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UNIQUENESS FOR THE POWER OF A MEROMORPHIC FUNCTION†

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ABSTRACT. In this paper, we investigate the uniqueness problem related to the power of a meromorphic functions sharing a small function with its derivative. The results in this paper improve and generalize some well known previous results.

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1. Introduction, definitions and results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. Let k be a positive integer or infinity and $a \in C \cup \{\infty\}$. Set $E(a, f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity k is counted k times in the set. If these zeros points are only counted once, then we denote the set by $\overline{E}(a, f)$. Let f and g be two nonconstant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM; if $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a, f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a, f)$ the set of distinct a-points of f with multiplicities not greater than k. It is assumed that the reader is familiar with the notations of Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $S(r, f)$ and so on, that can be found, for instance, in [2][7]. We denote by $N_{k)}(r, 1/(f - a))$ the counting function for zeros of $f - a$ with multiplicity less than or equal to k, and by $\overline{N}_{k)}(r, 1/(f - a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f - a))$ be the counting function for zeros of $f - a$ with multiplicity

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at least k and $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Let F and G be two nonconstant meromorphic functions such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $N_L\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-point of F where $p > q$; by $N_E^{1)}\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-point of F where $p = q = 1$; by $N_E^{(2}\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-point of F where $p = q \geq 2$. In the same way, we can define $N_L\left(r, \frac{1}{G-1}\right)$, $N_E^{1)}\left(r, \frac{1}{G-1}\right)$, $N_E^{(2}\left(r, \frac{1}{G-1}\right)$. Particularly, if F and G share 1 CM, then $N_L\left(r, \frac{1}{F-1}\right) = N_L\left(r, \frac{1}{G-1}\right) = 0$. With these notations, if F and G share 1 IM, it is easy to see that

$$\overline{N}\left(r, \frac{1}{F-1}\right) = N_E^{1)}\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + N_E^{(2}\left(r, \frac{1}{G-1}\right) = \overline{N}\left(r, \frac{1}{G-1}\right).$$

Mermorphic functions sharing values with their derivatives has become a subject of great interest in uniqueness theory recently. The paper by Rubel and Yang is the starting point of this topic, along with the following.

Theorem 1.1 ([5]). *Let f be a nonconstant entire function. If f and f' share two distinct finite values CM, then $f = f'$.*

Now one may ask the following question: Can we change the number 2 of shared values to 1 in the Theorem 1.1 ? The following counterexample from shows the answer is negative. Let $f = e^{e^z} \int_0^z e^{-e^t}(1-e^t)dt$. Clearly, f and f' share 1 CM but $f \neq f'$.

In order to get uniqueness theorem when a meromorphic function shared one finite value with its derivative, some additional condition might be needed. In 2003, Yu considered the uniqueness problems with deficiency condition and obtained the following result.

Theorem 1.2 ([8]). *Let f be a nonconstant entire function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > 3/4$, then $f \equiv f^{(k)}$.*

In 2010, Meng proved the following result.

Theorem 1.3 ([4]). *Let f be a nonconstant entire function, and let a be a small function of f such that $a(z) \not\equiv 0, \infty$. If $\overline{E}_{4)}(a, f) = \overline{E}_{4)}(a, f^{(k)})$ and $E_{2)}(a, f) = E_{2)}(a, f^{(k)})$ and $\delta_{2+k}(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

Recently, J.L. Zhang and L.Z. Yang considered f^n sharing a small function with its k-th derivative and got the following result.

Theorem 1.4 ([9]). *Let f be a nonconstant meromorphic function, let n, k be two positive integers satisfying $n > k + 1 + \sqrt{k+1}$. If f^n and $(f^n)^{(k)}$ share $a(z)$ CM, where $a(z)(\neq 0, \infty)$ is a small function of f, then $f^n = (f^n)^{(k)}$ and $f = ce^{\frac{\lambda}{n}z}$, where c, λ are constants and $\lambda^k = 1$.*

In 2012, X.B. Zhang proved the conclusion of Theorem 1.4 remains valid for more general meromorphic functions.

