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HEXAVALENT NORMAL EDGE-TRANSITIVE CAYLEY GRAPHS OF ORDER A PRODUCT OF THREE PRIMES

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ABSTRACT. The Cayley graph $\Gamma = Cay(G, S)$ is called normal edge-transitive if $N_A(R(G))$ acts transitively on the set of edges of Γ, where $A = Aut(\Gamma)$ and $R(G)$ is the regular subgroup of A. In this paper, we determine all hexavalent normal edge-transitive Cayley graphs on groups of order pqr, where $p > q > r > 2$ are prime numbers.

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1. Introduction

Xu was the first mathematician who proposed the concept of normal Cayley graph [15] and then Wang et al. [14] obtained all disconnected normal Cayley graphs. Recently, the normality of edge-transitive Cayley graphs is considered by mathematicians and one of the standard problems in this area is to determine the normal edge-transitivity of Cayley graphs with specific orders, see [2, 4, 11, 14]. Baik et al. in [2] studied normal edge-transitivity of Cayley graphs on abelian groups of valency at most five and Bosma et al. in [3] also considered the edge-transitive Cayley graphs of valency four on non-abelian simple groups. In [5, 12] authors obtained all tetravalent normal edge-transitive Cayley graphs on either a group of odd order or a finite non-abelian simple group. Recently, Kovács [11] classified all connected tetravalent non-normal arc-transitive Cayley graphs on dihedral groups and Darafsheh et al. [4] studied the normal edge-transitive Cayley graphs on non-abelian groups of order $4p$, where p is a prime number. In this paper, we consider the hexavalent normal edge-transitive Cayley graphs on groups of order pqr, where $p > q > r > 2$ are prime numbers.

Here, in the next section, we give the necessary definitions and some preliminary results. In section three, we compute the full automorphism group of group

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$F_{p,q}$ and then we determine the normality of related Cayley graph. Finally, in section four, we verify the normal edge-transitivity of hexavalent Cayley graphs of order pqr. Here, our notation is standard and mainly taken from the standard books of algebraic graph theory such as [7].

2. Definitions and Preliminaries

In this section, we introduce some basic notation and terminology used throughout the paper. All graphs considered here are finite, simple, undirected and connected. The vertex set, the edge set and the automorphism group of graph Γ are denoted by $V(\Gamma), E(\Gamma)$ and $Aut(\Gamma)$, respectively. For a finite group G, the generating subset S is symmetric if $1 \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = Cay(G, S)$ on G with respect to S has the vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{(g, sg) | g \in G, s \in S\}$. By this definition Γ always is connected. The Cayley graph $\Gamma = Cay(G, S)$ is normal if $G \trianglelefteq Aut(\Gamma)$.

The regular subgroup of $Aut(\Gamma)$ is $R(G) = \{\rho_g : G \to G, \rho_g(x) = xg, \forall x \in G\}$. One can prove easily that for every $g \in G$, $\rho_g \in Aut(\Gamma)$ and $R(G) \cong G$. Define now $Aut(G, S) = \{\alpha \in Aut(G), \alpha(S) = S\}$, then $Aut(G, S)$ is a subgroup of $Aut(G)$ which fixes the subset S. Let $A = Aut(\Gamma)$, a Cayley graph Γ is called normal edge-transitive or normal arc-transitive if $N_A(R(G))$ acts transitively on the set of edges or arcs of Γ, respectively. If Γ is normal edge-transitive, but not normal arc-transitive, then it is called normal half- arc-transitive. The main theorems of this paper are based on two following fundamental results:

Proposition 2.1. [2, 13], *Let $\Gamma = Cay(G, S)$ be a connected Cayley graph on S. Then Γ is normal edge-transitive if and only if $Aut(G, S)$ is either transitive on S or has two orbits in S in the form of T and T^{-1}, where T is a non-empty subset of S and $S = T \cup T^{-1}$.*

Corollary 2.2. *Let $\Gamma = Cay(G, S)$ and H be the subset of all involutions of the group G. If $\langle H \rangle \neq G$ and Γ is connected normal edge-transitive, then its valency is even.*

Corollary 2.3. *Let Γ is connected normal edge-transitive Cayley graph, then all elements of S have the same order.*

For given graphs Γ_1 and Γ_2 their Cartesian product $\Gamma_1 \square \Gamma_2$ is defined as the graph on the vertex set $V(\Gamma_1) \times V(\Gamma_2)$, where two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if and only if either $([u_1 = v_1$ and $u_2v_2 \in E(\Gamma_2)])$ or $([u_2 = v_2$ and $u_1v_1 \in E(\Gamma_1)])$. Let also G and H be two groups, then the direct product of G and H is denoted by $G \times H$.

