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ON THE WEIERSTRASS THEOREM OF A MAXIMAL SPACELIKE SURFACE

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Abstract. The purpose of this paper is to show how to represent a maximal spacelike surface in L^n in terms of its generalized Guass map.

1. Introduction

It is well known that in the theory of minimal surfaces in R^3, the classical Weierstrass representation formula has played a major role [9]. The formula shows that a minimal surface in R^3 can be represented by real parts of complex integrations of holomorphic functions. The classical result is extended to a minimal surface in R^n by Hoffman and Osserman. Since a maximal surface in L^n is a counterpart of a minimal surface in R^n, it is quite natural to ask if similar representation formula of a maximal surface in L^n can be obtained. The purpose of this paper is to show how to represent a maximal spacelike surface in L^n in terms of its generalized Guass map.

2. The main result

We begin with fixing our terminology and notation. Let $L^n = (R^n, g)$ denote Lorentzian n-space with the flat Lorentzian metric g of index 1. Let M be a connected smooth orientable 2 manifold, and $X : M \longrightarrow L^n$ be a smooth imbedding of M into L^n. Throughout this paper, we assume that X is a spacelike imbedding or M is a spacelike surface in L^n, that is, the pull back X^*g of the Lorentzian metric g via X is a positive definite metric on M.

Let $M = (M, \bar{g})$ be a spacelike surface in L^n with the induced metric $\bar{g} = X^*g$ so that $X : M \longrightarrow L^n$ is an isometric imbedding. By (u_1, u_2) we always denote isothermal coordinates compatible with the orientation on M. Then the metric $\bar{g}$ is expressed locally as

$$\bar{g} = \lambda^2((du_1)^2 + (du_2)^2), \lambda > 0. \tag{1}$$

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It is well known that (u_1, u_2) is defined around each point of M, and we may regard M as a Riemann surface by introducing a complex local coordinate $z = u_1 + iu_2$.

We shall define the generalized Gauss map using local coordinates. Let M be a spacelike surface in L^n, or a Riemann surface. Locally, if u_1 and u_2 are isothermal parameters in a neighborhood of p on M, then M is defined near p by a map $X(z) = (x_1(z), \dots, x_n(z)) \in L^n$, where $z = u_1 + iu_2$. Define the generalized Gauss map Ψ by

$$\Psi(z) = \frac{\partial X}{\partial u_1} + i\frac{\partial X}{\partial u_2},$$

where $\Psi(z) \in CP^{n-1}_+ = \{Z = (z_1, \dots, z_n) \in CP^{n-1} \mid g_c(Z, Z) > 0\}$. Here g_c denotes the flat Hermitian metric in C^n with the signature $(-, +, \dots, +)$. Let us think of the effect of choosing another isothermal parameters $\tilde{u_1}$, $\tilde{u_2}$, and $\tilde{z} = \tilde{u_1} + i\tilde{u_2}$. Since the change of coordinates on a Riemann surface is analytic, we know that

$$\frac{\partial X}{\partial \tilde{u}_1} + i\frac{\partial X}{\partial \tilde{u}_2} = (\frac{\partial X}{\partial u_1} + i\frac{\partial X}{\partial u_2})(\frac{\partial u_1}{\partial \tilde{u}_1} - i\frac{\partial u_1}{\partial \tilde{u}_2}) \ ,$$

which implies $\Psi(z) = \Psi(\tilde{z})$ in CP^{n-1}_+. Since the pair of vectors $\frac{\partial X}{\partial u_1}$, $\frac{\partial X}{\partial u_2}$ are orthogonal and equal in length in L^n, it follows that

$$\frac{\partial X}{\partial u_1} + i\frac{\partial X}{\partial u_2} \in Q^{n-2}_+ \ ,$$

where $Q^{n-2}_+ = \{(z_1, \dots, z_n) \in CP^{n-1}_+ \mid -{z_1}^2 + {z_2}^2 + \dots + {z_n}^2 = 0\}$. Consequently, the generalized Gauss map Ψ is given locally by

$$(u_1, u_2) \longrightarrow \frac{\partial X}{\partial u_1} + i\frac{\partial X}{\partial u_2} \in Q^{n-2}_+ \subset CP^{n-1}_+ \ . \tag{2}$$

We may represent the Gauss map locally by

$$\Psi(z) = (\overline{\phi_1}(z), \dots, \overline{\phi_n}(z)) \ ,$$

where $\phi_k = 2\frac{\partial x_k}{\partial z} = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}$. Denote $(\phi_1, \dots, \phi_n)$ by Φ. Then Ψ is holomorphic when Φ is antiholomorphic and Ψ is antiholomorphic when Φ is holomorphic. We will consider Φ as the Gauss map instead of Ψ.

