabla \phi|^{r-2}\nabla \phi) = \lambda |x|^{-(s+1)r+t}|\phi|^{r-2}\phi, \ x \in \Omega, \\ \phi = 0, \ x \in \partial\Omega, \end{cases} \tag{3}$$

For $r = p, s = a$ and $t = c_1$, let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2.1) such that $\phi_{1,p}(x) > 0$ in Ω and $||\phi_{1,p}||_\infty = 1$ and for $r = q, s = b$ and $t = c_2$, let $\phi_{1,q}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,q}$ of (2.1) such that $\phi_{1,q}(x) > 0$ in Ω and $||\phi_{1,q}||_\infty = 1$ (see [21, 26]). It can be shown that $\frac{\partial \phi_{1,r}}{\partial n} < 0$ on $\partial\Omega$ for $r = p, q$. Here n is the outward normal. This result is well known and hence, depending on Ω there exist positive constants $\epsilon, \delta, \sigma_p, \sigma_q$ such that

$$\lambda_{1,r}|x|^{-(s+1)r+t}\phi_{1,r}^r - |x|^{-sr}|\nabla \phi_{1,r}|^r \le -\epsilon, \ x \in \overline{\Omega}_\delta, \tag{4}$$

$$\phi_{1,r} \geq \sigma_r, \ x \in \Omega \setminus \overline{\Omega}_\delta, \tag{5}$$

with $r = p, q; s = a, b; t = c_1, c_2$ and $\overline{\Omega}_\delta = \{x \in \Omega | d(x, \partial\Omega) \leq \delta\}$(see [21]).

We will also consider the unique solution

$$\big(\zeta_p(x), \zeta_q(x)\big) \in W_0^{1,p}(\Omega, \|x\|^{-ap}) \times W_0^{1,q}(\Omega, \|x\|^{-bq})$$

for the system:

$$\begin{cases} -div(|x|^{-ap}|\nabla\zeta_p|^{p-2}\nabla\zeta_p) = |x|^{-(a+1)p+c_1}, \ x \in \Omega, \\ -div(|x|^{-bq}|\nabla\zeta_q|^{q-2}\nabla\zeta_q) \ = |x|^{-(b+1)q+c_2}, \ x \in \Omega, \\ \zeta_p = \zeta_q = 0, \ x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $\zeta_r(x) > 0$ in Ω and $\frac{\partial\zeta_r(x)}{\partial n} < 0$ on $\partial\Omega$, for $r = p, q$ (see [21]).

We shall establish our existence result via sub and super solutions method. A pair of nonnegative functions (ψ_1, ψ_2) and (z_1, z_2) are called a weak subsolution and weak supersolution of (1.1) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$\begin{aligned}
M_1\left(\int_\Omega |x|^{-ap}|\nabla\psi_1|^p\,dx\right)&\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w dx \\
&\leq \int_\Omega |x|^{-(a+1)p+c_1}\big(\alpha_1 A_1(x)f(\psi_2) + \beta_1 B_1(x)h(\psi_1)\big)w dx, \\
M_2\left(\int_\Omega |x|^{-bq}|\nabla\psi_2|^q\,dx\right)&\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w dx \\
&\leq \int_\Omega |x|^{-(b+1)q+c_2}\big(\alpha_2 A_2(x)g(\psi_1) + \beta_2 B_2(x)k(\psi_2)\big)w dx, \\
M_1\left(\int_\Omega |x|^{-ap}|\nabla z_1|^p\,dx\right)&\int_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1.\nabla w dx \\
&\geq \int_\Omega |x|^{-(a+1)p+c_1}\big(\alpha_1 A_1(x)f(z_2) + \beta_1 B_1(x)h(z_1)\big)w dx, \\
M_2\left(\int_\Omega |x|^{-bq}|\nabla z_2|^q\,dx\right)&\int_\Omega |x|^{-bq}|\nabla z_2|^{q-2}\nabla z_2.\nabla w dx \\
&\geq \int_\Omega |x|^{-(b+1)q+c_2}\big(\alpha_2 A_2(x)g(z_1) + \beta_2 B_2(x)k(z_2)\big)w dx
\end{aligned}$$

for all $x \in \Omega$ and $w \in W = \Big\{w \in C_0^\infty(\Omega) | w \geq 0, x \in \Omega\Big\}$.

