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PRACTICAL INVESTMENT STRATEGIES UNDER A MULTI-SCALE HESTON'S STOCHASTIC VOLATILITY MODEL

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Abstract. We study an optimization problem for HARA utility function under a multi-scale Heston's stochastic volatility model. We investigate a practical strategy that do not depend on the incorporated factor which is unobservable in the market.

1. Introduction

In this study, we suppose that an investor manages his or her initial wealth X_0 by investing in a financial market consisting of a risky asset and a risk-free asset whose price processes are given as follows. The price B_t of the risk-free asset at time t follows the ordinary differential equation (ODE)

$$dB_t = rB_t dt, \tag{1}$$

where $r > 0$ is a constant interest rate. The price S_t of risky one is given by the following stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = \mu(Y_t, Z_t)dt + f(Y_t)\sqrt{Z_t}dW_t^s, \tag{2}$$

where

$$dY_t = \frac{Z_t}{\epsilon}\beta(Y_t)dt + \sqrt{\frac{Z_t}{\epsilon}}\alpha(Y_t)dW_t^y, \tag{3}$$

$$dZ_t = \kappa(\theta - Z_t)dt + \sigma\sqrt{Z_t}dW_t^z. \tag{4}$$

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Here W^s, W^y and W^z are correlated Brownian motions in a filtered probability space $\big(\Omega, \mathcal{F}, \mathcal{F}_t, P\big)$ with correlation structure given by

$$\begin{aligned} d\big\langle W^s, W^y\big\rangle_t &= \rho_{sy} dt, \\ d\big\langle W^s, W^z\big\rangle_t &= \rho_{sz} dt, \\ d\big\langle W^y, W^z\big\rangle_t &= \rho_{yz} dt. \end{aligned}$$

The correlation coefficients ρ_{sy}, ρ_{sz} and ρ_{yz} are constants in $(-1, 1)$ satisfying $\rho_{sy}^2 + \rho_{sz}^2 + \rho_{yz}^2 - 2\rho_{sy}\rho_{sz}\rho_{yz} < 1$, so that the covariance matrix of the Brownian motions is guaranteed to be positive definite.

We will specify assumptions on the coefficients $\mu(y, z), f(y), \alpha(y)$ and $\beta(y)$ of our model later. We assume that given $Z_t = z$, the process Y_t in (1.3) is a mean-reverting process and $Y_t = Y^{(1)}_{t/\epsilon}$ in distribution, where $Y^{(1)}$ is an ergodic diffusion process with unique invariant distribution denoted by Φ (independent of ϵ) and has the infinitesimal generator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 = \frac{1}{2}\alpha^2(y)\frac{\partial^2}{\partial y^2} + \beta(y)\frac{\partial}{\partial y}. \tag{5}$$

We use the notation $\langle\cdot\rangle$ for averaging with respect to Φ, i.e.,

$$\langle g\rangle = \int g(y)\Phi(dy). \tag{6}$$

In this case we call (B_t, S_t) a financial market with a multiscale Heston's stochastic volatility model.

We assume that the investor dynamically manages his or her portfolio by allocating a fraction π_t of the wealth at time $t \in [0, T]$ in the risky asset, while the remaining amount is held in the risk-free asset earning the risk-free interest of r. Assuming the investment strategy π is self-financing, the associated wealth process X_t^π satisfies

$$dX_t = X_t\left\{r + \pi_t\left(\mu(Y_t, Z_t) - r\right)\right\}dt + \pi_t f(Y_t)\sqrt{Z_t}X_t dW_t^s. \tag{7}$$

