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MUIRHEAD'S AND HOLLAND'S INEQUALITIES OF MIXED POWER MEANS FOR POSITIVE REAL NUMBERS†

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ABSTRACT. We review weighted power means of positive real numbers and see their properties including the convexity and concavity for weights. We study the mixed power means of positive real numbers related to majorization of weights, which gives us an extension of Muirhead's inequality. Furthermore, we generalize Holland's conjecture to the power means.

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1. Introduction

For any vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ the $\mathbf{a}$-mean $[\mathbf{a}]$ of nonnegative real numbers $x_1, x_2, \ldots, x_n$ is defined by

$$[\mathbf{a}] = \frac{1}{n!} \sum_{\sigma} x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n},$$

where the sum is taken over all permutations σ on $\{1, 2, \ldots, n\}$. For example,

$$[(1, 0, \ldots, 0)] = \frac{1}{n} \sum_{j=1}^{n} x_j \text{ and } [(1/n, 1/n, \ldots, 1/n)] = \left(\prod_{j=1}^{n} x_j \right)^{1/n}$$

are the arithmetic mean and the geometric mean, respectively. One can see that for any probability vector $\omega = (w_1, w_2, \ldots, w_n)$ the ω-mean $[\omega]$ is a kind

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of mixed means, that is, the arithmetic mean of weighted geometric means of $x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}$.

For two vectors $\mathbf{a} = (a_1, a_2, \dots, a_n), \mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$, we say that $\mathbf{b}$ majorizes $\mathbf{a}$ if and only if

$$\sum_{j=1}^{k} a_j^{\downarrow} \leq \sum_{j=1}^{k} b_j^{\downarrow},$$
$$\sum_{j=1}^{n} a_j^{\downarrow} = \sum_{j=1}^{n} b_j^{\downarrow},$$

for all $k = 1, \dots, n-1$, where $a_j^{\downarrow}$ and $b_j^{\downarrow}$ are the elements of $\mathbf{a}$ and $\mathbf{b}$ sorted in decreasing order, respectively. Muirhead's inequality states in [9] that $[\mathbf{a}] \leq [\mathbf{b}]$ if and only if $\mathbf{b}$ majorizes $\mathbf{a}$ (see [1, 8] for more details and applications). In Section 3 we generalize the Muirhead's inequality to the power means that we review in Section 2.

F. Holland [2] introduced the following inequality for positive real numbers $x_1, x_2, \dots, x_n$,

$$\left(\prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{i} \right)^{n} \leq \frac{1}{n} \sum_{i=1}^{n} (x_1 \dots x_i)^{1/i}.$$

One can see also that each side is a kind of mixed means. That is, the left-hand side is the geometric mean of inductive arithmetic means of $x_1, x_2, \dots, x_n$, and vice versa for the right-hand side. In Section 4 we show the generalization of the Holland's inequality extended to power means.

The weighted power mean for positive definite Hermitian matrices are well defined from the matrix nonlinear equation. Furthermore, the Karcher mean (also known as the least square mean or Riemannian mean) has been shown as the limit of the power mean; see [7] for more details and properties. It would be interesting to show that our results are extended to the weighted power mean of positive definite matrices, so we discuss it in Section 5.

For convenience, we use the following notation: for any $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$

$$\begin{aligned}
\mathbf{x} \odot \mathbf{y} &:= (x_1 y_1, \dots, x_n y_n) \in \mathbb{R}^n, \\
\mathbf{x}^t &:= (x_1^t, \dots, x_n^t) \in \mathbb{R}^n \text{ for any } t \in \mathbb{R}, \\
\mathbf{x}_\sigma &:= (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in \mathbb{R}^n \text{ for any permutation } \sigma \text{ on } \{1, \dots, n\}, \\
\mathbf{x}_{\neq k} &:= (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-1} \text{ for some } k \in \{1, \dots, n\}.
\end{aligned}$$

2. Weighted power means

Let $\mathbb{R}_+^n = \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : x_j > 0 \text{ for all } j = 1, \dots, n\}$. Let $\omega = (w_1, \dots, w_n)$ be a probability vector; $w_j \geq 0$ for all $j = 1, \dots, n$ and

$\sum_{j=1}^{n} w_j = 1$. For any nonzero number p the weighted power mean of any vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, also known as the generalized mean or Hölder mean, is defined by

$$F_p(\omega; \mathbf{x}) := \left(\sum_{j=1}^{n} w_j x_j^p \right)^{1/p}.$$

One can see easily that for any vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$

$$F_p(\omega; \mathbf{x}) = \langle \omega, \mathbf{x}^p \rangle^{1/p} \tag{1}$$

for any $p \neq 0$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^n$. For $\omega = (1/n, \ldots, 1/n)$ we simply denote $F_p(\mathbf{x}) := F_p(\omega; \mathbf{x})$ for all p.