A key role in our arguments will be played by the following auxiliary result. Its proof is similar to those presented in [12], the reader can consult further the papers[1, 2, 3, 18].

Lemma 2.1. *Assume that $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous and increasing, and there exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \in \mathbb{R}_0^+$. If the functions*

$u, v \in W_0^{1,p}(\Omega, |x|^{-ap})$ satisfy

$$M\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx$$
$$\leq M\left(\int_\Omega |x|^{-ap}|\nabla v|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \qquad (6)$$

for all $\varphi \in W_0^{1,p}(\Omega, |x|^{-ap})$, $\varphi \geq 0$, then $u \leq v$ in Ω.

From Lemma 2.1 we can establish the basic principle of the sub and super solutions method for nonlocal systems. Indeed, we consider the following nonlocal system

$$\begin{cases} -M_1\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-(a+1)p+c_1}h(x,u,v), \text{ in } \Omega, \\ -M_2\left(\int_\Omega |x|^{-bq}|\nabla v|^q\,dx\right)\operatorname{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v) = |x|^{-(b+1)q+c_2}k(x,u,v), \text{ in } \Omega, \\ u = v = 0, \text{ on } \partial\Omega, \end{cases} \qquad (7)$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and $h, k : \overline{\Omega}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfy the following conditions

(HK1) $h(x,s,t)$ and $k(x,s,t)$ are Carathéodory functions and they are bounded if s, t belong to bounded sets.

(KH2) There exists a function $g : \mathbb{R} \to \mathbb{R}$ being continuous, nondecreasing, with $g(0) = 0$, $0 \leq g(s) \leq C(1 + |s|^{\min\{p,q\}-1})$ for some $C > 0$, and applications $s \mapsto h(x,s,t) + g(s)$ and $t \mapsto k(x,s,t) + g(t)$ are nondecreasing, for a.e. $x \in \Omega$.

If $u, v \in L^\infty(\Omega)$, with $u(x) \leq v(x)$ for a.e. $x \in \Omega$, we define

$$[u, v] = \Big\{w \in L^\infty(\Omega) : \ u(x) \leq w(x) \leq v(x) \text{ for a.e. } x \in \Omega\Big\}.$$

Using Lemma 2.1 and the method as in the proof of Theorem 2.4 of [21] (see also Section 4 of [10]), we can establish a version of the abstract sub and super solution method for our class of the operators as follows.

Proposition 2.2. *Let $M_1, M_2 : \mathbb{R}_0^+ \to \mathbb{R}^+$ be two functions satisfying the condition **(H1)**. Assume that the functions h, k satisfy the conditions (HK1) and (HK2). Assume that $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$, are respectively, a weak subsolution and a weak supersolution of system (7) with $\underline{u}(x) \leq \overline{u}(x)$ and $\underline{v}(x) \leq \overline{v}(x)$ for a.e. $x \in \Omega$. Then there exists a minimal (u_*, v_*) (and, respectively, a maximal (u^*, v^*)) weak solution for system (7) in the set $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. In particular, every weak solution $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ of system (7) satisfies $u_*(x) \leq u(x) \leq u^*(x)$ and $v_*(x) \leq v(x) \leq v^*(x)$ for a.e. $x \in \Omega$.*

3. Existence results

Now we are ready to state and proof our existence result as follows.