We assume that all coefficients of the above SDEs are $\mathcal{F}_t$–progressively measurable and that the system of SDEs (1.2) - (1.4) and (1.10) has unique strong solution. Given for a fixed parameter ϵ and a strategy π_t, we denote the solution of (1.10) by $(X^{\epsilon,\pi}(t))_{t\in[0,T]}$. The control function π_t is said to be admissible if it is $\mathcal{F}_t$–progressively measurable and satisfies

$$E\Big[\int_0^T \pi_t^2 f^2(Y_t) Z_t X_t^2 dt\Big] < \infty.$$

We denote the set of all admissible strategies by $\mathcal{A}$. We assume that $\mu(Y_t, Z_t) - r = \mu(Y_t)Z_t$, so that the market price of risk ζ_t is given by

$$\zeta_t = \frac{\mu(Y_t, Z_t) - r}{f(Y_t)\sqrt{Z_t}} = \frac{\mu(Y_t)}{f(Y_t)}\sqrt{Z_t} := \lambda(Y_t)\sqrt{Z_t}. \tag{8}$$

Since our financial market model (B_t, S_t) is very complicate, it is impossible to get the implicit form of the optimal investment strategy. So some approximations of the optimal strategy are studied (cf. [6, 7]). The major contribution of this work is to show that the portfolio optimization problem under the multi-scale stochastic volatility model built on the Heston's model can be treated similarly as the one considered in Fouque et al. [6] for HARA utility functions. This is accomplished by taking advantage of the explicit form of value function derived in Kraft [8] under the pure Heston's model.

The structure of this paper is as follows. In Section 2, we formulate our problem and derive the associated HJB equation and the asymptotic analysis method is applied to obtain explicit approximations to the value function and the optimal investment strategy for the HARA utility functions. Section 3 introduces a practical investment strategy that does not depend on the unobservable fast factor of volatility.

2. Formulation of the problem and theory background

In this section we formulate our stochastic optimization problem and derive the associated HJB equation. We define the value function corresponding to an investment strategy π by

$$V^{\epsilon,\pi}(t,x,y,z) = \mathbb{E}\left[U(X_T^{\epsilon,\pi})\middle|X_t^{\epsilon,\pi} = x, Y_t = y, Z_t = z\right].$$

for all $(t,x,y,z) \in [0,T] \times \mathbf{R^1} \times \mathbf{R^1} \times \mathbf{R^1}$, where U is a HARA utility function defined by

$$U(x;p,q,\eta) = \frac{1-p}{pq}\left(\frac{qx}{1-p} + \eta\right)^p, \quad q > 0, p < 1, p \neq 1 \tag{9}$$

and $\mathbb{E}[X|A]$ is the conditional expectation of a random variable X given an event A. The object of the investor is to find the optimal investment strategy π^* such that

$$V^{\epsilon,\pi^*}(t,x,y,z) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U(X_T^{\pi})\middle|X_t^{\pi} = x, Y_t = y, Z_t = z\right].$$

and the optimal value function

$$V^{\epsilon}(t,x,y,z) = V^{\epsilon,\pi^*}(t,x,y,z).$$

In fact, the optimal value function is the value function corresponding to the optimal investment strategy π^*. Then the associated Hamilton-Jacobi-Bellman (HJB) equation (cf. Øksendal [9]) for V^ϵ is given by, for $t \in [0,T]$, $x \in \mathbb{R}^+$,

$y \in \mathbb{R}$ and $z \in \mathbb{R}^+$,

$$V_t^\epsilon + \frac{z}{\epsilon}\mathcal{L}_0 V^\epsilon + rxV_x^\epsilon + \kappa(\theta - z)V_z^\epsilon + \frac{1}{2}\sigma^2 z V_{zz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{yz}\sigma\alpha(y)zV_{yz}^\epsilon + \sup_\pi \left[\frac{1}{2}\pi^2 f^2(y)zx^2 V_{xx}^\epsilon + \pi zx\Big(\mu(y)V_x^\epsilon + \rho_{sz}\sigma f(y)V_{xz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{sy}\alpha(y)f(y)V_{xy}^\epsilon\Big)\right] = 0 \tag{10}$$

where the terminal condition is given by

$$V^\epsilon(T, x, y, z) = U(x). \tag{11}$$

Maximizing the quadratic expression in π, the optimal investment strategy is given in feedback form by