The weighted arithmetic mean and the weighted harmonic mean are all known as the special examples of weighted power mean:

$$F_1(\omega; \mathbf{x}) = \sum_{j=1}^{n} w_j x_j = \mathbb{A}(\omega; \mathbf{x}),$$

$$F_{-1}(\omega; \mathbf{x}) = \left(\sum_{j=1}^{n} w_j x_j^{-1} \right)^{-1} = \mathbb{H}(\omega; \mathbf{x}).$$

The weighted power mean F_p when $p = 0$ can be defined as its limit as $p \to 0$, which is the weighted geometric mean. In other words,

$$F_0(\omega; \mathbf{x}) := \lim_{p \to 0} F_p(\omega; \mathbf{x}) = \prod_{j=1}^{n} x_j^{w_j} = \mathbb{G}(\omega; \mathbf{x}).$$

We list the properties of weighted power means.

Lemma 2.1. *Let $\omega = (w_1, \ldots, w_n)$ be a probability vector, $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$, and $p \in \mathbb{R}$. The following are satisfied.*

- (P1) $F_p(\omega; \mathbf{x}) = x$ *for* $\mathbf{x} = (x, \ldots, x) \in \mathbb{R}_+^n$.
- (P2) $F_p(\omega; \mathbf{x} \odot \mathbf{y}) = F_p(\omega; \mathbf{x}) F_p\left(\frac{1}{\langle \omega, \mathbf{x}^p \rangle} \omega \odot \mathbf{x}^p; \mathbf{y} \right)$.
- (P3) $F_p(\omega_\sigma; \mathbf{x}_\sigma) = F_p(\omega; \mathbf{x})$ *for any permutation* σ *on* $\{1, \ldots, n\}$.
- (P4) $F_p(\omega; \mathbf{x}^q)^{1/q} = F_{pq}(\omega; \mathbf{x})$ *for any* $q \neq 0$.
- (P5) $F_p(\omega; \mathbf{x}) \leq F_p(\omega; \mathbf{y})$ *if* $x_j \leq y_j$ *for all* $j = 1, \ldots, n$.
- (P6) $F_p(\omega; \mathbf{x}) \leq F_q(\omega; \mathbf{x})$ *for* $p \leq q$.
- (P7) *For any* $t \in [0, 1]$

$$(1-t) F_p(\omega; \mathbf{x}) + t F_p(\omega; \mathbf{y}) \leq F_p(\omega; (1-t)\mathbf{x} + t\mathbf{y}) \ \textit{if}\ p \leq 1,$$
$$(1-t) F_p(\omega; \mathbf{x}) + t F_p(\omega; \mathbf{y}) \geq F_p(\omega; (1-t)\mathbf{x} + t\mathbf{y}) \ \textit{if}\ p \geq 1.$$

- (P8) $F_p(\omega; \mathbf{x}) = F_p(w_1, \ldots, w_{n-1} + w_n; \mathbf{x}_{\neq n})$ *if* $x_{n-1} = x_n$.

(P9) $F_p(\omega;\mathbf{x}) = F_p\left(1-w_n, w_n; F_p\left(\frac{1}{1-w_n}\omega_{\neq n};\mathbf{x}_{\neq n}\right), x_n\right)$.

(P10) $F_p(\omega; a_1,\dots,a_{n-1},x) = x$ *if and only if* $x = F_p\left(\frac{1}{1-w_n}\omega_{\neq n}; a_1,\dots,a_{n-1}\right)$, *where all* $a_j \in \mathbb{R}_+$.

Remark 2.1. One can see that the idempotency (P1) follows inductively from (P8), and the homogeneity $F_p(\omega;\alpha\mathbf{x}) = \alpha F_p(\omega;\mathbf{x})$ follows from (P2). Furthermore, the arithmetic-geometric-harmonic mean inequality is a special case of monotonicity for parameters (P6). In other words,

$$\mathbb{H}(\omega;\mathbf{x}) = F_{-1}(\omega;\mathbf{x}) \le \mathbb{G}(\omega;\mathbf{x}) = F_0(\omega;\mathbf{x}) \le \mathbb{A}(\omega;\mathbf{x}) = F_1(\omega;\mathbf{x}).$$

