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REDUCED MODULES AND STRONGLY REGULAR RINGS

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Abstract. It is a well-known fact that a ring R is regular if and only if every left R-modules is flat. In this article we prove that a ring R is strongly regular if and only if every left R-modules is reduced if and only if every left-R modules is quasi-reduced.

1. Introduction

Throughout all rings are associative with identity and all modules are unitary. Let R be a ring. Note that an element $a \in R$ is nilpotent if $a^n = 0$ for some $n \geq 1$, and R is reduced if R has no nonzero nilpotent elements. R is semicommutative if for $a, b \in R$, $ab = 0$ implies that $aRb = 0$, and R is abelian if every idempotent $e = e^2 \in R$ is central. It is not difficult to show that reduced rings are semicommutative and semicommutative rings are abelian. It can be also proved that a ring R is semicommutative if and only if $l(x) = \{a \in R | ax = 0\}$ is a two-sided ideal if and only if $r(x) = \{a \in R | xa = 0\}$ is a two-sided ideal for any $x \in R$. A ring R is left *duo* if every left ideal of R is two-sided. Right duo ring is defined analogously. Clearly left or right duo rings are semicommutative.

Many properties of rings can be extended to modules. Due to Zhang [5] and Buhpang et al. [2], a left module ${}_RM$ is *reduced* if $ax = 0$ implies that $aM \cap Rx = (0)$ for $a \in R$, $x \in M$. M is *semicommutative* if $ax = 0$ implies $aRx = (0)$, and M is *abelian* if $(ea)x = (ae)x$ for any $a, e \in R$ with $e = e^2$ and $x \in M$.

Now we introduce a generalization of reducedness for modules. A module ${}_RM$ is said to be *quasi-reduced*(briefly *q-reduced*) if $a^n x = 0$ $(a \in R,\ x \in M,\ n \geq 1)$ implies $ax = 0$. Note that ${}_RM$ is q-reduced if and only if $ax = 0$ whenever $a^2 x = 0$ for $a \in R$, $x \in M$.

For modules we have the following.

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Proposition 1.1. *A module $_RM$ is semicommutative if and only if $l_R(x)$ is a two sided ideal of R for each $x \in M$, where $l_R(x) = \{r \in R | rx = 0\}$.*

Lemma 1.2. *Let R be a ring and M a left R-Module. Then*
(1) If M is reduced, then it is both q-reduced and semicommutative.
(2) If M is q-reduced or semicommutative, then it is abelian.

Proof. (1) Suppose M is a reduced module. If $a^2x = 0$ for $a \in R$, $x \in M$, then $a(ax) = a^2x = 0$. This means that $R(ax) \cap aM = (0)$. Since $ax \in R(ax) \cap aM$, we have $ax = 0$. Thus M is q-reduced. Now $ax = 0$ implies that $arx \in Rx \cap aM = (0)$ for all $r \in R$. Thus, $aRx = (0)$, this proves that M is semicommutative.
(2) Let $e = e^2 \in R$. Suppose M is q-reduced. Since $(ere-er)^2 = (ere-re)^2 = 0$ for all $r \in R$, we get $(ere - er)x = (ere - re)x = 0$ for any $x \in M$. Thus $(re)x = (ere)x = (er)x$. Now if M is semicommutative, then $er(1 - e)x = (1 - e)rex = 0$ for all $e = e^2$, $r \in R$ and $x \in M$, since $e(1 - e) = (1 - e)e = 0$. Thus, $(re)x = (ere)x = (er)x$ for all $r \in R$ and $x \in M$. □

Theorem 1.3. *A left module M over a ring R is reduced if and only if it is both q-reduced and semicommutative.*

Proof. The only if part is given by Lemma 1.2(1). Suppose that $_RM$ is q-reduced and semicommutative. If $ax = 0$ with $a \in R$, $x \in M$ and $y \in Rx \cap aM$, then $y = bx = az$ for some $b \in R$ and $z \in M$. Since M is semicommutative, and $ax = 0$, we get $ay = abx = 0$, hence $a^2z = ay = 0$. This implies that $y = az = 0$ since M is q-reduced. Therefore $Rx \cap aM = (0)$. □

The concepts of semicommutativity and quasi-reducedness are independent.