$$\pi^*(t, x, y, z) = -\frac{\lambda(y)V_x^\epsilon + \rho_{sz}\sigma V_{xz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{sy}\alpha(y)V_{xy}^\epsilon}{f(y)xV_{xx}^\epsilon}, \tag{12}$$

where λ is the function defined in (8). Substituting this optimal strategy into (10) yields

$$V_t^\epsilon + \frac{z}{\epsilon}\mathcal{L}_0 V^\epsilon + rxV_x^\epsilon + \kappa(\theta - z)V_z^\epsilon + \frac{1}{2}\sigma^2 z V_{zz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{yz}\sigma\alpha(y)zV_{yz}^\epsilon - \frac{z\Big(\lambda(y)V_x^\epsilon + \rho_{sz}\sigma V_{xz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{sy}\alpha(y)V_{xy}^\epsilon\Big)^2}{2V_{xx}^\epsilon} = 0. \tag{13}$$

Assumption 2.1. As in Fouque et al [6], we assume that the value function $V^\epsilon(t, x, y, z)$ is strictly increasing, strictly concave in x for each $t \in [0, T), y \in \mathbb{R}$ and $z \in \mathbb{R}^+$, and is smooth enough on the domain $[0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$. We also assume that it is the unique solution for the HJB equation (10) with terminal condition (11).

We now use asymptotic analysis developed in Fouque et al [3] to obtain approximations to the value function and optimal investment strategy for the HARA utility function defined in (9).

We begin by expanding the optimal value function $V^\epsilon(t, x, y, z)$ in powers of $\sqrt{\epsilon}$ as

$$V^\epsilon(t, x, y, z) = V^{(0)}(t, x, y, z) + \sqrt{\epsilon}V^{(1)}(t, x, y, z) + \epsilon V^{(2)}(t, x, y, z) + \cdots, \tag{14}$$

for any small positive parameter $\epsilon < 1$. Then we substitute this expansion into (13) and successively compare powers of ϵ. Collecting the ϵ^{-1} order terms gives

$$z\mathcal{L}_0 V^{(0)} - \frac{1}{2}\rho_{sy}^2\alpha^2(y)z\frac{\left(V_{xy}^{(0)}\right)^2}{V_{xx}^{(0)}} = 0. \tag{15}$$

As in [6], a well-behaved solution $V^{(0)}$ of (15) must be independent of y, and then we choose $V^{(0)} = V^{(0)}(t, x, z)$.

Next, comparing the $\epsilon^{-\frac{1}{2}}$ order terms in the (13) leads to

$$\mathcal{L}_0 V^{(1)} = 0,$$

since $V^{(0)}$ is independent of y. Similarly, we choose $V^{(1)}$ to be independent of y. Using the y–independence of $V^{(0)}$ and $V^{(1)}$, the order one terms in (13) are collected as

$$V_t^{(0)} + z\mathcal{L}_0 V^{(2)} + rxV_x^{(0)} + \kappa(\theta - z)V_z^{(0)} + \frac{1}{2}\sigma^2 z V_{zz}^{(0)} \\ - \frac{z\Big(\lambda(y)V_x^{(0)} + \rho_{sz}\sigma V_{xz}^{(0)}\Big)^2}{2V_{xx}^{(0)}} = 0. \tag{16}$$

Viewing (16) as a Poisson equation for $V^{(2)}$ in y, the centering condition is given by

$$\left\langle V_t^{(0)} + rxV_x^{(0)} + \kappa(\theta - z)V_z^{(0)} + \frac{1}{2}\sigma^2 z V_{zz}^{(0)} - \tau(t, x, y, z) \right\rangle = 0,$$

where $\langle \cdot \rangle$ is the averaging operator defined in (6) and

$$\tau(t, x, y, z) = \frac{z\Big(\lambda(y)V_x^{(0)} + \rho_{sz}\sigma V_{xz}^{(0)}\Big)^2}{2V_{xx}^{(0)}}..$$