By using the definition of weighted power means and Lemma 2.1 (P8) we have

Lemma 2.2. *Let* $\omega = (w_1,\dots,w_m)$ *and* $\mu^{(i)} = (\mu_1^{(i)},\dots,\mu_n^{(i)})$ *be probability vectors for* $i = 1,\dots,m$. *For any vectors* $\mathbf{x}^{(i)} = (x_1^{(i)},\dots,x_n^{(i)})$ *in* $\mathbb{R}_+^n$,

$$\begin{aligned}&F_p(\omega; F_p(\mu^{(1)};\mathbf{x}^{(1)}),\dots,F_p(\mu^{(m)};\mathbf{x}^{(m)}))\\ &= F_p((w_1\mu_1^{(1)},\dots,w_1\mu_n^{(1)},\dots,w_n\mu_1^{(m)},\dots,w_n\mu_n^{(m)}); (x_1^{(1)},\dots,x_n^{(1)},\dots,x_1^{(m)},\dots,x_n^{(m)})).\end{aligned}$$

In particular,

$$F_p(\omega; F_p(\mu^{(1)};\mathbf{x}),\dots,F_p(\mu^{(m)};\mathbf{x})) = F_p\left(\left(\sum_{k=1}^m w_k\mu_1^{(k)},\dots,\sum_{k=1}^m w_k\mu_n^{(k)}\right);\mathbf{x}\right).$$

In Lemma 2.1 (P7) we have seen the joint concavity and convexity of weighted power means for variables: for any $t\in[0,1]$ and $\mathbf{x},\mathbf{y}\in\mathbb{R}_+^n$

$$\begin{aligned}(1-t)F_p(\omega;\mathbf{x}) + tF_p(\omega;\mathbf{y}) &\le F_p(\omega;(1-t)\mathbf{x}+t\mathbf{y}) \text{ if } p\le 1,\\ (1-t)F_p(\omega;\mathbf{x}) + tF_p(\omega;\mathbf{y}) &\ge F_p(\omega;(1-t)\mathbf{x}+t\mathbf{y}) \text{ if } p\ge 1.\end{aligned}$$

We show the joint concavity and convexity of weighted power means for weights.

Proposition 2.3. *Let* ω,μ *be probability vectors,* $t\in[0,1]$ *and* $\mathbf{x}\in\mathbb{R}_+^n$. *Then*

$$\begin{aligned}F_p((1-t)\omega+t\mu;\mathbf{x}) &\le (1-t)F_p(\omega;\mathbf{x}) + tF_p(\mu;\mathbf{x}) \textit{ for } p\le 1,\\ F_p((1-t)\omega+t\mu;\mathbf{x}) &\ge (1-t)F_p(\omega;\mathbf{x}) + tF_p(\mu;\mathbf{x}) \textit{ for } p\ge 1.\end{aligned}$$

Proof. If $p(\neq 0)\le 1$ it is enough to show that

$$F_p\left(\frac{\omega+\mu}{2};\mathbf{x}\right) \le \frac{F_p(\omega;\mathbf{x}) + F_p(\mu;\mathbf{x})}{2},$$

since the map $\omega\mapsto F_p(\omega;\mathbf{x})$ is continuous. By the fact that the real-valued function $f(x) = x^r$ for $r\ge 1$ or $r<0$ is convex, we obtain

$$\left\langle\frac{\omega+\mu}{2},\mathbf{x}^p\right\rangle^{1/p} = \left[\frac{\langle\omega,\mathbf{x}^p\rangle + \langle\mu,\mathbf{x}^p\rangle}{2}\right]^{1/p} \le \frac{\langle\omega,\mathbf{x}^p\rangle^{1/p} + \langle\mu,\mathbf{x}^p\rangle^{1/p}}{2}.$$

If $p=0$ we can prove it by taking the limit as $p\to 0$ in the inequality.

The second inequality for the case of $p \geq 1$ is proved for the concavity of the real-valued function $f(x) = x^r$ for $0 < r \leq 1$. $\square$

3. Mixed power means with majorization of weights

In this section we investigate the properties of mixed weighted power means related to a majorization of weights. We see that our result is a generalization of Muirhead's inequality.

Let $\omega = (w_1, \dots, w_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ be probability vectors. We say that μ majorizes ω (or ω is majorized by μ), denoted by $\omega \prec \mu$, if and only if

$$\sum_{j=1}^{k} w_j^{\downarrow} \leq \sum_{j=1}^{k} \mu_j^{\downarrow}$$

for all $k = 1, \dots, n-1$, where $w_j^{\downarrow}$ and $\mu_j^{\downarrow}$ are the elements of ω and μ sorted in decreasing order, respectively. One can easily see that $\sum_{j=1}^{n} w_j^{\downarrow} = 1 = \sum_{j=1}^{n} \mu_j^{\downarrow}$.