Theorem 1.5 ([12]). *Let f be a nonconstant meromorphic function, let n, k be two positive integers satisfying $n \geq m + k + 4$ where m is a nonnegative integer. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + ... + a_1 w + a_0$. If $f^n P(f)$ and $[f^n P(f)]^{(k)}$ share $a(z)$ CM, where $a(z)(\neq 0, \infty)$ is a small function of f, then $f^n P(f) = [f^n P(f)]^{(k)}$ and $f = ce^{\frac{\lambda}{n}z}$, where c, λ are constants and $\lambda^k = 1$.*

It is natural to ask whether the sharing nature of the small function $a(z)$ can be reduced to IM in Theorem 1.5. Considering this question, we prove the following results.

Theorem 1.6. *Let f be a nonconstant meromorphic function, let n, k be two positive integers satisfying $n \geq m + 2k + 14$ where m is a nonnegative integer. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + ... + a_1 w + a_0$, where $a_m \neq 0$. If $f^n P(f)$ and $[f^n P(f)]^{(k)}$ share $a(z)$ IM, where $a(z)(\neq 0, \infty)$ is a small function of f, then $f = ce^{\frac{\lambda}{n}z}$, where c, λ are constants and $\lambda^k = 1$.*

Corollary 1.7. *Let f be a nonconstant entire function, let n, k be two positive integers satisfying $n \geq m + k + 8$ where m is a nonnegative integer. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + ... + a_1 w + a_0$, where $a_m \neq 0$. If $f^n P(f)$ and $[f^n P(f)]^{(k)}$ share $a(z)$ IM, where $a(z)(\neq 0, \infty)$ is a small function of f, then $f = ce^{\frac{\lambda}{n}z}$, where c, λ are constants and $\lambda^k = 1$.*

We also prove the following results.

Theorem 1.8. *Let f be a nonconstant meromorphic function, let n, k be two positive integers satisfying $n > \frac{3}{2}k + m + 8$ where m is a nonnegative integer. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + ... + a_1 w + a_0$, where $a_m \neq 0$. If*

$$\overline{E}_{4)}(a(z), f^n P(f)) = \overline{E}_{4)}(a(z), [f^n P(f)]^{(k)})$$

and

$$E_{2)}(a(z), f^n P(f)) = E_{2)}(a(z), [f^n P(f)]^{(k)}),$$

where $a(z)(\neq 0, \infty)$ is a small function of f, then $f = ce^{\frac{\lambda}{n}z}$, where c, λ are constants and $\lambda^k = 1$.

Corollary 1.9. *Let f be a nonconstant entire function, let n, k be two positive integers satisfying $n \geq k+m+4$ where m is a nonnegative integer. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + ... + a_1 w + a_0$, where $a_m \neq 0$. If*

$$\overline{E}_{4)}(a(z), f^n P(f)) = \overline{E}_{4)}(a(z), [f^n P(f)]^{(k)})$$

and

$$E_{2)}(a(z), f^n P(f)) = E_{2)}(a(z), [f^n P(f)]^{(k)}) \,,$$

where $a(z)(\neq 0, \infty)$ is a small function of f, then $f = ce^{\frac{\lambda}{n}z}$, where c, λ are constants and $\lambda^k = 1$.

2. Some Lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \,.$$

Lemma 2.1 ([6])**.** *Let f be a nonconstant meromorphic function, and let $a_1, a_2, ..., a_n$ be finite complex numbers, $a_n \neq 0$. Then*

$$T(r, a_n f^n + \cdots + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f) \,.$$

Lemma 2.2 ([11])**.** *Let f be a nonconstant meromorphic function, k be a positive integer, then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f) \,,$$

where $N_p\left(r, \frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Clearly $\overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right)$.