Then it follows that

$$V_t^{(0)} + rxV_x^{(0)} + \kappa(\theta - z)V_z^{(0)} + \frac{1}{2}\sigma^2 z V_{zz}^{(0)} - \frac{1}{2}\tilde{\lambda}^2 z \frac{\Big(V_x^{(0)}\Big)^2}{V_{xx}^{(0)}} \\ - \rho_{sz}\sigma\bar{\lambda} z \frac{V_x^{(0)} V_{zx}^{0}}{V_{xx}^{(0)}} - \frac{1}{2}\rho_{sz}^2 \sigma^2 z \frac{\Big(V_{zx}^{(0)}\Big)^2}{V_{xx}^{(0)}} = 0, \tag{17}$$

with $\bar{\lambda} = \langle \lambda \rangle$ and $\tilde{\lambda}^2 = \langle \lambda^2 \rangle$. From the expansion (14) of V, we have the terminal condition

$$V^{(0)}(T, x, z) = U(x). \tag{18}$$

Remark 1. We observe that when $\lambda(y) = 1$, (17) reduces to the HJB equation corresponding to the pure Heston model where its solution can be computed explicitly (see, for example,[8]).

It is possible to obtain a closed-form solution for the PDE (17) with terminal condition (18) for HARA utility functions. Next result is given by Kim and Veng [7], which can be viewed as a generalization of Proposition 3.1 in [11].

Theorem 2.1. *([7]). For the HARA utility function U given in (9), the PDE (17) with the terminal condition (18) has an explicit solution*

$$V^{(0)}(t,x,z) = \frac{1-p}{pq}\Big(\frac{qx}{1-p}e^{r(T-t)}+\eta\Big)^p e^{A(t)+B(t)z}, \tag{19}$$

where $A(t)$ and $B(t)$ are C^1 real-valued functions given as follows. Denote

$$\Gamma = \frac{2}{\sigma^2\left(\frac{p}{1-p}\rho_{sz}^2+1\right)},$$

$$m_{1,2} = K \pm \tfrac{1}{2}\sqrt{\Delta},$$

$$K = \tfrac{1}{2}\left(\kappa - \tfrac{p}{1-p}\rho_{sz}\sigma\bar{\lambda}\right),$$

$$\Delta = \frac{p^2\rho_{sz}^2\sigma^2\left(\bar{\lambda}^2-\tilde{\lambda}^2\right)+\kappa^2(1-p)^2-2p(1-p)\rho_{sz}\sigma\bar{\lambda}\kappa-p(1-p)\sigma^2\tilde{\lambda}^2}{(1-p)^2}.$$

Case 1: If $\Delta \neq 0$. Then

$$\begin{aligned} A(t) &= \kappa\theta\Gamma\left[m_1(T-t) - \ln\left(\frac{1-\frac{m_1}{m_2}e^{(T-t)\sqrt{\Delta}}}{1-\frac{m_1}{m_2}}\right)\right], \\ B(t) &= \Gamma\frac{m_1\left(1-e^{\sqrt{\Delta}(T-t)}\right)}{1-\frac{m_1}{m_2}e^{\sqrt{\Delta}(T-t)}}, \end{aligned}$$

Case 2: If $\Delta = 0$ and $0 < 1+KT$. Then

$$\begin{aligned} A(t) &= \kappa\theta\Gamma\left(K(T-t) - \ln\left(1-K(t-T)\right)\right), \\ B(t) &= \Gamma\frac{K^2(t-T)}{K(t-T)-1}. \end{aligned}$$

Remark 2. (1) The condition $0 < 1+KT$ appearing in the case $\Delta = 0$ is imposed to assure that $A(t)$ is well-defined and bound (and then so is $V^{(0)}$) on $[0,T]$.
(2) We observe from Theorem 2.1 that $V^{(0)}$ satisfies

$$V_z^{(0)}(t,x,z) = B(t)V^{(0)}(t,x,z) \tag{20}$$

for all $(t,x,z) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}^+$. This property is very important for us to derive an explicit expression for the first order correction term $V^{(1)}$.