For given graphs Γ_1 and Γ_2 we define their direct product $\Gamma_1 \boxtimes \Gamma_2$ as the graph on the vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if and only if $u_1v_1 \in E(\Gamma_1)$ and $u_2v_2 \in E(\Gamma_2)$.

Theorem 2.4. [1] *Let* $\Gamma_1 = Cay(G, S_1)$ *and* $\Gamma_2 = Cay(H, S_2)$ *be two Cayley graphs. Then the Cartesian product* $\Gamma_1 \Box \Gamma_2$ *is the Cayley graph of the direct product* $G \times H$ *with the generating subset* $(S_1, 1) \cup (1, S_2)$.

Theorem 2.5. [1] *Let* $\Gamma_1 = Cay(G, S_1)$ *and* $\Gamma_2 = Cay(H, S_2)$ *be two Cayley graphs, then the direct product* $\Gamma_1 \boxtimes \Gamma_2$ *is the Cayley graph of* $G \times H$ *with the generating subset* $S_1 \times S_2$.

Theorem 2.6. *Let* $\Gamma_1 = Cay(G, S_1)$, $\Gamma_2 = Cay(H, S_2)$ *and* $gcd(|G|, |H|) = 1$, *then*

i) $Aut(G \times H, S) = Aut(G, S_1) \times Aut(H, S_2)$ *where* $S = (S_1, 1) \cup (1, S_2)$.
ii) $Aut(G \times H, S_1 \times S_2) = Aut(G, S_1) \times Aut(H, S_2)$.

Proof. *i*) Since G, H are finite and $gcd(|G|, |H|) = 1$, it is a well-known fact that

$$Aut(G \times H) = Aut(G) \times Aut(H)$$

and hence

$$Aut(G \times H, S) = \{\sigma = (\alpha, \beta) \in Aut(G) \times Aut(H); \sigma(S) = S\}.$$

But for all $s \in S$, $s = (x, 1)$ or $s = (1, y)$ where $x \in S_1$ and $y \in S_2$. On the other hand, $(\alpha, \beta)(x, 1) = (\alpha(x), 1)$ and $(\alpha, \beta)(1, y) = (1, \beta(y))$. It's immediate that $(\alpha, \beta)(x, 1),\ (\alpha, \beta)(1, y) \in S$ if and only if $\alpha(x) \in S_1$ and $\beta(y) \in S_2$ for all $x \in S_1,\ y \in S_2$ if and only if $\alpha \in Aut(G, S_1)$ and $\beta \in Aut(G, S_2)$. This implies that

$$Aut(G \times H, S) = Aut(G, S_1) \times Aut(H, S_2).$$

ii) It is not difficult to see that for $S = S_1 \times S_2$ we have

$$Aut(G \times H, S) = \{(\alpha, \beta) \in Aut(G) \times Aut(H) : (\alpha, \beta)(S) = S\}.$$

This means that for for all $(x, y) \in S$ we have

$$\begin{aligned} Aut(G \times H, S) &= \{(\alpha, \beta) \in Aut(G) \times Aut(H) : (\alpha, \beta)(x, y) \in S_1 \times S_2\} \\ &= \{(\alpha, \beta) \in Aut(G) \times Aut(H) : (\alpha(x), \beta(y)) \in S_1 \times S_2\} \\ &= \{(\alpha, \beta) \in Aut(G) \times Aut(H) : \alpha(x) \in S_1, \beta(y) \in S_2\} \\ &= \{(\alpha, \beta) \in Aut(G) \times Aut(H) : \alpha \in Aut(G, S_1), \beta \in (H, S_2)\} \\ &= Aut(G, S_1) \times Aut(H, S_2). \end{aligned}$$

□

It is a well-known fact that, if H acts on Γ where K acts on Ω and $\Gamma \cap \Omega = \emptyset$, then $H \times K$ acts on disjint union $\Gamma \dot{\cup} \Omega$ as follows:

$$x^{(h,k)} = \begin{cases} x^h & x \in \Gamma \\ x^k & x \in \Omega \end{cases}.$$

This yields that It is easy to see that

$$|Fix_{\Gamma \dot{\cup} \Omega}((g, h))| = |Fix_{\Gamma}(g)| + |Fix_{\Omega}(h)|. \tag{1}$$