We have the useful characterization of majorization for probability vectors modified from [3, Theorem 4.3.33].

Lemma 3.1. *Let ω and μ be probability vectors. Then the following are equivalent.*

(a) *μ majorizes ω.*

(b) *There exists a probability vector $(c_1, \dots, c_{n!})$ such that*

$$\omega = \sum_{k=1}^{n!} c_k \mu_{\tau_k}, \tag{2}$$

where τ_k are permutations on $\{1, \dots, n\}$ for $k = 1, \dots, n!$.

We see how the majorization of weights is related to the mixed power means.

Lemma 3.2. *Let ω and μ be probability vectors such that $\omega \prec \mu$. Let σ_i be distinct permutations on $\{1, \dots, n\}$, where $i = 1, \dots, n!$. For any $\mathbf{y} \in \mathbb{R}_+^n$,*

(1) *$F_r(\mathbb{A}(\omega; \mathbf{y}_{\sigma_1}), \dots, \mathbb{A}(\omega; \mathbf{y}_{\sigma_{n!}})) \leq F_r(\mathbb{A}(\mu; \mathbf{y}_{\sigma_1}), \dots, \mathbb{A}(\mu; \mathbf{y}_{\sigma_{n!}}))$ if $r \geq 1$,*

(2) *$F_r(\mathbb{A}(\omega; \mathbf{y}_{\sigma_1}), \dots, \mathbb{A}(\omega; \mathbf{y}_{\sigma_{n!}})) \geq F_r(\mathbb{A}(\mu; \mathbf{y}_{\sigma_1}), \dots, \mathbb{A}(\mu; \mathbf{y}_{\sigma_{n!}}))$ if $r \leq 1$.*

Proof. Note that $f(x) = x^r$ is convex for $r \in (-\infty, 0) \cup [1, \infty)$ and concave for $r \in (0, 1]$, respectively. So we have

$$\langle \omega, \mathbf{y}_{\sigma_j} \rangle^r = \left\langle \sum_{k=1}^{n!} c_k \mu_{\tau_k}, \mathbf{y}_{\sigma_j} \right\rangle^r \leq \sum_{k=1}^{n!} c_k \langle \mu_{\tau_k}, \mathbf{y}_{\sigma_j} \rangle^r, \ r \in (-\infty, 0) \cup [1, \infty),$$

$$\langle \omega, \mathbf{y}_{\sigma_j} \rangle^r = \left\langle \sum_{k=1}^{n!} c_k \mu_{\tau_k}, \mathbf{y}_{\sigma_j} \right\rangle^r \geq \sum_{k=1}^{n!} c_k \langle \mu_{\tau_k}, \mathbf{y}_{\sigma_j} \rangle^r, \ r \in (0, 1],$$

where $\omega = \sum_{k=1}^{n!} c_k \mu_{\tau_k}$ for some probability vector $(c_1, \dots, c_{n!})$ and permutations $\tau_1, \dots, \tau_{n!}$ as in Lemma 3.1. Then

$$\left[\sum_{j=1}^{n!} \frac{1}{n!} \langle \omega, \mathbf{y}_{\sigma_j} \rangle^r \right]^{1/r} \leq \left[\sum_{j=1}^{n!} \frac{1}{n!} \sum_{k=1}^{n!} c_k \langle \mu_{\tau_k}, \mathbf{y}_{\sigma_j} \rangle^r \right]^{1/r}, \ r \in [1, \infty),$$

$$\left[\sum_{j=1}^{n!} \frac{1}{n!} \langle \omega, \mathbf{y}_{\sigma_j} \rangle^r \right]^{1/r} \geq \left[\sum_{j=1}^{n!} \frac{1}{n!} \sum_{k=1}^{n!} c_k \langle \mu_{\tau_k}, \mathbf{y}_{\sigma_j} \rangle^r \right]^{1/r}, \ r \in (-\infty, 0) \cup (0, 1].$$

Here, we have

$$\begin{aligned}
\sum_{j=1}^{n!} \frac{1}{n!} \sum_{k=1}^{n!} c_k \langle \mu_{\tau_k}, \mathbf{y}_{\sigma_j} \rangle^r &= \sum_{k=1}^{n!} c_k \sum_{j=1}^{n!} \frac{1}{n!} \langle \mu_{\tau_k}, \mathbf{y}_{\sigma_j} \rangle^r \\
&= \sum_{k=1}^{n!} c_k \sum_{j=1}^{n!} \frac{1}{n!} \langle \mu, \mathbf{y}_{\sigma_j \tau_k^{-1}} \rangle^r \\
&= \sum_{k=1}^{n!} c_k \sum_{j=1}^{n!} \frac{1}{n!} \langle \mu, \mathbf{y}_{\sigma_j} \rangle^r \\
&= \sum_{j=1}^{n!} \frac{1}{n!} \langle \mu, \mathbf{y}_{\sigma_j} \rangle^r.
\end{aligned}$$

The third equality follows from the fact that $\{\sigma_j \tau : j = 1, \dots, n!\} = \{\sigma_j : j = 1, \dots, n!\}$ for any fixed permutation τ on $\{1, \dots, n\}$.