Theorem 3.1. *Assume **(H1)-(H5)** hold. Then there exists a positive weak solution of system (1.1) when $\alpha_1 a_1 + \beta_1 b_1$ and $\alpha_2 a_2 + \beta_2 b_2$ are large enough.*

Proof. Since f, g, h, k are continuous and nondecreasing, we have $f(s), g(s), h(s), k(s) \geq -k_0$ for all $s \geq 0$ and for some$k_0 > 0$. Choose $r > 0$ such that

$$r \leq \mathrm{Min}\Big\{|x|^{-(a+1)p+c_1}, |x|^{-(b+1)q+c_2}\Big\}, \forall x \in \overline{\Omega}_\delta.$$

Now we let:

$$\begin{cases} \psi_1(x) = \left[\frac{(\alpha_1 a_1+\beta_1 b_1)k_0 r}{\epsilon m_{1,\infty}}\right]^{\frac{1}{p-1}} \left(\frac{p-1}{p}\right) \phi_{1,p}^{\frac{p}{p-1}}(x) \\ \psi_2(x) = \left[\frac{(\alpha_2 b_2+\beta_2 b_2)k_0 r}{\epsilon m_{2,\infty}}\right]^{\frac{1}{q-1}} \left(\frac{q-1}{q}\right) \phi_{1,q}^{\frac{q}{q-1}}(x) \end{cases} \tag{8}$$

for all $x \in \Omega$ and verify that (ψ_1, ψ_2) is a weak subsolution of (1.1). Let $w \in W$, then a calculation shows that

$$\begin{aligned} &M_1\left(\int_\Omega |x|^{-ap}|\nabla\psi_1|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w dx \\ &= M_1\left(\int_\Omega |x|^{-ap}|\nabla\psi_1|^p\,dx\right)\left(\frac{(\alpha_1 a_1+\beta_1 b_1)k_0 r}{\epsilon m_{1,\infty}}\right) \\ &\times \int_\Omega |x|^{-ap}\phi_{1,p}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}.\nabla w dx \\ &= M_1\left(\int_\Omega |x|^{-ap}|\nabla\psi_1|^p\,dx\right)\left(\frac{(\alpha_1 a_1+\beta_1 b_1)k_0 r}{\epsilon m_{1,\infty}}\right) \\ &\times \int_\Omega |x|^{-ap}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\big[\nabla(\phi_{1,p}w) - \nabla\phi_{1,p}w\big]dx, \\ &\leq \left(\frac{(\alpha_1 a_1+\beta_1 b_1)k_0 r}{\epsilon}\right)\int_\Omega \Big[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\Big]w dx. \end{aligned}$$

Similarly

$$\begin{aligned} &M_2\left(\int_\Omega |x|^{-bq}|\nabla\psi_2|^q\,dx\right)\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w dx \\ &\leq \left(\frac{(\alpha_2 a_2+\beta_2 b_2)k_0 r}{\epsilon}\right)\int_\Omega \Big[\lambda_{1,q}|x|^{-(b+1)q+c_2}\phi_{1,q}^q - |x|^{-bq}|\nabla\phi_{1,q}|^q\Big]w dx. \end{aligned}$$

First we consider the case $x \in \overline{\Omega}_\delta$. We have

$$\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p \leq -\epsilon,\ x \in \overline{\Omega}_\delta.$$

Since $\psi_1, \psi_2 \geq 0$ in Ω it follows that

$$-k_0 r \leq \mathrm{Min}\Big\{|x|^{-(a+1)p+c_1}f(\psi_2), |x|^{-(a+1)p+c_1}h(\psi_1)\Big\},\ x \in \overline{\Omega}_\delta.$$

Hence, we have

$$\begin{aligned} &\left(\frac{(\alpha_1 a_1+\beta_1 b_1)k_0 r}{\epsilon}\right)\int_{\overline{\Omega}_\delta} \Big[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\Big]w dx \\ &\leq -(\alpha_1 a_1+\beta_1 b_1)k_0 r\int_{\overline{\Omega}_\delta} w dx \end{aligned}$$

$$\leq \int_{\overline{\Omega}_\delta} |x|^{-(a+1)p+c_1}\Big(\alpha_1 A_1(x)f(\psi_2)+\beta_1 B_1(x)h(\psi_1)\Big)wdx.$$

A similar argument shows that

$$\left(\frac{(\alpha_2 a_2+\beta_2 b_2)k_0 r}{\epsilon}\right)\int_{\overline{\Omega}_\delta}\Big[\lambda_{1,q}|x|^{-(b+1)p+c_2}\phi_{1,q}^q-|x|^{-bq}|\nabla\phi_{1,q}|^q\Big]wdx$$
$$\leq \int_{\overline{\Omega}_\delta} |x|^{-(b+1)q+c_2}\Big(\alpha_2 A_2(x)g(\psi_1)+\beta_2 B_2(x)k(\psi_2)\Big)wdx.$$