Suppose the number of orbits of two actions $(G|\Gamma)$ and $(H|\Omega)$ are t and s, respectively. According to Burnside-Cauchi Theorem and by using Eq.(1), we can verify that the number of orbits of $G \times K$ under the action $(H \times K|\Gamma\dot{\cup}\Omega)$ is $t+s$. Also, $H \times K$ acts on $\Gamma \times \Omega$ by $(a,b)^g = (a^g, b^g)$ and then

$$Fix_{\Gamma\times\Omega}((g,h)) = Fix_{\Gamma}(g) \times Fix_{\Omega}(h).$$

This implies that number of orbits of the action $(G \times H|\Gamma \times \Omega)$ is ts. Thus, we can deduce the following proposition:

Proposition 2.7. *Let $(G|X)$ and $(H|Y)$ be two transitive actions, then the action $(G \times H|X \times Y)$ is transitive while $(G \times H|X\dot{\cup}Y)$ has two orbits.*

Corollary 2.8. *Let $\Gamma_1 = Cay(G,S_1)$, $\Gamma_2 = Cay(H,S_2)$ be normal edge-transitive Cayley graphs and $S = S_1 \times S_2$. Suppose at least one of graphs Γ_1 or Γ_2 is normal arc-transitive, then $\Gamma_1 \boxtimes \Gamma_2$ is normal edge transitive.*

Proof. Use Proposition 2.7. □

Corollary 2.9. *Let $\Gamma_1 = Cay(G,S_1), \Gamma_2 = Cay(H,S_2)$ be normal edge-transitive Cayley graphs but not arc-transitive and $S = (S_1,1) \cup (1,S_2)$, then $\Gamma = Cay(G \times H, S)$ is not normal edge-transitive.*

Proof. First assume that Γ_1 and Γ_2 are normal edge-transitive and two actions $Aut(G,S_1)$ on S_1 and $Aut(H,S_2)$ on S_2 are transitive, respectively. Let $S = (S_1,1) \cup (1,S_2)$, by applying Proposition 2.7, it follows that $Aut(G \times H, S)$ has two orbits on S such as T_1 and T_2 where $T_1^{-1} \neq T_2$. Now by using Proposition 2.1, it follows that $Aut(G \times H, S)$ is not normal edge-transitive. If the action $Aut(G,S_1)$ on S_1 or the action $Aut(H,S_2)$ on S_2 has two orbits by a similar way, one can prove that $\Gamma_1 \Box \Gamma_2$ is not normal edge-transitive and this completes the proof. □

3. Main results

In the begining of this section, we compute the full automorphism group of the Frobenius group. This group plays a significant role in computing the next results of this paper. In general, a Frobenius group of order pq (p is prime and $q|p-1$) is a group of order pq by the following presentation:

$$F_{p,q} = \langle a, b : \ a^p = b^q = 1, b^{-1}ab = a^u \rangle,$$

where u is an element of order q in multiplicative group $\mathbb{Z}_p^*$.

Lemma 3.1. *We have*

$$|Aut(F_{p,q})| = p(p-1).$$

Proof. By considering the presentation of $F_{p,q}$, one can see that all elements of this group are as follows:

$$1, a, \cdots, a^{p-1},$$
$$b, ba, \cdots, ba^{p-1},$$

$$\vdots$$
$$b^{q-1}, b^{q-1}a, \cdots, b^{q-1}a^{p-1}.$$

Let $\alpha \in Aut(F_{p,q})$, necessarily $\alpha(\langle a\rangle) = \langle a\rangle$ and then there exist $1 \leq i \leq p-1$ such that $\alpha(a) = a^i$. We claim that $\alpha(b) = ba^i$, where $1 \leq i \leq p-1$. Suppose $\alpha(b) = b^r a^s$ and $\alpha(a) = a^i$ $(1 \leq s, i \leq p-1$ and $1 \leq r \leq q-1)$. Since $\alpha(bab^{-1}) = \alpha(a^u)$ where $o(u) = q$ in $\mathbb{Z}_p^*$, one can see that $b^r a^s a^i a^{-s} b^{-r} = a^{ui}$ and so $a^{u^r i} = a^{ui}$. This leads us to conclude that $r = 1$ and thus $\alpha(b) = ba^s$. One can verify that $|Aut(F_{p,q})| \leq p(p-1)$. Consequently, all automorphisms of $Aut(F_{p,q})$ are of form $\alpha_{i,j}$, where $\alpha_{i,j}(a) = a^i$ and $\alpha_{i,j}(b) = ba^j$. On the other hand, all such automorphisms are distinct and for $\alpha_{r,s}, \alpha_{i,j} \in Aut(F_{p,q})$, $\alpha_{r,s}\alpha_{i,j} = \alpha_{ir,s+jr}$. Hence, $\alpha_{r,s}\alpha_{i,j} \in Aut(F_{p,q})$ and thus $|Aut(F_{p,q})| \geq p(p-1)$. This completes the proof. □