For $r = 0$, we take the limit of the second inequality as $r \to 0$. Thus, we proved. $\square$

Theorem 3.3. *Let ω and μ be probability vectors such that $\omega \prec \mu$. Let σ_i be distinct permutations on $\{1, \dots, n\}$, where $i = 1, \dots, n!$. For any $\mathbf{x} \in \mathbb{R}_+^n$ and $p \leq q$,*

(1) $F_q(F_p(\omega; \mathbf{x}_{\sigma_1}), \dots, F_p(\omega; \mathbf{x}_{\sigma_{n!}})) \leq F_q(F_p(\mu; \mathbf{x}_{\sigma_1}), \dots, F_p(\mu; \mathbf{x}_{\sigma_{n!}}))$,
(2) $F_p(F_q(\omega; \mathbf{x}_{\sigma_1}), \dots, F_q(\omega; \mathbf{x}_{\sigma_{n!}})) \geq F_p(F_q(\mu; \mathbf{x}_{\sigma_1}), \dots, F_q(\mu; \mathbf{x}_{\sigma_{n!}}))$.

Proof. We first prove them for $0 < p \leq q$ ($p < 0 < q$, respectively). For $0 < p \leq q$ ($p < 0 < q$, respectively), (1) is equivalent that

$$\left[\sum_{k=1}^{n!} \frac{1}{n!} \langle \omega, \mathbf{x}_{\sigma_k}^p \rangle^{q/p} \right]^{p/q} \underset{(\geq)}{\leq} \left[\sum_{k=1}^{n!} \frac{1}{n!} \langle \mu, \mathbf{x}_{\sigma_k}^p \rangle^{q/p} \right]^{p/q}.$$

Replacing $\mathbf{x}^p$ by $\mathbf{y}$ and q/p by $r \geq 1$ ($r < 0$, respectively), it is equivalent that

$$\left[\sum_{k=1}^{n!} \frac{1}{n!} \langle \omega, \mathbf{y}_{\sigma_k} \rangle^r \right]^{1/r} \underset{(\geq)}{\leq} \left[\sum_{k=1}^{n!} \frac{1}{n!} \langle \mu, \mathbf{y}_{\sigma_k} \rangle^r \right]^{1/r}$$

which is proved by Lemma 3.2. Similarly, (2) is proved for $0 < p \leq q$ ($p < 0 < q$, respectively).

For $p \leq q < 0$,

$$\begin{aligned} F_q(F_p(\omega; \mathbf{x}_{\sigma_1}), \ldots, F_p(\omega; \mathbf{x}_{\sigma_{n!}})) &= F_{-q}(F_{-p}(\omega; \mathbf{x}_{\sigma_1}^{-1}), \ldots, F_{-p}(\omega; \mathbf{x}_{\sigma_{n!}}^{-1}))^{-1} \\ &\leq F_{-q}(F_{-p}(\mu; \mathbf{x}_{\sigma_1}^{-1}), \ldots, F_{-p}(\mu; \mathbf{x}_{\sigma_{n!}}^{-1}))^{-1} \\ &= F_q(F_p(\mu; \mathbf{x}_{\sigma_1}), \ldots, F_p(\mu; \mathbf{x}_{\sigma_{n!}})). \end{aligned}$$

The inequality follows from (2) for $0 < -q \leq -p$. Similarly, (2) is proved for $p \leq q < 0$.