Example 1.4. Let F be a field and $R = \dfrac{F[x]}{(x^2)}$, where (x^2) is the ideal generated by $x^2 \in F[x]$. Then R is commutative, hence semicommutative. The left regular module $_RR$ is a semicommutative which is not q-reduced, because $\overline{x}^2\overline{1} = \overline{x}^2 = 0$ but $\overline{x}\overline{1} = \overline{x} \neq 0$ where $\overline{x} = x+(x^2)$ and $\overline{1} = 1+(x^2)$. Hence $_RR$ is not q-reduced.

Example 1.5. Let F be a field and $R = F < x, y >$ be the free algebra in two noncommuting indeterminates x and y. Put $I = Ry$ be the left ideal of R generated by y. Then $M = R/Ry$ is a q-reduced module which is not semicommutative. To prove this, we need some steps.

Step 1. For f, g and $h \in R$, if $f + fgx + hy = 0$ and $f \neq 0$, then $g = 0$.

Proof. Note that f has no constant term so we can write $f = f_1x + f_2y$ with $f_1, f_2 \in R$. From the equality $f + fgx + hy = 0$, we get $f_1x = -(fg)x$. If

$f_1 \neq 0$, then $fg \neq 0$ and $\deg f_1 < \deg f_1 x \leq \deg f \leq \deg fg$ =degf_1. This is a contradiction, so $f_1 = 0$ and hence $fg = 0$ and so $g = 0$. □

Step 2. For f, g and $h \in R$, if $f + fxg + hyg = 0$, then $f = 0$.

Proof. Assume $f \neq 0$. Then $f = f' g$ with $f' \neq 0$, $g \neq 0$ where $f' = -(fx + hy)$. Thus $0 = f + fxg + hyg = (f' + f' gx + hy)g$, and hence $f' + f' gx + hy = 0$ since $g \neq 0$. Since $f' \neq 0$ it follows from Step 1 that $g = 0$, a contradiction. So $f = 0$ (and $hyg = 0$). □

Step 3. Let r,$s \in R$. If $rs \in I$ and $r \neq 0$, then either $s \in I$ or $r \in I$ and $s = a + gy$ for some $a \in F$, $g \in R$.

Proof. Let $r = a + f_1 x + f_2 y$, $s = b + g_1 x + g_2 y$ where $a, b \in F$ and $f_i, g_i \in R (i = 1, 2)$. Then $0 \equiv rs = ab + (bf_1 + ag_1 + f_1 x g_1 + f_2 y g_1)x + (bf_2 + ag_2 + f_1 x g_2 + f_2 y g_2)y$ $\equiv ab + (bf_1 + ag_1 + f_1 x g_1 + f_2 y g_1)x \pmod{I}$.
Since polynomials in I have zero constant terms, we have $ab = 0$ and

$$bf_1 + ag_1 + f_1 x g_1 + f_2 y g_1 = 0 \ (*)$$

Case 1. If $b = 0$, then $(*)$ can be rewritten as $0 = ag_1 + f_1 x g_1 + f_2 y g_1 = rg_1$. Then $g_1 = 0$. Hence, $s = b + g_1 x + g_2 y = g_2 y \in I$.

Case 2. If $b \neq 0$, then $a = 0$ and we get $bf_1 + f_1 x g_1 + (f_2 y) g_1 = 0$ from $(*)$. Hence $f_1 + f_1 x(\frac{1}{b} g_1) + f_2 y(\frac{1}{b} g_1) = 0$, it follows from Step 2 that $f_1 = 0$. Since $f \neq 0$, we get $f_2 \neq 0$ and $g_1 = 0$. Therefore $r = a + f_1 x + f_2 y = f_2 y \in I$ and $s = b + g_1 x + g_2 y = b + g_2 y$. □

Step 4. If $r \in R$ and $r^2 \in I$, then $r \in I$.

Proof. Take $s = r$ and apply Step 3. □

Step 5. If $r, s \in R$ and $r^2 s \in I$, then $rs \in I$.

Proof. If $r = 0$, then there is nothing to prove. So we may assume $r \neq 0$ (so $r^2 \neq 0$). By Step 3, either $s \in I$ or $r^2 \in I$ and $s = a + gy$ for some $a \in F$ and $g \in R$. If $s \in I$, then clearly $rs \in I$. If $r^2 \in I$ and $s = a + gy$, then by Step 4, $r \in I$, and hence $rs = r(a + gy) \in I$. □

Step 6. ${}_R M$ is q-reduced.