On the other hand, on $\Omega\setminus\overline{\Omega}_\delta$, since $\phi_{1,p}\geq\sigma_p, \phi_{1,q}\geq\sigma_q$ for some $0<\sigma_p,\sigma_q<1$, if $\alpha_1 a_1+\beta_1 b_1$ and $\alpha_2 a_2+\beta_2 b_2$ are large, then by **(H3)** we have

$$f(\psi_2),h(\psi_1),g(\psi_1),k(\psi_2)\geq\frac{k_0 r}{\epsilon}\ \mathrm{Max}\Big\{\lambda_{1,p},\lambda_{1,q}\Big\}.$$

Hence

$$\left(\frac{(\alpha_1 a_1+\beta_1 b_1)k_0 r}{\epsilon}\right)\int_{\Omega\setminus\overline{\Omega}_\delta}\Big[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p-|x|^{-ap}|\nabla\phi_{1,p}|^p\Big]wdx$$
$$\leq\left(\frac{(\alpha_1 a_1+\beta_1 b_1)k_0 r}{\epsilon}\right)\int_{\Omega\setminus\overline{\Omega}_\delta}\lambda_{1,p}|x|^{-(a+1)p+c_1}wdx$$
$$\leq \int_{\Omega\setminus\overline{\Omega}_\delta} |x|^{-(a+1)p+c_1}\Big(\alpha_1 A_1(x)f(\psi_2)+\beta_1 B_1(x)h(\psi_1)\Big)wdx.$$

Similarly

$$\left(\frac{(\alpha_2 a_2+\beta_2 b_2)k_0 r}{\epsilon}\right)\int_{\Omega\setminus\overline{\Omega}_\delta}\Big[\lambda_{1,q}|x|^{-(b+1)q+c_2}\phi_{1,q}^q-|x|^{-bq}|\nabla\phi_{1,q}|^q\Big]wdx$$
$$\leq \int_{\Omega\setminus\overline{\Omega}_\delta} |x|^{-(b+1)q+c_2}\Big(\alpha_2 A_2(x)g(\psi_1)+\beta_2 B_2(x)k(\psi_2)\Big)wdx.$$

Hence

$$M_1\left(\int_\Omega |x|^{-ap}|\nabla\psi_1|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla wdx$$
$$\leq\int_\Omega |x|^{-(a+1)p+c_1}\Big(\alpha_1 A_1(x)f(\psi_2)+\beta_1 B_1(x)h(\psi_1)\Big)wdx,$$
$$M_2\left(\int_\Omega |x|^{-bq}|\nabla\psi_2|^q\,dx\right)\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla wdx$$
$$\leq\int_\Omega |x|^{-(b+1)q+c_2}\Big(\alpha_2 A_2(x)g(\psi_1)+\beta_2 B_2(x)k(\psi_2)\Big)wdx,$$

i.e., (ψ_1,ψ_2) is a weak subsolution of (1.1).

Now, we will prove there exists a N large enough so that

$$\big(z_1(x),z_2(x)\big)=\left(N\zeta_p(x),\left[\frac{\alpha_2\|A_2\|_\infty+\beta_2\|B_2\|_\infty}{m_2}\right]^{\frac{1}{q-1}}\big[g(N\|\zeta_p\|_\infty)\big]^{\frac{1}{q-1}}\zeta_q(x)\right),$$

is a supersolution of (1.1). A calculation shows that:

$$\begin{aligned}
& M_1\left(\int_\Omega |x|^{-ap}|\nabla z_1|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1.\nabla w dx\\
&= M_1\left(\int_\Omega |x|^{-ap}|\nabla z_1|^p\,dx\right)N^{p-1}\int_\Omega |x|^{-ap}|\nabla\zeta_p|^{p-2}\nabla\zeta_p.\nabla w dx\\
&\geq m_1 N^{p-1}\int_\Omega |x|^{-ap}|\nabla\zeta_p|^{p-2}\nabla\zeta_p.\nabla w dx\\
&= m_1 N^{p-1}\int_\Omega |x|^{-(a+1)p+c_1} w dx.
\end{aligned}$$