For $p = 0$ or $q = 0$ we take the limit of the inequalities (1) and (2) as $p \to 0$ or $q \to 0$. Thus, we proved. □

Corollary 3.4. *Let ω be any probability vector, and $\mathbf{x} \in \mathbb{R}_+^n$. Let σ_i be distinct permutations on $\{1, \ldots, n\}$, where $i = 1, \ldots, n!$. For any $p \leq q$*

$$\begin{aligned} F_p(\mathbf{x}) \leq F_q(F_p(\omega; \mathbf{x}_{\sigma_1}), \ldots, F_p(\omega; \mathbf{x}_{\sigma_{n!}})) \leq F_q(\mathbf{x}), \\ F_p(\mathbf{x}) \leq F_p(F_q(\omega; \mathbf{x}_{\sigma_1}), \ldots, F_q(\omega; \mathbf{x}_{\sigma_{n!}})) \leq F_q(\mathbf{x}). \end{aligned} \tag{3}$$

Proof. Note that $\nu = (1/n, 1/n, \ldots, 1/n) \prec \omega \prec \mu = (1, 0, \ldots, 0)$ for any probability vector ω. We further have

$$F_q(F_p(\nu; \mathbf{x}_{\sigma_1}), \ldots, F_p(\nu; \mathbf{x}_{\sigma_{n!}})) = F_q(F_p(\mathbf{x}), \ldots, F_p(\mathbf{x})) = F_p(\mathbf{x})$$

by Lemma 2.1 (P1), and

$$\begin{aligned} &F_q(F_p(\mu; \mathbf{x}_{\sigma_1}), \ldots, F_p(\mu; \mathbf{x}_{\sigma_{n!}})) \\ &= F_q(1/n!, \ldots, 1/n!; x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n) = F_q(\mathbf{x}) \end{aligned}$$

by Lemma 2.1 (P8). Therefore, by Theorem 3.3

$$\begin{aligned} F_p(\mathbf{x}) &= F_q(F_p(\nu; \mathbf{x}_{\sigma_1}), \ldots, F_p(\nu; \mathbf{x}_{\sigma_{n!}})) \\ &\leq F_q(F_p(\omega; \mathbf{x}_{\sigma_1}), \ldots, F_p(\omega; \mathbf{x}_{\sigma_{n!}})) \\ &\leq F_q(F_p(\mu; \mathbf{x}_{\sigma_1}), \ldots, F_p(\mu; \mathbf{x}_{\sigma_{n!}})) = F_q(\mathbf{x}). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} F_p(\mathbf{x}) &= F_p(F_q(\mu; \mathbf{x}_{\sigma_1}), \ldots, F_q(\mu; \mathbf{x}_{\sigma_{n!}})) \\ &\leq F_p(F_q(\omega; \mathbf{x}_{\sigma_1}), \ldots, F_q(\omega; \mathbf{x}_{\sigma_{n!}})) \\ &\leq F_p(F_q(\nu; \mathbf{x}_{\sigma_1}), \ldots, F_q(\nu; \mathbf{x}_{\sigma_{n!}})) = F_q(\mathbf{x}). \end{aligned}$$

□

Remark 3.1. Let $p = 0 < q = 1$. Then Theorem 3.3 (1) gives us

$$\mathbb{A}(\mathbb{G}(\omega; \mathbf{x}_{\sigma_1}), \ldots, \mathbb{G}(\omega; \mathbf{x}_{\sigma_{n!}})) \leq \mathbb{A}(\mathbb{G}(\mu; \mathbf{x}_{\sigma_1}), \ldots, \mathbb{G}(\mu; \mathbf{x}_{\sigma_{n!}})) \tag{4}$$

if μ majorizes ω. This is the Muirhead's inequality [9], and so Theorem 3.3 is its generalization to the weighted power means.

4. Holland's conjecture extended to power means

F. Holland presented in [2] a conjecture: For any positive real numbers $x_1, x_2, \ldots, x_n$, the following inequality holds:

$$\left(\prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{i}\right)^n \leq \frac{1}{n} \sum_{i=1}^{n} (x_1 \ldots x_i)^{1/i}. \tag{5}$$

K. Kedlaya [5] has given a proof of (5). In this section we consider the Holland's conjecture extended to power means: For nonzero numbers $p \leq q$,

$$F_{p,q}(x_1, \ldots, x_n) \geq F_{q,p}(x_1, \ldots, x_n), \tag{6}$$

in which

$$\begin{aligned} F_{p,q}(x_1, \ldots, x_n) &= F_p\left(x_1, F_q(x_1, x_2), \ldots, F_q(x_1, x_2, \ldots, x_n)\right) \\ &= \left[\sum_{j=1}^{n} \frac{1}{n} \left(\frac{x_1^q + x_2^q + \cdots + x_j^q}{j}\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}. \end{aligned}$$

We introduce the following lemma from [5] which is useful for our results.