Theorem 3.2. *We have*

$$Aut(F_{p,q}) \cong F_{p,p-1}.$$

Proof. Consider two following maps:

$$\alpha : \begin{cases} a \to a^{t^{-1}} \\ b \to b \end{cases}, \beta : \begin{cases} a \to a \\ b \to ba \end{cases},$$

where t is an element of order $p-1$ in $\mathbb{Z}_p^*$ and t^{-1} is it's inverse. One can prove that α, β are distinct automorphisms of $F_{p,q}$ where $o(\alpha) = p$, $o(\beta) = p-1$ and

$$\begin{aligned} \alpha\beta(a) &= \alpha(a^{t^{-1}}) = a^{t^{-1}} = \beta\alpha^t(a), \\ \alpha\beta(b) &= \alpha(b) = ba. \end{aligned}$$

On the other hand, $\beta\alpha^t(b) = \beta(ba^t) = b(a^{t^{-1}})^t = ba$ and so $\alpha\beta(b) = \beta\alpha^t(b)$. Hence, for all $x \in F_{p,q}$, $\alpha\beta(x) = \beta\alpha^t(x)$ and thus $\beta^{-1}\alpha\beta = \alpha^t$. Let also

$$H = \langle \alpha, \beta : \alpha^p = \beta^{p-1} = id, \beta^{-1}\alpha\beta = \alpha^v\rangle,$$

where $v^{p-1} \equiv 1 \ (mod\ p)$. Then $H \cong F_{p,p-1}$ is a subgroup of $Aut(F_{p,q})$ of order $p(p-1)$ and by Lemma 3.1, the proof is completed. □

3.1. Normal edge-transitive Cayley graphs. Here, we study the normal edge-transitivity of Cayley graph $\Gamma = Cay(G, S)$ where $G \cong F_{p,q}$. According to Corollary 2.3, all elements of S have the same order. Further, we have:

Lemma 3.3. *For $1 \leq i \leq p-1$, $a^i \notin S$.*

Proof. Suppose in the contrary that $a^i \in S$ and assume that $Aut(F_{p,q}, S)$ acts transitively on S. Since only $a^i (1 \leq i \leq p-1)$ has order p, according to Proposition 2.1 and Corollary 2.3, $S \subseteq \langle a\rangle$ and thus $F_{p,q} \subseteq \langle a\rangle$, a contradiction. Now suppose the action $Aut(F_{p,q}, S)$ on S has two orbits such that $S = T \cup T^{-1}$. Since $a^i \in S$, without loss of generality, we can take $a^i \in T$ and $a^{-i} \in T^{-1}$. By a similar way, we can show $F_{p,q} \subseteq \langle a\rangle$, a contradiction. □

Now suppose R be the subgroup of $\mathbb{Z}_p^*$ consisting of the powers of u, thus $|R| = q$. Write $r = (p-1)/q$ and choose coset representatives $v_1, \cdots, v_r$ for R in $\mathbb{Z}_p^*$. Then, the conjugacy classes of $F_{p,q}$ are as follows [9]:

$$\{1\},$$
$$(a^{v_i})^G = \{a^{v_i r} : r \in R\} \ (1 \le i \le r),$$
$$(b^n)^G = \{a^m b^n : \ 0 \le m \le p-1\}(1 \le n \le q-1).$$

Theorem 3.4. *The Cayley graph $Cay(F_{p,q}, S)$ is tetravalent normal edge-transitive if and only if*

$$S = \{b^i a^m, b^i a^n, (b^i a^m)^{-1}, (b^i a^n)^{-1}\},$$

where $1 \le i \le q-1, \ \le n, m \le p$ and p, q are prime numbers.

