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ON DEGENERATE q-TANGENT POLYNOMIALS OF HIGHER ORDER

C.S. RYOO

Abstract. In this paper, we introduce degenerate tangent numbers $\mathcal{T}_{n,q}^{(k)}(\lambda)$ and tangent polynomials $\mathcal{T}_{n,q}^{(k)}(x,\lambda)$ of higher order. Finally, we obtain interesting properties of these numbers and polynomials.

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1. Introduction

L. Carlitz introduced the degenerate Bernoulli polynomials(see [1]). Feng Qi *et al.*[2] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. T. Kim studied the Barnes' type multiple degenerate Bernoulli and Euler polynomials(see [3]), C. S. Ryoo worked in the area of the tangent numbers and tangent polynomials(see [4, 5, 6, 7, 8]). Recently, Ryoo introduced the degenerate tangent numbers and tangent polynomials(see [6]). In this paper, we introduce degenerate tangent numbers $\mathcal{T}_{n,q}^{(k)}(\lambda)$ and tangent polynomials $\mathcal{T}_{n,q}^{(k)}(x,\lambda)$ of higher order. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ denotes the set of complex numbers. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

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the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows(see [3]):

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0) \tag{1.2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [9])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.4}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.5}$$

First, using multiple of p-adic integral, we introduce q-tangent polynomials of higher order $T_{n,q}^{(k)}(x)$: For $k \in \mathbb{N}$, we define

$$\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!} = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}} q^{x_1+\cdots+x_k} e^{(x+2x_1+2x_2+\cdots+2x_k)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k). \tag{1.6}$$

By (1.2), tangent polynomials of higher order, $T_{n,q}^{(k)}(x)$ are defined by means of the following generating function

$$\left(\frac{2}{qe^{2t}+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!}. \tag{1.7}$$

When $x = 0$, $T_{n,q}^{(k)}(0) = T_{n,q}^{(k)}$ are called the q-tangent numbers of higher order(see [5]). In [5], we studied tangent numbers $T_{n,q}^{(k)}$ polynomials $T_{n,q}^{(k)}(x)$ of higher order and investigate their properties.

Theorem 1.1. *For positive integers n and $k \in \mathbb{N}$, we have*

$$T_{n,q}^{(k)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-times} q^{x_1+\cdots+x_k}(x + 2x_1 + \cdots + 2x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k),$$

$$T_{n,q}^{(k)} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-times} q^{x_1+\cdots+x_k}(2x_1 + \cdots + 2x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).$$

2. Degenerate tangent polynomials of higher order

In this section, we assume that $q \in \mathbb{C}_p$. We introduce degenerate tangent polynomials of higher order, $\mathcal{T}_{n,q}^{(k)}(x,\lambda)$. We use the notation

$$\sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} = \sum_{k_1,\ldots,k_n=0}^{m}.$$

Let us assume that $t, \lambda \in \mathbb{Z}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$. Now, using multiple of p-adic integral, we introduce q-tangent polynomials $\mathcal{T}_{n,q}^{(k)}(x,\lambda)$ of higher order : For $k \in \mathbb{N}$, we define

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k)}(x,\lambda)\frac{t^n}{n!}\\ &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} q^{x_1+\cdots+x_k}(1+\lambda t)^{\frac{(x+2x_1+2x_2+\cdots+2x_k)}{\lambda}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).\end{aligned} \tag{2.1}$$

When $x = 0$, $\mathcal{T}_{n,q}^{(k)}(0,\lambda) = \mathcal{T}_{n,q}^{(k)}(\lambda)$ are called the degenerate q-tangent numbers of higher order. By (1.2) and (2.1), we get

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} q^{x_1+\cdots+x_k}(1+\lambda t)^{\frac{(x+2x_1+2x_2+\cdots+2x_k)}{\lambda}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)\\ &= \left(\frac{2}{q(1+\lambda t)^{2/\lambda}+1}\right)^k (1+\lambda t)^{x/\lambda}.\end{aligned} \tag{2.2}$$