For convenience, we recall the following notation introduced in [6]

$$R(t,x) = -\frac{V_x^{(0)}(t,x,z)}{V_{xx}^{(0)}(t,x,z)}, \tag{21}$$

$$D_j = R^j(t,x)\frac{\partial^j}{\partial x^j}, \quad j = 1,2,\ldots. \tag{22}$$

We notice that R is well defined as $V^{(0)}$ is strictly concave. From (20), direct computation shows that R is independent of z, and that is reason we wrote $R(t,x)$ rather than $R(t,x,z)$. We also introduce the linear operator $\mathcal{L}_{t,x,z}(\lambda_1,\lambda_2)$ defined by

$$\begin{aligned}\mathcal{L}_{t,x,z}(\lambda_1,\lambda_2) = {} & \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \kappa(\theta - z)\frac{\partial}{\partial z} + \frac{1}{2}\sigma^2 z\frac{\partial^2}{\partial z^2} \\ & + \left(\lambda_1^2 + \rho_{sz}\sigma\lambda_2 B(t)\right) zD_1 \\ & + \frac{1}{2}\left(\lambda_1^2 + 2\rho_{sz}\sigma\lambda_2 B(t) + \rho_{sz}^2\sigma^2 B^2(t)\right) zD_2 \\ & + \rho_{sz}\sigma\left(\lambda_2 + \rho_{sz}\sigma B(t)\right) zD_1\frac{\partial}{\partial z}.\end{aligned} \tag{23}$$

Then it follows that (17) can be written as

$$\mathcal{L}_{t,x,z}\big(\tilde{\lambda},\bar{\lambda}\big)V^{(0)} = 0. \tag{24}$$

Similarly, we can rewrite (16) using the operator $\mathcal{L}_{t,x,z}$ as

$$z\mathcal{L}_0 V^{(2)} + \mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big)V^{(0)} = 0. \tag{25}$$

Then it follows from (24) and (25) that

$$\mathcal{L}_0 V^{(2)} = -\frac{1}{z}\Big(\mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big) - \mathcal{L}_{t,x,z}\big(\tilde{\lambda},\bar{\lambda}\big)\Big)V^{(0)}$$

Hence,

$$V^{(2)} = -\frac{1}{z}\mathcal{L}_0^{-1}\Big(\mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big) - \mathcal{L}_{t,x,z}\big(\tilde{\lambda},\bar{\lambda}\big)\Big)V^{(0)}, \tag{26}$$

where $\mathcal{L}_0^{-1}$ is the inverse operator of $\mathcal{L}_0$.

Now we proceed to derive the first order correction term $V^{(1)}$. Collecting the order $\sqrt{\epsilon}$ terms in the expansion of the PDE (13) and using of (20), (21) and (22), we get

$$z\mathcal{L}_0 V^{(3)} + \mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big)V^{(1)} + z\mathcal{L}_1 V^{(2)} = 0, \tag{27}$$

where

$$\mathcal{L}_1 = \rho_{yz}\sigma\alpha(y)\frac{\partial^2}{\partial y\partial z} + \rho_{sy}\alpha(y)\Big(\lambda(y) + \rho_{sz}\sigma B(t)\Big)R\frac{\partial^2}{\partial x\partial y}.$$

Viewing (27) as a Poisson equation for $V^{(3)}$ in y, the centering condition is given by

$$\Big\langle \mathcal{L}_{t,x,z}(\lambda(y),\lambda(y))V^{(1)} + z\mathcal{L}_1 V^{(2)}\Big\rangle = 0.$$