By **(H3)-(H5)** we can choose N large enough so that

$$\begin{aligned}
m_1 N^{p-1} &\geq \alpha_1\|A_1\|_\infty f\left(\left[\frac{\alpha_2\|A_2\|_\infty+\beta_2\|B_2\|_\infty}{m_2}\right]^{\frac{1}{q-1}}\|\zeta_q\|_\infty\big[g(N\|\zeta_p\|_\infty)\big]^{\frac{1}{q-1}}\right)\\
&\quad+\beta_1\|B_1\|_\infty h(N\|\zeta_p\|_\infty)\\
&\geq \alpha_1 f\left(\left[\frac{\alpha_2\|A_2\|_\infty+\beta_2\|B_2\|_\infty}{m_2}\right]^{\frac{1}{q-1}}\zeta_q(x)\big[g(N\|\zeta_p\|_\infty)\big]^{\frac{1}{q-1}}\right)\\
&\quad+\beta_1 h(N\zeta_p(x))\\
&= \alpha_1 A_1(x) f(z_2)+\beta_1 B_1(x) h(z_1).
\end{aligned}$$

Hence

$$\begin{aligned}
& M_1\left(\int_\Omega |x|^{-ap}|\nabla z_1|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1.\nabla w dx\\
&\quad\geq \int_\Omega |x|^{-(a+1)p+c_1}\Big(\alpha_1 A_1(x) f(z_2)+\beta_1 B_1(x) h(z_1)\Big) w dx.
\end{aligned}$$

Next, by **(H3)**, **(H5)** and for N large enough we have

$$\begin{aligned}
g(N\|\zeta_p\|_\infty) &\geq k\left(\left[\frac{\alpha_2\|A_2\|_\infty+\beta_2\|B_2\|_\infty}{m_2}\right]^{\frac{1}{q-1}}\big[g(N\|\zeta_p\|_\infty)\big]^{\frac{1}{q-1}}\|\zeta_q\|_\infty\right)\\
&\geq k(z_2).
\end{aligned}$$

Hence

$$\begin{aligned}
& M_2\left(\int_\Omega |x|^{-bq}|\nabla z_2|^q\,dx\right)\int_\Omega |x|^{-bq}|\nabla z_2|^{q-2}\nabla z_2.\nabla w dx\\
&= M_2\left(\int_\Omega |x|^{-bq}|\nabla z_2|^q\,dx\right)\left(\frac{\alpha_2\|A_2\|_\infty+\beta_2\|B_2\|_\infty}{m_2}\right) g\big(N\|\zeta_p\|_\infty\big)\\
&\times\int_\Omega |x|^{-bq}|\nabla\zeta_q|^{q-2}\nabla\zeta_q.\nabla w dx\\
&\geq \Big(\alpha_2\|A_2\|_\infty+\beta_2\|B_2\|_\infty\Big) g\big(N\|\zeta_p\|_\infty\big)\int_\Omega |x|^{-(b+1)q+c_2}\, w dx
\end{aligned}$$

$$\geq \int_{\Omega} |x|^{-(b+1)q+c_2} \Big[\alpha_2 \|A_2\|_\infty g(z_1) + \beta_2 \|B_2\|_\infty g\big(N\|\zeta_p\|_\infty\big) \Big] w dx$$
$$\geq \int_{\Omega} |x|^{-(b+1)q+c_2} \Big[\alpha_2 A_2(x) g(z_1) + \beta_2 B_2(x) k(z_2) \Big] w dx,$$

This relations show that (z_1, z_2) is a weak supersolution of (1.1) with $\psi_i \leq z_i$ for $i = 1, 2$ and N large enough. Thus, by Proposition 2.2 there exists a positive weak solution (u, v) of (1.1) such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$. This completes the proof of Theorem 3.1.

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