Lemma 4.1. *The vector* $\mathbf{a}(i,j) = (a_1(i,j), a_2(i,j), \ldots, a_n(i,j))$ *given by*

$$\begin{aligned} a_k(i,j) &= \binom{n-i}{j-k}\binom{j-1}{k-1} \Big/ \binom{n-1}{j-1} \\ &= \frac{(n-i)!(n-j)!(i-1)!(j-1)!}{(n-1)!(k-1)!(n-i-j+k)!(i-k)!(j-k)!} \end{aligned}$$

for $i, j = 1, 2, \ldots, n$ *satisfies the following.*

(1) $a_k(i,j) \geq 0$ *for all* i, j, k.
(2) $a_k(i,j) = 0$ *for all* $k > \min\{i, j\}$.
(3) $a_k(i,j) = a_k(j,i)$ *for all* i, j, k.
(4) $\displaystyle\sum_{k=1}^{n} a_k(i,j) = 1$ *for all* i, j.
(5) $\displaystyle\sum_{i=1}^{n} a_k(i,j) = \begin{cases} n/j & \text{for } k \leq j, \\ 0 & \text{for } k > j. \end{cases}$

From Lemma 4.1 (1) and (4), one can see that $\mathbf{a}(i,j)$ is a probability vector. We also state the following inequalities due to Minkowski (see [1], p. 31).

Lemma 4.2. *Let* x_{ij} *be positive real numbers for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$. *Then*

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} x_{ij}^p\right)^{1/p} \geq \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{m} x_{ij}\right)^p\right)^{1/p} \quad \text{if } p \in [1, \infty), \tag{7}$$

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_{ij}^{p}\right)^{1/p} \leq \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m} x_{ij}\right)^{p}\right)^{1/p} \quad \text{if } p \in (-\infty,0)\cup(0,1]. \quad (8)$$

The equalities hold if and only if $p = 1$ or the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are proportional, where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im})$.

Proposition 4.3. *Let $x_1, \ldots, x_n \in \mathbb{R}_+$. Then we have*

$$F_{p,1}(x_1,\ldots,x_n) \leq F_{1,p}(x_1,\ldots,x_n) \quad \text{if } p \in [1,\infty), \quad (9)$$

$$F_{p,1}(x_1,\ldots,x_n) \geq F_{1,p}(x_1,\ldots,x_n) \quad \text{if } p \in (-\infty,1]. \quad (10)$$

Proof. We denote $\mathbb{A}(i,j) = \mathbb{A}(\omega; x_1,\ldots,x_n)$, $F_p(i,j) = F_p(\omega; x_1,\ldots,x_n)$ the weighted arithmetic and power means obtained by setting $\omega = \mathbf{a}(i,j)$, respectively. Note that $\mathbb{A}(i,j) = F_1(i,j)$.

We first prove the case of $p \geq 1$. Using properties (2), (5) in Lemma 4.1 and the fact that $\mathbb{A}(i,j) \leq F_p(i,j)$ for $p > 1$ from Lemma 2.1 (P6), we have

$$\frac{x_1 + x_2 + \cdots + x_j}{j} = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_k(i,j)\right)x_k = \frac{1}{n}\sum_{i=1}^{n}\mathbb{A}(i,j) \leq \frac{1}{n}\sum_{i=1}^{n} F_p(i,j).$$

Taking the power mean F_p on both sides over j and Lemma 2.1 (P5) yield

$$F_{p,1}(x_1,\ldots,x_n) = \left[\sum_{j=1}^{n}\frac{1}{n}\left(\frac{x_1+\cdots+x_j}{j}\right)^{p}\right]^{1/p} \leq \left[\sum_{j=1}^{n}\frac{1}{n}\left(\sum_{i=1}^{n}\frac{1}{n}F_p(i,j)\right)^{p}\right]^{1/p}.$$

By (7), we obtain

$$\left[\sum_{j=1}^{n}\frac{1}{n}\left(\sum_{i=1}^{n}\frac{1}{n}F_p(i,j)\right)^{p}\right]^{1/p} \leq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{n}F_p(i,j)^p\right)^{1/p}.$$

Furthermore, by Lemma 4.1 (3) and (5),

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{n}F_p(i,j)^p\right)^{1/p} &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}a_k(i,j)x_k^p\right)^{1/p}\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\frac{x_k^p}{n}\sum_{j=1}^{n}a_k(i,j)\right)^{1/p}\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_1^p+\cdots+x_i^p}{i}\right)^{1/p}\\
&= F_{1,p}(x_1,\ldots,x_n).
\end{aligned}$$

The proof for $p \in (-\infty,0)\cup(0,1]$ follows the same lines as the proof for $p \geq 1$, except that another Minkowski inequality (8) is applied instead. For $p = 0$ we take the limit of the inequality (10) as $p \to 0$. $\square$