Since $V^{(1)}$ does not depend on y, we deduce that

$$\mathcal{L}_{t,x,z}\big(\tilde{\lambda},\bar{\lambda}\big)V^{(1)} = -z\Big\langle \mathcal{L}_1 V^{(2)}\Big\rangle.$$

Substituting $V^{(2)}$, given by (26) , into this equation yields

$$\mathcal{L}_{t,x,z}(\tilde{\lambda},\bar{\lambda})V^{(1)} = \mathcal{A}V^{(0)}, \tag{28}$$

where

$$\mathcal{A} = z\left\langle \mathcal{L}_1 \frac{1}{z}\mathcal{L}_0^{-1}\left(\mathcal{L}_{t,x,z}(\lambda(y),\lambda(y)) - \mathcal{L}_{t,x,z}(\tilde{\lambda},\bar{\lambda})\right)\right\rangle.$$

From the expansion (14), the PDE (28) has the terminal condition

$$V^{(1)}(T,x,z) = 0. \tag{29}$$

With $V^{(0)}$ given in Theorem 2.1, we can derive $V^{(1)}$ explicitly in terms of $V^{(0)}$ as in the following theorem. The following result is given in [7].

Theorem 2.2. *([7]). The linear PDE (28) with terminal condition (29) has a solution of the form*

$$V^{(1)}(t,x,z) = \Big(g_1(t) + g_2(t)z\Big)V^{(0)}(t,x,z), \tag{30}$$

where the functions $g_1(t)$ and $g_2(t)$ are given by

$$\begin{aligned} g_1(t) &= \kappa\theta \int_t^T g_2(s)ds,, \ \ g_2(t) = -\int_t^T C(t,s)b(s)ds, \\ b(t) &= P\Big(P\rho_{sy}F_3 + (\rho_{yz}\sigma F_1 + P\rho_{sy}\rho_{sz}\sigma(F_1+F_4))B(t) \\ &\qquad\qquad + \big(\rho_{yz} + P\rho_{sy}\rho_{sz}\big)\rho_{sz}\sigma^2 F_2 B^2(t)\Big), \end{aligned}$$

and the function $C(t,s)$ is defined as follows:

Case 1: If $\Delta \neq 0$, then

$$C(t,s) = e^{\sqrt{\Delta}(s-t)}\left(\frac{1-\frac{m_1}{m_2}e^{\sqrt{\Delta}(T-s)}}{1-\frac{m_1}{m_2}e^{\sqrt{\Delta}(T-t)}}\right)^2.$$

Case 2: If $\Delta = 0$ and $0 < 1 + KT$, then

$$C(t,s) = \left(\frac{1-K(s-T)}{1-K(t-T)}\right)^2.$$

Here, $B(t), K, m_1, m_2$ and Δ are defined as in Theorem 3.1.

Since we have computed an asymptotic approximation for the value function, we can proceed to derive that for the optimal strategy π^* defined in (12). Like the value function, we look for the optimal strategy π^* of the form

$$\pi^*(t,x,y,z) = \pi^{*(0)} + \sqrt{\epsilon}\pi^{*(1)} + \epsilon\pi^{*(2)} + \cdots,$$

and we are interested to derive the terms $\pi^{(0)}$ and $\pi^{(1)}$ explicitly.

Substituting the expansion (14) for V^ϵ into (12) and using the results of Theorem 2.1, Theorem 2.2 and the fact that $D_2V^{(0)} = -D_1V^{(0)}$, we get

$$\pi^{*(0)} = \frac{1}{qf(y)}\left(\lambda(y) + \rho_{sz}\sigma B(t)\right)\left(\frac{q}{1-p} + \frac{\eta e^{r(t-T)}}{x}\right) \tag{31}$$

and

$$\pi^{*(1)} = \frac{1}{qf(y)} \left[\rho_{sz}\sigma g_2(t) - P\rho_{sy}\alpha(y)\left(\phi'(y) + \rho_{sz}\sigma B(t)\psi'(y)\right)\right] \times \left(\frac{q}{1-p} + \frac{\eta e^{r(t-T)}}{x}\right). \tag{32}$$