Proof. First, we show $o(b^i a^j) = q$, $(1 \le i \le q-1$ and $1 \le j \le p)$. To do this, note that

$$(b^i a^j)^2 \quad = \quad b^{2i}(b^{-i} a^j b^i) a^j = b^{2i} a^{j(1+u^i)}.$$

This yields that $(b^i a^j)^q = b^{qi} a^{j(1+u^i+\cdots+u^{(q-1)i})} = e$ and so $o(b^i a^j) = q$. Suppose $\Gamma = Cay(F_{p,q}, S)$ is normal edge-transitive. A according to Proposition 2.1, $Aut(F_{p,q}, S)$ has at most two orbits. First, suppose $Cay(F_{p,q}, S)$ be transitive on S. According to Corollary 2.2 , $|S|$ is even and $S \subseteq \{b^i a^j \ 1 \le i \le q, 1 \le j \le p\}$. Let $S = \{b^i a^j, b^m a^n, (b^i a^j)^{-1}, (b^m a^n)^{-1}\}$. Since, $Aut(F_{p,q}, S)$ on S is transitive, there exist $\beta \in Aut(F_{p,q})$ such that $\beta(b^i a^j) = b^m a^n$. This leads us to verify $m = i$ which yields that $S = \{b^i a^m, b^i a^n, (b^i a^m)^{-1}, (b^i a^n)^{-1}\}$. On the other hand, an automorphism of $Aut(F_{p,q})$ maps $b^i a^j$ to $a^{-j} b^{-i}$ and it follows that $i = q$, a contradiction. Hence, $Aut(F_{p,q}, S)$ is not transitive on S. Finally, suppose $S = T \cup T^{-1}$ where $T = \{b^i a^j, b^m a^n\}$. Clearly, there exists $\alpha \in Aut(F_{p,q})$ such that $\alpha(b^i a^j) = b^m a^n$ or $\alpha(b^i a^j) = a^{-n} b^{-m}$. If $\alpha(b^i a^j) = b^m a^n$, then $i = n$ and necessarily $T = \{b^i a^j, b^i a^n\}$. If $\alpha(b^i a^j) = a^{-n} b^{-m}$, then $m = -i$ and so $T = \{b^i a^j, a^{-n} b^{-i}\}$. In general, in this case we have:

$$S = \{b^i a^m, b^i a^n, (b^i a^m)^{-1}, (b^i a^n)^{-1}\},$$

and so $Aut(F_{p,q}, S)$ has two orbits on S. Conversely, if S is as above, then $Cay(F_{p,q}, S)$ is normal edge-transitive and this completes the proof. □

4. Normal edge-transitive Cayley graph of order pqr

Let G be a group of order pqr, where $p > q > r > 2$ are prime numbers, the aim of this section is to compute the normal edge-transitivity of G. In [8] the presentations of groups of order $pqr(p > q > r)$ are introduced and in [6] it is proved that all groups of order pqr are isomorphic to exactly one of the following presentations:

- $G_1 = \mathbb{Z}_{pqr}$,
- $G_2 = F_{p,qr}(qr|p-1)$,
- $G_3 = \mathbb{Z}_r \times F_{p,q}(q|p-1)$,
- $G_4 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$,

- $G_5 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$,
- $G_{i+5} = \langle a, b, c : a^p = b^q = c^r = 1, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^i}\rangle$, where $r|p-1, q-1, o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \leq i \leq r-1)$.

By using Corollary 2.8 and Theorem 3.4, the Cayley graph $\Gamma_i = Cay(G_i, S_i)$ $(1 \leq i \leq 5)$ is normal edge-transitive, but the main problem is finding a normal symmetric generating subset with exactly four elements. To do this consider the following cases:

- $G_1 = \mathbb{Z}_{pqr}$, let x, y are two generator of this group, then a symmetric normal generating subset of G_1 is $S = \{x, y, x^{-1}, y^{-1}\}$ and hence the automorphism group $\varphi : G_1 \to G_1$ with $\varphi(a) = a^{-1}$ (for all $a \in G_1$) is in $Aut(G_1, S)$. This means that Γ_1 is tetravalent normal edge-transitive.
- $G_2 = F_{p,qr}(qr|p-1)$, according to Theorem 3.4, Γ_2 is tetravalent normal edge-transitive.
- $G_3 = \mathbb{Z}_r \times F_{p,q}(q|p-1)$, this group has the following presentation:

$$\langle a, b, c : a^p = b^q = c^r = 1, b^{-1}ab = a^u, bc = cb, ac = ca\rangle,$$