Proof. For r, $s \in R$, if $r^2(s + I) = 0$ then $r^2 s \in I$. By step 5, $rs \in I$ and hence $r(s + I) = rs + I = 0$. □

Step 7. ${}_R M$ is not semicommutative.

Proof. Note that $y(1+I)=0$, but $yx(1+I)\neq 0$ in M. Hence ${}_RM$ is not semicommutative. □

2. Properties of rings and modules

Proposition 2.1. *Let R be a ring. Then*
(1) R is a reduced ring if and only if ${}_RR$ is a reduced module if and only if ${}_RR$ is a q-reduced module.
(2) R is a semicommutative ring if and only if ${}_RR$ is a semicommutative module.
(3) R is an abelian ring if and only if ${}_RR$ is an abelian module.

Proof. (1) Suppose R is a reduced ring and $ax=0$ for $a,\, x\in R$. If $y\in Rx\cap aR$, then $y=bx=az$ for some $b,\, z\in R$. Thus, $xy=xbx=xaz$. Note that $xa=0$, thus $xbx=xaz=0$. Since $(bx)^2=0$ and R is reduced, we obtain $y=bx=0$ and hence $Rx\cap aR=(0)$. If ${}_RR$ is reduced, then ${}_RR$ is q-reduced by Lemma 1.2 (1). Now if ${}_RR$ is q-reduced and $r^2=0$ for $r\in R$, then $r^21=r^2=0$, hence $r=r1=0$. Thus R is a reduced ring.
Proofs of (2) and (3) are obvious from the definitions. □

For a left module ${}_RM$, the annihilator $l_R(M)=\cap\,\{l_R(x)|x\in M\}$ is a two-sided ideal of R. M is said to be *faithful* if $l_R(M)=(0)$. A ring R with a faithful and irreducible left module is called a left *primitive ring.* For example, every matrix ring over a division ring is left primitive.

Corollary 2.2. *Let R be a ring. Then*
(1) R is reduced if and only if R has a faithful and reduced module if and only if R has a faithful and q-reduced module.
(2) R is semicommutative if and only if R has a faithful and semicommutative module.
(3) R is abelian if and only if R has a faithful and abelian module.

Proof. Note that if R is a reduced(resp., semicommutative, abelian) ring, then ${}_RR$ is both faithful and reduced(resp., semicommutative, abelian).
(1) Since the only if parts are obvious, it suffices to show that if R has a faithful and q-reduced module, then it is reduced. Let ${}_RM$ be a faithful and q-reduced R-module. If $a\in R$ and $a^2=0$, then $a^2M=(0)$. Since M is q-reduced and faithful, $aM=0$ and hence $a=0$.
(2) Let ${}_RM$ be a faithful and semicommutative module. If $a,\ b\in R$ and $ab=0$, then $abM=(0)$. Thus $(aRb)M=(0)$ and so $aRb=0$.
(3) Let ${}_RM$ be a faithful and abelian module. If $e=e^2$, $r\in R$, then
$(er-re)M=(0)$. Thus $er=re$, since M is faithful. □

A ring R is said to be *prime* if $aRb=0$ implies either $a=0$ or $b=0$. Primitive rings are prime. For modules over a prime or primitive ring, we have the following.

Proposition 2.3. *Let R be a prime ring. Then the following are equivalent.*
(1) R is a domain.
(2) There is a faithful module ${}_RM$ which is reduced.
(3) There is a faithful module ${}_RM$ which is q-reduced.
(4) There is a faithful module ${}_RM$ which is semicommutative.

Proof. (1) $\Longrightarrow$ (2) If R is a domain, then ${}_RR$ is a faithful and reduced R-module.
(2) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (4) are by Lemma 1.2(1).
(3) $\Longrightarrow$ (1) Suppose ${}_RM$ is a faithful and q-reduced module. If $ab = 0$ $(a,\ b \in R)$, then $(bRa)^2M = 0$. Since ${}_RM$ is q-reduced, we have $(bRa)M = 0$. Thus $bRa = 0$, so $a = 0$ or $b = 0$.
(4) $\Longrightarrow$ (1) Suppose ${}_RM$ is a faithful and semicommutative module. If $a,\ b \in R$ such that $ab = 0$, then $a(bM) = (ab)M = 0$. Thus, $(aRb)M = aR(bM) = 0$ since M is semicommutative. This implies $aRb = 0$ and so $a = 0$ or $b = 0$. □