By (2.1) and (2.2), degenerate q-tangent polynomials of higher order, $\mathcal{T}_{n,q}^{(k)}(x,\lambda)$ are defined by means of the following generating function

$$\left(\frac{2}{q(1+\lambda t)^{2/\lambda}+1}\right)^k (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k)}(x,\lambda)\frac{t^n}{n!}. \tag{2.3}$$

Thus, by (2.3) and (1.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathcal{T}_{n,q}^{(k)}(x,\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\left(\frac{2}{q(1+\lambda t)^{2/\lambda}+1}\right)^k(1+\lambda t)^{x/\lambda}\\ &= \left(\frac{2}{qe^{2t}+1}\right)^k e^{xt} \\ &= \sum_{n=0}^{\infty}T_{n,q}^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 2.1. *For positive integers n, we have*

$$\lim_{\lambda\to 0}\mathcal{T}_{n,q}^{(k)}(x,\lambda)=T_{n,q}^{(k)}(x).$$

By (2.1), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(k)}(x,\lambda)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}q^{x_1+\cdots+x_k}\left(\frac{x+2x_1+\cdots+2x_k}{\lambda}\right)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\frac{\lambda^n t^n}{n!}.\end{aligned} \tag{2.5}$$

By (1.5) and (2.5), we have the following theorem.

Theorem 2.2. *For positive integers n and $k\in\mathbb{N}$, we have*

$$\begin{aligned}&\mathcal{T}_{n,q}^{(k)}(x,\lambda)\\ &=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-times}q^{x_1+\cdots+x_k}\left(x+2x_1+\cdots+2x_k|\lambda\right)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).\end{aligned} \tag{2.6}$$

Corollary 2.3. *For positive integers n, we have*

$$\mathcal{T}_{n,q}^{(k)}(\lambda)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{x_1+\cdots+x_k}(2x_1+\cdots+2x_k|\lambda)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).$$

We observe that

$$(2x_1+\cdots+2x_k|\lambda)_n=\sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)(2x_1+\cdots+2x_k)^l, \tag{2.7}$$

Thus, by (2.5), (2.7), and Theorem 1.1, we have the following theorem.

Theorem 2.4. *For positive integers n and $k \in \mathbb{N}$, we have*

$$\mathcal{T}_{n,q}^{(k)}(x,\lambda) = \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) T_{l,q}^{(k)}(x).$$

By (2.3) and (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k)}(x,\lambda)\frac{t^n}{n!} &= \left(\frac{2}{q(1+\lambda t)^{2/\lambda}+1}\right)^k (1+\lambda t)^{x/\lambda}\\ &= \left(\sum_{m=0}^{\infty} \mathcal{T}_{m,q}^{(k)}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l \frac{t^l}{l!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,q}^{(k)}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\mathcal{T}_{n,q}^{(k)}(x,\lambda) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,q}^{(k)}(\lambda)(x|\lambda)_{n-l}.$$

From (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k)}(x+y,\lambda)\frac{t^n}{n!} &= \left(\frac{2}{q(1+\lambda t)^{2/\lambda}+1}\right)^k (1+\lambda t)^{(x+y)/\lambda}\\ &= \left(\frac{2}{q(1+\lambda t)^{2/\lambda}+1}\right)^k (1+\lambda t)^{x/\lambda}(1+\lambda t)^{y/\lambda}\\ &= \left(\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k)}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n \frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,q}^{(k)}(x,\lambda)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Therefore, by (2.8), we have the following theorem.

Theorem 2.6. *For $n \in \mathbb{Z}_+$, we have*

$$\mathcal{T}_{n,q}^{(k)}(x+y,\lambda) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,q}^{(k)}(x,\lambda)(y|\lambda)_{n-l}.$$

From Theorem 2.6, we note that $\mathcal{T}_{n,q}^{(k)}(x,\lambda)$ is a Sheffer sequence.