where $o(u) = q$ in $\mathbb{Z}_p^*$. Let $\Gamma_3 = Cay(G_3, S_2)$ and Γ_3 is tetravalent normal edge-transitive. Clearly a^i, b^j, $c^k \notin S$ and hence we can suppose $S \subseteq \{c^ib^ja^k, (c^ib^ja^k)^{-1}, c^xb^la^s, (c^xb^la^s)^{-1}\}$ where $1 \leq i, x \leq r-1$, $1 \leq j, l \leq p-1$ and $1 \leq s, k \leq q-1$. It is easy to see that $G_3 = \langle S\rangle$. Since $gcd(r, pq) = 1$, one can see that $Aut(G_3) \cong Aut(\mathbb{Z}_r) \times Aut(F_{p,q})$. This implies that for every $\alpha \in Aut(G_3)$, we have $\alpha(c) = c^i (1 \leq i \leq r-1)$. On the other hand, $\alpha|_{F_{p,q}} \in Aut(F_{p,q})$ is as given in Theorem 3.4. In other words, all automorphisms of $Aut(G_3)$ are as follows:

$$\alpha : \begin{cases} a \to a^{t^{-1}} \\ b \to b \\ c \to c^f \end{cases}, \beta : \begin{cases} a \to a \\ b \to ba \\ c \to c^f \end{cases},$$

where t is an element of order $p-1$ in $\mathbb{Z}_p^*$, t^{-1} is it's inverse and $f \in \{1, 2, \cdots, r-1\}$. This implies that

$$Aut(G_3) = \mathbb{Z}_{r-1} \times F_{p,p-1}.$$

Suppose there is an automorphism $\theta \in Aut(G_3)$ where $\theta(c^ib^ja^k) = (c^ib^ja^k)^{-1}$. Hence, $c^{if}b^ja^{kt^{-1}} = c^{-i}a^{-k}b^{-j}$ or $c^{if}(ba)^ja^k = c^{-i}a^{-k}b^{-j}$ which implies that $j = 0$, a contradiction. This yields that the action of $Aut(G_3, S)$ on S is not transitive. Thus, necessarily $\theta(c^ib^ja^k) = c^xb^ma^n$. In other words, $c^{if}(ba)^ja^k = c^xb^ma^n$ or $c^{if}b^ja^{kt^{-1}} = c^xb^ma^n$. So, by these conditions, we can conclude that $if \equiv x(mod\ r)$, $j = m$ and either $k \equiv nt(mod\ p)$ or $u^{j-1} + \cdots + u + 1 + k \equiv n(mod\ p)$. Thus

$$S \subseteq \{c^ib^ja^k, (c^ib^ja^k)^{-1}, c^xb^ja^n, (c^xb^ja^n)^{-1}\},$$

where $k \equiv nt(mod\ p)$ or $u^{j-1} + \cdots + u + 1 + k \equiv n(mod\ p)$.
- For $G_4 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$ and $G_5 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$ the conditions are similar to the last case.

- Let $\Gamma = Cay(G_6, S_6)$ and Γ is tetravalent normal edge-transitive, according to Corollary 2.2, its valency is even. Hence, a minimal generating normal symmetric subset of G_6 has four elements. Suppose $S = \{c^ib^ja^k, (c^ib^ja^k)^{-1}, c^xb^la^s, (c^xb^la^s)^{-1}\}$, where $1 \leq i, x \leq r-1$, $1 \leq j, l \leq p-1$ and $1 \leq s, k \leq q-1$. First we show that $i = x = 1$. In [6], it is proved that for $k \geq 1$, we have

$$Aut(G_{5+i}) = \{\alpha : \alpha(a) = a^z, \alpha(b) = b^g, \alpha(c) = cb^ea^{h(u-1)}\},$$

where $1 \leq g \leq q-1, 0 \leq h \leq p-1, 0 \leq e \leq q-1, 1 \leq z \leq p-1$ and u is the p-th primitive root of unity. It is not difficult to see that then $i = x$. Suppose now that there is an automorphism $\alpha \in Aut(G_6)$ such that $\alpha(c^ib^ja^k) = c^xb^la^s$, where

$$\alpha : \begin{cases} a \to a^m \\ b \to b^n \\ c \to cb^ya^t \end{cases} . \tag{2}$$

This implies that $\alpha(c^ib^ja^k) = (cb^ya^t)^ib^{yj}a^{kt} = c^ib^la^s$. On the other hand,

$$cb^ya^t \underbrace{\cdots}_{i\ times} cb^ya^t = c^i \tag{3}$$

and so $c^{i^2}b^\theta a^\gamma = c^i$, where $1 \leq \theta \leq q-1$, $1 \leq \gamma \leq p-1$ are functions of variables u, v. It follows that $c^{i^2-i} = 1$ and hence $i = 1$. But $\{c^ib^ja^k, (c^ib^ja^k)^{-1}, c^xb^la^s, (c^xb^la^s)^{-1}\}$ generates G_6 if $j = k = 0$ and so $\Gamma = Cay(G_6, S)$ is tetravalent normal edge-transitive if

$$S = \{c, c^{-1}, ca^lb^s, (ca^lb^s)^{-1}\},$$

where $1 \leq l \leq p-1$ and $1 \leq s \leq q-1$. So, we proved the following theorem.