Lemma 2.4. *Let ${}_RM$ be a faithful and irreducible module. If M is q-reduced or semicommutative, then $l_R(x) = 0$ for all $0 \neq x \in M$.*

Proof. Case 1. Suppose ${}_RM$ is q-reduced. Assume on the contrary that $I = l_R(x) \neq (0)$ for some $0 \neq x \in M$. Since ${}_RM$ is faithful and irreducible, $Iy = M$ for some $y \in M$. Now $x \in M = Iy$, so $x = ay$ for some $a \in I = l_R(x)$. Since M is q-reduced and $a^2y = ax = 0$, we have $x = ay = 0$, contradiction. So $l_R(x) = (0)$ for all $0 \neq x \in M$.
Case 2. Suppose ${}_RM$ is semicommutative and $0 \neq x \in M$, $a \in l_R(x)$. Then $ax = 0$, and hence $aM = a(Rx) = aRx = (0)$. Hence $a = 0$. □

Theorem 2.5. *Let R be a left primitive ring with a faithful and irreducible module ${}_RM$. Then the following are equivalent.*
(1) R is a division ring.
(2) M is reduced.
(3) M is q-reduced.
(4) M is semicommutative.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) $\Rightarrow$ (4) are obvious, since a vector space over a division ring is a reduced module.
(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1). Let $0 \neq a \in R$; then $ax \neq 0$ for some $0 \neq x \in M$. So $M = Rax$, hence $x = bax$ for some $b \in R$. Now $1 - ba \in l_R(x) = (0)$. Therefore $ba = 1$ and so R is a division ring. □

By left-right symmetry, we have the following.

Corollary 2.6. *For a ring R, the following are equivalent.*
(1) R is a division ring.
(2) R has a faithful and irreducible right R-module which is reduced.

(3) R has a faithful and irreducible right R-module which is q-reduced.
(4) R has a faithful and irreducible right R-module which is semicommutative.

3. Modules over strongly regular rings

In this section, we prove that a ring R is strongly regular if and only if every left(or right) R-module is reduced. A ring R is (*von Neumann*) *regular* if for each $a \in R$, $a = aba$ for some $b \in R$. R is *strongly regular* if for each $a \in R$, $a = ba^2$ for some $b \in R$. It is well-known that R is strongly regular if and only if it is abelian and regular [3, Theorem 3.5].

Note that a left R-module M is flat if the tensor functor $- \otimes_R M$ is left exact.

Theorem 3.1. *For a ring R, the following are equivalent.*
(1) R is regular.
(2) Every left R-module is flat.

Proof. See [4, Proposition 5.4.4]. □

Lemma 3.2. *Let R be a strongly regular ring. Then*
(1) For each $a \in R$, there exists a unique element $b \in R$ such that $ab = ba$, and $a = a^2b = ba^2$. Moreover, ab is a central idempotent.
(2) R is reduced.
(3) R is left duo.

Proof. (1) See [1, Lemma 1]. (2) Let $a \in R$ with $a^2 = 0$. Choose $b \in R$ such that $a = ba^2$, hence $a = ba^2 = 0$ and R is reduced.
(3) Let I be a left ideal of R and $a \in I$, $r \in R$. Choose $b \in R$ such that $a = ba^2$, and $ab = ba$ is central. Then $ar = (ba^2)r = (ba)(ar) = (ar)(ba) \in I$, so I is an ideal. □

Next theorem is a main result of this article.

Theorem 3.3. *For a ring R, the following are equivalent.*
(1) R is strongly regular.
(2) Every left R-module is reduced.
(3) Every left R-module is q-reduced.
(4) Every principal left R-module is q-reduced.

Proof. (1) $\Rightarrow$ (2) Suppose that R is strongly regular. First we claim that every left R-module is q-reduced. To see this let M be a left R-module and $a^2x = 0$ for $a \in R$ and $x \in M$. Then $ax = (ba^2)x = b(a^2x) = 0$ for some $b \in R$, and hence ${}_RM$ is a q-reduced module. Now R is left duo by Lemma 3.2(3), so every left R-module is semicommutative. Therefore every left R-module is reduced.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (1) For $a \in R$, let $M = R/Ra^2$ and $x = 1 + Ra^2 \in M$. Then M

is principal, hence is a $q-$reduced left R-module by assumption (4). Since $a^2x = 0$, we have $ax = 0$. This implies that $a \in Ra^2$, hence $a = ba^2$ for some $b \in R$. □

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