Theorem 2.1. *Let M be a spacelike surface in L^n, and Φ the Gauss map on M. Then Φ is holomorphic if and only if M is maximal.*

Proof. For a maximal surface M in L^n defined by an isometric imbedding $X : M \longrightarrow L^n$, we know that

$$0 = 2\lambda^2 H = \frac{\partial^2 X}{(\partial u_1)^2} + \frac{\partial^2 X}{(\partial u_2)^2} \ ,$$

where $\lambda^2 = g(\frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_i})$ and $u = (u_1, u_2)$ is an isothermal coordinate on M. Therefore each x_k is harmonic and $\phi_k = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}$ is analytic. For any k and

j, $\frac{\phi_k}{\phi_j}$ is analytic around p when $\phi_k(p) \neq 0$ and thus

$$\Phi : z \longrightarrow (\phi_1, \ldots, \phi_n) \in CP_{+}^{n-1}$$

is holomorphic.

Convesely, suppose Φ is holomorphic. In other words, $\frac{\phi_k}{\phi_j}$ is always holomorphic whenever the denominators do not vanish. Since $\frac{\partial X}{\partial u_1}$ and $\frac{\partial X}{\partial u_2}$ are linearly independent at any point z_0, not all $\phi_j(z_0)$ cannot vanish. Say $\phi_j(z_0) \neq 0$. Then $\gamma_k(z) = \phi_k(z)/\phi_j(z)$ is analytic near z_0. Set $\mu(z) = 1/\phi_j(z)$. Then

$$0 = \frac{\partial \mu}{\partial \bar{z}} \phi_k + \mu \frac{\partial \phi_k}{\partial \bar{z}} \quad .$$

But then

$$\begin{aligned} \frac{\partial^2 x_k}{(\partial u_1)^2} + \frac{\partial^2 x_k}{(\partial u_2)^2} &= 4 \frac{\partial^2 x_k}{\partial z \partial \bar{z}} \\ &= 2 \frac{\partial}{\partial \bar{z}} \left(2 \frac{\partial x_k}{\partial z}\right) \\ &= 2 \frac{\partial}{\partial \bar{z}} (\phi_k) \\ &= -2 \frac{1}{\mu} \frac{\partial \mu}{\partial \bar{z}} \phi_k \quad . \end{aligned}$$

Let

$$-\frac{2}{\mu} \frac{\partial \mu}{\partial \bar{z}} = f(z) + i g(z) \quad ,$$

where f and g are real. Since

$$\frac{\partial^2 x_k}{(\partial u_1)^2} + \frac{\partial^2 x_k}{(\partial u_2)^2} = -\frac{2}{\mu} \frac{\partial \mu}{\partial \bar{z}} \phi_k$$

is real, imaginary part of

$$\frac{\partial^2 x_k}{(\partial u_1)^2} + \frac{\partial^2 x_k}{(\partial u_2)^2} = (f(z) + i g(z)) \phi_k$$

must vanish. Hence

$$\frac{\partial^2 X}{(\partial u_1)^2} + \frac{\partial^2 X}{(\partial u_2)^2} = f \frac{\partial X}{\partial u_1} + g \frac{\partial X}{\partial u_2} \quad .$$

Note that $f \frac{\partial X}{\partial u_1} + g \frac{\partial X}{\partial u_2} \in T_{z_0} M$. Since

$$\frac{\partial^2 X}{(\partial u_1)^2} + \frac{\partial^2 X}{(\partial u_2)^2} = 2\lambda^2 H \in T_{z_0} M \cap T_{z_0}^{\perp} M$$

and $T_{z_0} M$ is nondegenerate, H at the given point is zero. □

We now turn to the representation of a maximal surface in terms of its Gauss map. We start with the special case of simply connected surfaces.

Let $X : M \longrightarrow L^n$ be simply connected maximal surface defined by an imbedding. Since M is simply connected Riemann surface, by the uniformization theorem, we may view M as the unit disk, the unit sphere, or the complex plane. When $M \cong S^2$, X is constant since each x_k is harmonic on the compact Riemann surface. Therefore every simply connected maximal surface is considered to be an imbedded submanifold of a simply connected domain in the complex plane.

Theorem 2.2. *Let D be a simply connected domain in the complex plane. Define an 1-1 smooth map*

$$X : D \longrightarrow L^n \;\; (n > 3)$$

in one of the following ways:

Case 1. X is the direct sum into L^2 and R^{n-2}, where $(x_3, \dots, x_n)$ defines a (immersed) minimal surface in isothermal parameters in D, and x_1, x_2 are harmonic functions such that $x_1 - x_2$ is constant in D.