Theorem 4.1. *Let $G_1, \cdots G_{d+5} (1 \leq d \leq r-1)$ be all groups of order pqr where $p > q > r > 2$ are prime numbers and $1 \leq f \leq r-1$. The Cayley graph $\Gamma_i = Cay(G_i, S)$ is tetravalent normal edge-transitive if and only if*

(1) $G_1 = \mathbb{Z}_{pqr}$ *and* $S = \{x, y, x^{-1}, y^{-1}\}$*, where* $G_1 = \langle x, y \rangle$*,*
(2) $G_2 = F_{p,qr}(qr|p-1)$ *and* $S = \{b^ia^m, b^ia^n, (b^ia^m)^{-1}, (b^ia^n)^{-1}\}$*,*
(3) $G_3 = \mathbb{Z}_r \times F_{p,q}(q|p-1)$ *and* $S = \{c^ib^ja^k, (c^ib^ja^k)^{-1}, c^xb^ja^n, (c^xb^ja^n)^{-1}\}$*, where* $if \equiv x(mod\ r)$*,* $j = m$ *and either* $k \equiv nt(mod\ r)$ *or* $u^{j-1} + \cdots + u + 1 + k \equiv n(mod\ r)$*,*
(4) $G_4 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$ *and* $S = \{c^ib^ja^k, (c^ib^ja^k)^{-1}, c^xb^ja^n, (c^xb^ja^n)^{-1}\}$*, where* $if \equiv x(mod\ p)$*,* $j = m$ *and either* $k \equiv nt(mod\ p)$ *or* $u^{j-1} + \cdots + u + 1 + k \equiv n(mod\ p)$*,*
(5) $G_5 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$ *and* $S = \{c^ib^ja^k, (c^ib^ja^k)^{-1}, c^xb^ja^n, (c^xb^ja^n)^{-1}\}$*, where* $if \equiv x(mod\ q)$*,* $j = m$ *and either* $k \equiv nt(mod\ q)$ *or* $u^{j-1} + \cdots + u + 1 + k \equiv n(mod\ q)$*,*

(6) $G_{d+5} = \langle a, b, c : a^p = b^q = c^r = 1, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^i} \rangle$, *where* $r|p-1, q-1, o(u) = r$ *in* $\mathbb{Z}_q^*$ *and* $o(v) = r$ *in* $\mathbb{Z}_p^*$ $(1 \leq d \leq r-1)$ *and* $S = \{c, c^{-1}, ca^l b^s, (ca^l b^s)^{-1}\}$, *where* $1 \leq l \leq p-1$ *and* $1 \leq s \leq q-1$.

Theorem 4.2. *Let* $G = F_{p,qr}$, *then* $\Gamma = Cay(G, S)$ *is hexavalent normal edge-transitive if and only if*

$$S = \{b^i a^m, b^i a^n, b^i a^o, (b^i a^m)^{-1}, (b^i a^n)^{-1}, (b^i a^o)^{-1}\},$$

where $1 \leq m, n, o \leq p$ *and* $1 \leq i \leq q$.

Proof. Let $\theta \in Aut(G_2)$ such that $\theta(a) = a^f$ and $\theta(b) = ba^h$, where $1 \leq f \leq p-1$ and $0 \leq h \leq p-1$. Let $T = \{b^i a^j, b^m a^n, b^o a^s\}$, if $\theta(b^i a^j) = b^m a^n$, then $(ba^h)^i a^{jf} = b^i a^{v^i h + v^{i-1} h + ... + h + jf} = b^m a^n$ and so $i = m$. Similarly, if $\theta(b^m a^n) = b^o a^s$, then $m = o$. On the other hand, the following relations hold:

$$v^i h + v^{i-1} h + ... + h + jf \equiv n (mod\ p) \Rightarrow (j-n)f \equiv (n-s)(mod\ p),$$
$$v^i h + v^{i-1} h + ... + h + nf \equiv s (mod\ p) \Rightarrow (n-s)f \equiv (s-j)(mod\ p),$$
$$v^i h + v^{i-1} h + ... + h + sf \equiv j (mod\ p) \Rightarrow (s-j)f \equiv (j-n)(mod\ p).$$