Lemma 2.3 ([10])**.** *Suppose that two nonconstant meromorphic function F and G share 1 and ∞ IM. Let H be given as above. If $H \not\equiv 0$, then*

$$\begin{aligned} T(r, F) + T(r, G) &\leq 3\overline{N}(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) \\ &+ N_E^{1)}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right) \\ &+ 3N_L\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G) \,. \end{aligned}$$

Lemma 2.4 ([1])**.** *Let F, G be two nonconstant meromorphic functions such that $\overline{E}_{4)}(1, F) = \overline{E}_{4)}(1, G)$, $E_{2)}(1, F) = E_{2)}(1, G)$ and $H \not\equiv 0$, then*

$$\begin{aligned} T(r, F) + T(r, G) \leq 2\{&N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) \\ &+ N_2(r, G)\} + S(r, F) + S(r, G) \,. \end{aligned}$$

Lemma 2.5 ([3]). *Let f be a nonconstant meromorphic function and let k be a positive integer. Then*

$$N_2\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{2+k}\left(r, \frac{1}{f}\right) + S(r, f)\,.$$

3. Proof of Theorem 1.6

Let

$$F = \frac{f^n P(f)}{a}, G = \frac{[f^n P(f)]^{(k)}}{a}\,. \tag{1}$$

Then it is easy to verify F and G share 1 and ∞ IM. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.3 that

$$\begin{aligned}T(r,F) + T(r,G) \leq 3\overline{N}(r,F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right)\\ + N_E^{1)}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right)\\ + 3N_L\left(r, \frac{1}{G-1}\right) + S(r,F) + S(r,G)\,.\end{aligned} \tag{2}$$

Since

$$\begin{aligned}N_E^{1)}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{F-1}\right)\\ + 2N_L\left(r, \frac{1}{G-1}\right) \leq N\left(r, \frac{1}{G-1}\right) \leq T(r,G) + O(1)\,.\end{aligned} \tag{3}$$

We get from (2) and (3) that

$$\begin{aligned}T(r,F) \leq 3\overline{N}(r,F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right)\\ + 2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + S(r,F)\,.\end{aligned} \tag{4}$$

It's obvious that

$$\begin{aligned}2N_L\left(r, \frac{1}{F-1}\right) \leq 2N\left(r, \frac{F}{F'}\right) \leq 2N\left(r, \frac{F'}{F}\right) + S(r,f)\\ \leq 2\overline{N}(r,F) + 2\overline{N}\left(r, \frac{1}{F}\right) + S(r,f)\,,\end{aligned} \tag{5}$$

$$\begin{aligned}N_L\left(r, \frac{1}{G-1}\right) \leq N\left(r, \frac{G}{G'}\right) \leq N\left(r, \frac{G'}{G}\right) + S(r,f)\\ \leq \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f)\,.\end{aligned} \tag{6}$$

Combining (4), (5) and (6), we deduce

$$T(r,F) \le 6\overline{N}(r,F) + N_4\left(r, \frac{1}{F}\right) + N_3\left(r, \frac{1}{G}\right) + S(r,f)\,. \tag{7}$$

Then we have from Lemma 2.1 and Lemma 2.2

$$\begin{aligned} T(r,F) &= (n+m)T(r,f) + S(r,f) \le 6\overline{N}(r,f) + 4\overline{N}\left(r, \frac{1}{f}\right) \\ &+N_4\left(r, \frac{1}{P(f)}\right) + k\overline{N}(r,f) + N_{k+3}\left(r, \frac{1}{f^n P(f)}\right) + S(r,f) \\ &\le 6\overline{N}(r,f) + 4\overline{N}\left(r, \frac{1}{f}\right) + mT(r,f) + k\overline{N}(r,f) \\ &+(k+3)\overline{N}\left(r, \frac{1}{f}\right) + mT(r,f) + S(r,f)\,, \end{aligned} \tag{8}$$

that is,

$$(n-m-2k-13)T(r,f) \le S(r,f)\,, \tag{9}$$

which contradicts with $n \ge m+2k+14$. Therefore $H \equiv 0$. By integration, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B\,, \tag{10}$$

where $A \ne 0$ and B are constants. From (10) we have

$$G = \frac{(B-A)F + (A-B-1)}{BF - (B+1)}\,. \tag{11}$$

We discuss the following three cases.