Case 2. Let ψ be an arbitrary holomorphic functions in D, $\psi \not\equiv 0$, and let $g_1, \dots, g_{n-2}$ be arbitrary meromorphic functions in D such that at any p in D, the maximum order of pole at p of $g_1, \dots, g_{n-2}$ is greater than or equal to the order of pole of $\sum_{k=1}^{n-2} g_k^2$ and the same as the order of zero of ψ at p. Futhermore,

$$\sum_{k=1}^{n-2} |g_k - \overline{g_k}|(p) > 0$$

wherever $\psi(p) \neq 0$.

Set

$$\begin{aligned} \Phi &= (\phi_1, \dots, \phi_n) \\ &= \tfrac{\psi}{2}\left(\textstyle\sum_{k=1}^{n-2} g_k^2 + 1, \sum_{k=1}^{n-2} g_k^2 - 1, 2g_1, \dots, 2g_{n-2}\right) \;, \end{aligned} \tag{3}$$

and let

$$x_k = Re \int \phi_k \;\;, \quad k = 1, \dots, n \;. \tag{4}$$

Then the map $X : D \longrightarrow L^n$ defines a maximal surface in terms of isothermal parameters in D.

Conversely, every simply connected maximal surface in L^n is obtained by the above construction.

Remark 1. The two cases of the theorem are mutually exclusive. Let $\psi = \phi_1 - \phi_2$, $\phi_3 = g_1\psi, \dots, \phi_n = g_{n-2}\psi$. If $x_1 - x_2$ is constant, then $\phi_1 \equiv \phi_2$, i.e. $\psi \equiv 0$. Hence the assumption $\psi \not\equiv 0$ guarantees $x_1 - x_2$ is not constant.

Proof. Let $u = (u_1, u_2)$ be a coordinate in D. We begin with case 1. Since $x_1 - x_2 \equiv$ constant,

$$g(\frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_i}) = \sum_{k=3}^{n} (\frac{\partial x_k}{\partial u_i})^2 > 0 \tag{5}$$

by the regularity of $(x_3, \dots, x_n)$. Since $(x_3, \dots, x_n) : D \longrightarrow R^{n-2}$ defines a minimal surface in isothermal parameters u_1, u_2 in D, $\phi_3, \dots, \phi_n$ are analytic functions such that

$${\phi_3}^2 + \ldots + {\phi_n}^2 = 0 \;\;. \tag{6}$$

Since x_1, x_2 are harmonic functions such that $x_1 - x_2 \equiv$ constant, we have analytic functions $\phi_1, \dots, \phi_k$ such that $\phi_1 \equiv \phi_2$ and

$$-{\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2 + \ldots + {\phi_n}^2 = 0 \ . \tag{7}$$

Futhermore,

$$\begin{aligned} g_c(\Phi, \Phi) &= \sum_{k=3}^{n} |\phi_k|^2 \\ &= \sum_{k=3}^{n} \{(\frac{\partial x_k}{\partial u_1})^2 + (\frac{\partial x_k}{\partial u_2})^2\} > 0 \ . \end{aligned} \tag{8}$$

Therefore $X : D \longrightarrow L^n$ defines a maximal surface in an isothermal coordinate in D and its Gauss map is Φ.

As for the case 2, we know ϕ_k's are analytic everywhere and

$$\phi_k = \frac{\partial x_k}{\partial u_1} - i \frac{\partial x_k}{\partial u_2} \tag{9}$$

from (4), where $u = (u_1, u_2)$ is a coordinate in D. Direct computation using (3) shows Φ lies in the quadric $Q^{n-2} \in CP^{n-1}$. This means

$$g(\frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_1}) = g(\frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_2}) \qquad \text{and} \qquad g(\frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2}) = 0 \ .$$

We also want to show $g_c(\Phi, \Phi) > 0$. When $\psi(p) \neq 0$, all $g_1, \dots, g_{n-2}$ are analytic and $g_c(\Phi, \Phi) = \frac{|\psi|^2}{2}(\sum_{k=1}^{n-2} |g_k - \overline{g_k}|^2) > 0$ near p. When $\psi(p) = 0$, $g_c(\Phi, \Phi) \geq \sum_{k=1}^{n-2} |\psi g_k|^2$ and at least one $(\psi g_k)(p) \neq 0$. Therefore $g_c(\Phi, \Phi) > 0$ everywhere. From this fact we obtain

$$g(\frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_i}) > 0 \ , \ g(\frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2}) = 0 \ ,$$

which implies that $\frac{\partial X}{\partial u_1}$ and $\frac{\partial X}{\partial u_2}$ are linearly independent spacelike vectors. Hence $X : D \longrightarrow L^n$ defines a maximal surface in L^n in terms of an isothermal coordinate in D.