In other words, if $f^3 \equiv 1(mod\ p)$, then $Aut(G_2, S)$ has two orbits on S. Here, we show that there is no $\alpha \in Aut(G)$ which $\alpha(b^i a^j) = a^{-m} b^{-i}$. Suppose in the contrary that there is such α, then $b^i a^{v^i h + v^{i-1} h + ... + h + jf} = b^{qr-i} a^{-mv^{qr-i}}$ and thus $-2i \equiv 0\ (mod\ qr)$, a contradiction. □

Theorem 4.3. *Let* $G_3 = \mathbb{Z}_r \times F_{p,q}$, *then* $\Gamma = Cay(G, S)$ *is hexavalent normal edge-transitive if and only if*

$$S = \{c^i b^j a^k, (c^i b^j a^k)^{-1}, c^x b^j a^d, (c^x b^j a^d)^{-1}, c^s b^j a^h, (c^s b^j a^h)^{-1}\}.$$

Proof. In the proof of Theorem 4.1, we showed that $Aut(G_3) \cong \mathbb{Z}_{r-1} \times F_{p,p-1}$. Let $S = \{c^i b^j a^k, (c^i b^j a^k)^{-1}, c^x b^l a^d, (c^x b^l a^d)^{-1}, c^s b^g a^h, (c^s b^g a^h)^{-1}\}$ and suppose θ is an automorphism on $Aut(G_3)$ such that $\theta(c^i b^j a^k) = c^x b^l a^d$, then $c^{fi} b^j a^{kt^{-1}} = c^x b^l a^d$. Hence, $if \equiv x(mod\ r)$, $j \equiv l(mod\ g)$ and $kt^{-1} \equiv d(mod\ p)$. Clearly, if $\theta(c^x b^l a^d) = c^s b^g a^h$, then $l = g = h$. On the other hand, $if \equiv x(mod\ r)$, $xf \equiv s(mod\ r)$ and $sf \equiv i(mod\ r)$ yields that $f^3 \equiv 1(mod\ r)$. Let $A = c^i b^j a^k$, $B = c^x b^j a^d$ and $C = c^s b^j a^h$, by a similar way, we can prove that $t^3 \equiv 1(mod\ p)$. If $f^3 \equiv 1(mod\ r)$ and $u + j - 1 \equiv 0(mod\ p)$, then the action $Aut(G_3, S)$ on S has two orbits T, T^{-1}. It should be noted that there is no automorphisms α or $\beta \in Aut(G_3)$ which $\alpha(A) = A^{-1}$ or B^{-1} and $\beta(A) = A^{-1}$ or B^{-1}, because in the other cases $j = 0$, a contradiction. Similarly, we can prove that there is no automorphism α or β with $\alpha(A) = B^{-1}$ or $\beta(A) = B^{-1}$. This completes the proof. □

Theorem 4.4. *Let* $G = G_{i+5}$, *then* $Cay(G, S)$ *is hexavalent normal edge-transitive if and only if*

$$S = \{c^i b^j a^k, (c^i b^j a^k)^{-1}, c^i b^l a^s, (c^i b^l a^s)^{-1}, c^i b^m a^n, (c^i b^m a^n)^{-1}\}.$$

Proof. Let $\alpha \in Aut(G)$ and $\alpha(S) = S$, where

$$S = \{c^i b^j a^k, (c^i b^j a^k)^{-1}, c^x b^l a^s, (c^x b^l a^s)^{-1}, c^o b^m a^n, (c^o b^m a^n)^{-1}\}.$$

According to Theorem 4.1, $i = 1$. We have, $e+gk \equiv s(mod\, q)$, $e+gs \equiv m(mod\, q)$ and $e + gm \equiv k(mod\ q)$ which results that $g^3 \equiv 1(mod\ q)$. On the other hand, $h(u-1) + jf \equiv l(mod\ p)$, $h(u-1) + lf \equiv n(mod\ p)$ and $h(u-1) + mf \equiv j(mod\ p)$ which verify that $f^3 \equiv 1(modp)$. In addition there is no $\alpha \in Aut(G)$ such that $\alpha(cb^j a^k) = (cb^j a^k)$. If in the contrary, there exist such an α, then $cb^{ig+l} a^{kf+h(u-1)} = c^{r-1} b^{-ju^{r-1}} a^{-kv^{i(r-1)}}$ and so $r - 2 \equiv 0(mod\ r)$. By a similar way, it can be shown that there is no $\alpha \in Aut(G)$ with $\alpha(cb^j a^k) = (cb^l a^s)^{-1}$ and so S has two orbits under the action of $Aut(G, S)$. This completes the proof. □

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