Case I. Suppose that $B \ne 0, -1$. From (11), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) = \overline{N}(r,G)\,. \tag{12}$$

From the second fundamental theorem, we have

$$\begin{aligned} (n+m)T(r,f) &= T(r,F) + S(r,f) \le \overline{N}(r,F) \\ &+\overline{N}\left(r. \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) + S(r,f) \\ &\le 2\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + mT(r,f) + S(r,f)\,, \end{aligned} \tag{13}$$

which contradicts with $n \ge m+2k+14$.

Case II. Suppose that $B = 0$. From (11), we have

$$G = AF - (A-1)\,. \tag{14}$$

If $A \neq 1$, from (14) we obtain

$$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{15}$$

From the second fundamental theorem and Lemma 2.2, we have

$$\begin{aligned}(n+m)T(r,f) &= T(r,F) + S(r,f) \leq \overline{N}(r,F) \\ &+\overline{N}\left(r.\frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + mT(r,f) + k\overline{N}(r,f) \\ &+(k+1)\overline{N}\left(r, \frac{1}{f}\right) + mT(r,f) + S(r,f),\end{aligned} \tag{16}$$

which contradicts with $n \geq m+2k+14$. Thus $A = 1$. From (14) we have $F = G$. Then $f^n P(f) = [f^n P(f)]^{(k)}$. Proceeding as the proof of Lemma 7 in [12], we can prove $m = 0$ and $f = ce^{\frac{\lambda}{n}z}$, where c, λ are constants and $\lambda^k = 1$.

Case III. Suppose that $B = -1$. From (11) we have

$$G = \frac{(A+1)F - A}{F}. \tag{17}$$

If $A \neq -1$, we obtain from (17) that

$$\overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{18}$$

By the same reasoning discussed in Case II, we obtain a contradiction. Hence $A = -1$. From (17), we get $FG = 1$, that is

$$f^n P(f)[f^n P(f)]^{(k)} = a^2. \tag{19}$$

From (19) we obtain $N\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{P(f)}\right) = S(r,f)$. So

$$\begin{aligned}2(n+m)T(r,f) &\leq T(r, f^{2n}P(f)^2) + S(r,f) = T\left(r, \frac{a^2}{f^{2n}P(f)^2}\right) + S(r,f) \\ &\leq m\left(r, \frac{a^2}{f^{2n}P(f)^2}\right) + N\left(r, \frac{a^2}{f^{2n}P(f)^2}\right) + S(r,f) = S(r,f),\end{aligned} \tag{20}$$

which is a contradiction. This completes the proof of Theorem 1.6.

4. Proof of Theorem 1.8

Let

$$F = \frac{f^n P(f)}{a}, G = \frac{[f^n P(f)]^{(k)}}{a}. \tag{21}$$

Then it is easy to verify $\overline{E}_{4)}(1,F)=\overline{E}_{4)}(1,G)$ and $E_{2)}(1,F)=E_{2)}(1,G)$. Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 2.4 and Lemma 2.5 that

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F) \\ &+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\}+S(r,f) \\ \leq 8\overline{N}(r,f)+2N_2&\left(r,\frac{1}{f^nP(f)}\right)+N_2\left(r,\frac{1}{[f^nP(f)]^{(k)}}\right) \\ +T(r,G)-T(r,F)&+N_{2+k}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f)\,. \end{aligned} \tag{22}$$

By Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned} 2T(r,F)&=2(n+m)T(r,f)+S(r,f)\leq 8\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{f}\right) \\ &+2N_2\left(r,\frac{1}{P(f)}\right)+k\overline{N}(r,f)+2N_{2+k}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f) \\ &\leq 8\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{f}\right)+2N_2\left(r,\frac{1}{P(f)}\right)+k\overline{N}(r,f) \\ &+2(2+k)\overline{N}\left(r,\frac{1}{f}\right)+2N_{2+k}\left(r,\frac{1}{P(f)}\right)+S(r,f)\,. \end{aligned} \tag{23}$$

It follows that

$$(2n-3k-2m-16)T(r,f)\leq S(r,f)\,, \tag{24}$$

which contradicts with $n>\frac{3}{2}k+m+8$. Therefore $H\equiv 0$. Similar to the arguments in Theorem 1.6, we see that Theorem 1.8 holds.

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