For the converse, given a simply connected maximal surface in L^n, by introducing isothermal parameters, the surface may be represented by an imbedding $X : D \longrightarrow L^n$, where D is a simply connected domain in the complex plane. The function defined by (9) will be analytic and satisfy $-{\phi_1}^2 + {\phi_2}^2 + \ldots + {\phi_n}^2 = 0$ and $g_c(\Phi, \Phi) > 0$. There are two possibilities:

1. If $\phi_1 \equiv \phi_2$, then $\sum_{k=3}^{n} {\phi_k}^2 \equiv 0$. Since $g_c(\Phi, \Phi) = \sum_{k=3}^{n} |\phi_k|^2 > 0$, the nonconstant map $(x_3, \dots, x_n) : D \longrightarrow R^{n-2}$ defines an imbedded minimal surface in isothermal parameters in D. Futhermore, x_1 and x_2 are harmonic maps such that $x_1 - x_2 \equiv$ constant. This is just the case 1.

2. If $\phi_1 \not\equiv \phi_2$, then the map $\psi = \phi_1 - \phi_2$ ia an analytic map with only isolated zeros. Define

$$g_k = \frac{\phi_{k+2}}{\psi} \ \ , k = 1, \dots, n-2. \tag{10}$$

The function g_k's are meromorphic and can only have poles where ψ vanishes. At a point p where $\psi(p) \neq 0$, it follows that

$$\begin{aligned} \Phi &= (\phi_1, \ldots, \phi_n) \\ &= \tfrac{\psi}{2}(\textstyle\sum_1^{n-2} g_k^2 + 1, \sum_1^{n-2} g_k^2 - 1, 2g_1, \ldots, 2g_{n-2}) \ , \end{aligned} \tag{11}$$

lies on the subset of Q^{n-2}, and $\frac{|\psi|}{2} \sum_{k=1}^{n-2} |g_k - \overline{g_k}| > 0$ at p. Since (3) holds everywhere except some isolated points, by continuity, it must holds at those isolated points where ψ vanishes. Finally, we will show that ψ and g_1, ... , g_{n-2} satisfy all the hypotheses. Suppose $\psi(p) = 0$. From the definition of ψ and g_k's, it is clear that the order of zero of ψ at p is greater than or equal to the maximum order of pole of g_1, ... , g_{n-2} at p. If the order of zero of ψ at p was greater than the maximum order of pole of g_1, ... , g_{n-2} at p, then all ψg_k's would be zero at p, and $g_c(\Phi, \Phi) = 0$, a contradiction. Hence the order of zero of ψ at p is exactly same as the maximum order of pole of g_1, ... , g_{n-2} at p. If the order of pole of $\sum_{k=1}^{n-2} {g_k}^2$ was greater than the maximum order of pole of g_1, ... , g_{n-2} at p, then $\psi(\sum_{k=1}^{n-2} {g_k}^2) = 2\phi_1 - \psi$ could not be analytic at p, a contradiction. Hence the order of pole of $\sum_{k=1}^{n-2} {g_k}^2$ at p is less than or equal to the maximum order of pole of g_1, ... , g_{n-2}. This completes the proof. □

We next modify the theorem to give a representation formula of arbitrary maximal surfaces. We begin with a Riemann surface S_o and define an 1-1 map $X : S_o \longrightarrow L^n$ in one of two ways. Case 1 is exactly as in Theorem 2. In case 2, we again choose n-2 arbitrary meromorphic functions g_k on S_o, but in place of the function ψ we choose an analytic differential α on S_o which is locally of the form $\alpha = \psi(z)dz$ in terms of a complex parameter z on S_o. If we then define ϕ_k locally by (3), we will obtain global differentials $\alpha_k = \phi_k(z)dz$ on S_o and may then set

$$x_k = Re \int \alpha_k \ , \tag{12}$$

where the integral is taken along a path from a fixed point to a variable point in S_o. We must add the condition

$$Re \int_C \alpha_k = 0$$

for any closed curve C on S_o, so that (12) defines a single-valued map $X : S_o \longrightarrow L^n$. This map will then define a maximal surface in L^n provided the hypotheses in case 2 of Theorem 2 are satisfied. Conversely, every maximal surface in L^n is represented in one of these two forms. The proof is modeled exactly on that of Theorem 2.

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