By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (2.3), we obtain

$$\begin{aligned}\left(\frac{2}{qe^{2t}+1}\right)^k e^{xt} &= \sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(k)}(x,\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n\frac{1}{n!}\\ &= \sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(k)}(x,\lambda)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\lambda^m\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{T}_{n,q}^{(k)}(x,\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned}\tag{2.9}$$

Thus, by (2.9) and (1.7), we have the following theorem.

Theorem 2.7. *For $n\in\mathbb{Z}_+$, we have*

$$T_{m,q}^{(k)}(x)=\sum_{n=0}^{m}\lambda^{m-n}\mathcal{T}_{n,q}^{(k)}(x,\lambda)S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (1.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty}T_{n,q}^{(k)}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} &= \left(\frac{2}{q(1+\lambda t)^{2d/\lambda}+1}\right)^k(1+\lambda t)^{x/\lambda}\\ &= \sum_{m=0}^{\infty}\mathcal{T}_{m,q}^{(k)}(x,\lambda)\frac{t^m}{m!},\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}T_{n,q}^{(k)}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!}\\ &=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{T}_{n,q}^{(k)}(x,\lambda)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}.\end{aligned}\tag{2.11}$$

Thus, by (2.10) and (2.11), we have the following theorem.

Theorem 2.8. *For $n\in\mathbb{Z}_+$, we have*

$$\mathcal{T}_{m,q}^{(k)}(x,\lambda)=\sum_{n=0}^{m}\lambda^{m-n}T_{n,q}^{(k)}(x)S_1(m,n).$$

Finally, we obtain distribution relation of degenerate tangent polynomials of higher order as follows: For $d \in \mathbb{N}$ with $d \equiv 1(\bmod 2)$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k)}(x,\lambda)\frac{t^n}{n!} \\
&= \left(\frac{2}{q(1+\lambda t)^{2/\lambda}+1}\right)\cdots\left(\frac{2}{q(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} \\
&= \left(\frac{2}{q(1+\lambda t)^{2d/\lambda}+1}\right)^k \\
&\quad\times \sum_{a_1,\cdots,a_k=0}^{d-1} (-1)^{a_1+\cdots+a_k} q^{a_1+\cdots+a_k} (1+\lambda t)^{\left(\frac{2a_1+\ldots+2a_k+x}{d}\right)(dt)}.
\end{aligned}$$

From the above, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k)}(x,\lambda)\frac{t^n}{n!} \\
&= \sum_{a_1,\ldots,a_k=0}^{d-1} (-1)^{a_1+\cdots+a_k} q^{a_1+\cdots+a_k} \sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k)}\left(\frac{2a_1+\ldots+2a_k+x}{d}, \frac{\lambda}{d}\right)\frac{(dt)^n}{n!}.
\end{aligned}$$

By comparing coefficients of $\dfrac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 2.9. *For $m \in \mathbb{N}$ with $m \equiv 1(\bmod 2)$, we have*

$$\mathcal{T}_{n,q}^{(k)}(x,\lambda) = m^n \sum_{a_1,\ldots,a_k=0}^{m-1} (-1)^{a_1+\cdots+a_k} q^{a_1+\cdots+a_k} \mathcal{T}_{n,q}^{(k)}\left(\frac{2a_1+\ldots+2a_k+x}{m}, \frac{\lambda}{m}\right).$$

Letting $\lambda \to 0$ in Theorem 2.9 gives the theorem

$$T_{n,q}^{(k)}(x) = m^n \sum_{a_1,\ldots,a_k=0}^{m-1} (-1)^{a_1+\cdots+a_k} q^{a_1+\cdots+a_k} T_{n,q}^{(k)}\left(\frac{2a_1+\cdots+2a_k+x}{m}\right)$$

which was proved by Ryoo [5, Theorem 2.4].

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C.S. Ryoo received Ph.D. degree from Kyushu University. His research interests focus on the numerical verification method, scientific computing and p-adic functional analysis.

Department of Mathematics, Hannam University, Daejeon, 306-791, Korea
e-mail:ryoocs@hnu.kr