Theorem 4.4. *Let $x_1, \dots, x_n \in \mathbb{R}_+$. For $p \leq q$, we have*

$$F_{p,q}(x_1, \dots, x_n) \geq F_{q,p}(x_1, \dots, x_n). \tag{11}$$

Proof. We first prove the case for $q > 0$. Replacing x_i by x_i^p and p by q/p in Proposition 4.3, we have

$$\sum_{j=1}^{n} \frac{1}{n} \left(\frac{x_1^q + \cdots + x_j^q}{j} \right)^{p/q} \geq \left(\sum_{j=1}^{n} \frac{1}{n} \left(\frac{x_1^p + \cdots + x_j^p}{j} \right)^{q/p} \right)^{p/q} \quad \text{for } 0 < p \leq q,$$

$$\sum_{j=1}^{n} \frac{1}{n} \left(\frac{x_1^q + \cdots + x_j^q}{j} \right)^{p/q} \leq \left(\sum_{j=1}^{n} \frac{1}{n} \left(\frac{x_1^p + \cdots + x_j^p}{j} \right)^{q/p} \right)^{p/q} \quad \text{for } p < 0 < q.$$

These inequalities are equivalent with (11).

For $p \leq q < 0$, the inequality (11) for $0 < -q \leq -p$ and $F_{p,q}(\mathbf{x}) = F_{-p,-q}(\mathbf{x}^{-1})^{-1}$ from Lemma 2.1 (P4) imply

$$F_{p,q}(\mathbf{x}) = F_{-p,-q}(\mathbf{x}^{-1})^{-1} \geq F_{-q,-p}(\mathbf{x}^{-1})^{-1} = F_{q,p}(\mathbf{x}). \tag{12}$$

For $p = 0$ or $q = 0$ we take the limit of the inequalities (11) and (12) as $p \to 0$ or $q \to 0$. □

Remark 4.1. Holland's conjecture states that for any positive real numbers $x_1, x_2, \dots, x_n$, the following inequality holds:

$$F_{0,1}(x_1, \dots, x_n) \leq F_{1,0}(x_1, \dots, x_n).$$

We can see that Theorem 4.4 is a generalization of the above inequality, since

$$F_{p,q}(x_1, \dots, x_n) \geq F_{q,p}(x_1, \dots, x_n)$$

holds also for $p = 0 < 1 = q$.

5. Further Research

For the n-tuple $\mathbf{A} = (A_1, \dots, A_n)$ of positive definite matrices and $p \neq 0$ one might consider

$$\left(\sum_{j=1}^{n} w_j A_j^p \right)^{1/p}$$

as the weighted matrix power mean from the same definition of weighted power mean for positive real numbers, however, it does not satisfy the monotonicity for variables. Y. Lim and M. Pálfia have suggested in [7] a successful definition of the weighted matrix power mean $F_p(\omega; A_1, \dots, A_n)$ such as a unique positive definite solution $X > 0$ of the nonlinear matrix equation

$$\begin{aligned} X &= \sum_{j=1}^{n} w_j X \#_p A_j, \text{ if } p \in (0,1], \\ X &= \left[\sum_{j=1}^{n} w_j X^{-1} \#_{-p} A_j^{-1}\right], \text{ if } p \in [-1,0), \end{aligned} \tag{13}$$

where $A\#_p B = A^{1/2}(A^{-1/2}BA^{-1/2})^p A^{1/2}$ is the weighted geometric mean. Furthermore, it has been shown that the limit of the matrix power mean as $p \to 0$ is the Karcher mean (also known as the least square mean or Riemannian mean)

$$\arg\min_{X>0} \sum_{j=1}^{n} w_j \delta(X, A_j)^2, \tag{14}$$

where δ is the Riemannian trace distance. See [4, 6, 10] for more information and related properties of the Karcher mean.

It would be interesting to see that our results can be extended to the weighted matrix power means. For instance, the following could be considered for $0 < p \le q \le 1$.

(1) If $\omega \prec \mu$,

$$F_q(F_p(\omega;\mathbf{A}_{\sigma_1}),\dots,F_p(\omega;\mathbf{A}_{\sigma_{n!}})) \le F_q(F_p(\mu;\mathbf{A}_{\sigma_1}),\dots,F_p(\mu;\mathbf{A}_{\sigma_{n!}})),$$
$$F_p(F_q(\omega;\mathbf{A}_{\sigma_1}),\dots,F_q(\omega;\mathbf{A}_{\sigma_{n!}})) \ge F_p(F_q(\mu;\mathbf{A}_{\sigma_1}),\dots,F_q(\mu;\mathbf{A}_{\sigma_{n!}})).$$

(2) $F_{p,q}(A_1,\dots,A_n) \ge F_{q,p}(A_1,\dots,A_n)$.

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