3. Practical strategies

The fast factor Y is not directly observable in the market and requires complicated techniques to be filtered from the return data of the risky asset. In this section, we wish to find a practical strategy that does not depend on this hidden level. As in [6], we apply asymptotic analysis to obtain this suboptimal strategy. Let $\mathcal{B}$ be the set of all admissible strategies of the form

$$\bar{\pi} = \bar{\pi}^{(0)} + \sqrt{\epsilon}\bar{\pi}^{(1)} + \epsilon\bar{\pi}^{(2)} + \cdots, \tag{33}$$

where the principal terms $\bar{\pi}^{(0)}$ and $\bar{\pi}^{(1)}$ do not depend on y. Then we have the following results for this constrained optimization problem. The investment strategy $\bar{\pi}^*$ satisfying

$$V^{\epsilon,\bar{\pi}^*}(t,x,y,z) = \sup_{\bar{\pi}\in\mathcal{B}} \mathbb{E}\left[U(X_T^{\bar{\pi}})\middle|X_t^{\bar{\pi}} = x, Y_t = y, Z_t = z\right].$$

is called suboptimal.

Theorem 3.1. *We denote $\bar{V}$ the value function corresponding to the optimization problem in which investment strategies are of the form* (33)*. Moreover, we suppose that $\bar{V}$ can be expanded as*

$$\bar{V} = \bar{V}^{(0)} + \sqrt{\epsilon}\bar{V}^{(1)} + \epsilon\bar{V}^{(2)} + \cdots. \tag{34}$$

For the HARA utility function defined in (9)*, the principal terms $\bar{\pi}^{*(0)}$ and $\bar{\pi}^{*(1)}$ of the suboptimal strategy $\bar{\pi}^*$ are given by*

$$\bar{\pi}^{*(0)} = \frac{1}{q}\Big(\bar{\mu} + \bar{\rho}_{sz}\sigma\bar{B}(t)\Big)\left(\frac{q}{1-p} + \frac{\eta e^{r(t-T)}}{x}\right), \tag{35}$$

$$\bar{\pi}^{*(1)} = \frac{1}{q}\left[\bar{\rho}_{sz}\sigma\bar{g}_2(t) + P\rho_{sy}\Big(\tilde{F}_4C^2(t) - \tilde{F}_5C(t) - \rho_{sz}\sigma\tilde{F}_6C(t)\bar{B}(t)\Big)\right] \times \left(\frac{q}{1-p} + \frac{\eta e^{r(t-T)}}{x}\right), \tag{36}$$

where the function $C(t)$ is defined by

$$C(t) = \Big(\bar{\mu} + \bar{\rho}_{sz}\sigma\bar{B}(t)\Big), \tag{37}$$

and the group parameters $\tilde{F}_i$ are defined by

$$\tilde{F}_k = \big\langle\alpha\gamma_k'\big\rangle,\ \tilde{F}_{k+3}big\langle\alpha f\gamma_k'\big\rangle,\ k = 1,2,3. \tag{38}$$

Here the functions $\gamma_i(y)$ are solutions of the following equations

$$\mathcal{L}_0\gamma_1 = \frac{1}{2}\left(f^2(y)-\tilde{f}^2\right), \tag{39}$$

$$\mathcal{L}_0\gamma_2 = \mu(y)-\bar{\mu}, \tag{40}$$

$$\mathcal{L}_0\gamma_3 = f(y)-\bar{f}. \tag{41}$$

The corresponding leading order value function $\bar{V}^{(0)}$ is given explicitly as

$$\bar{V}^{(0)}(t,x,z) = \frac{1-p}{pq}\Big(\frac{qx}{1-p}e^{r(T-t)}+\eta\Big)^p e^{\bar{A}(t)+\bar{B}(t)z}, \tag{42}$$

where $\bar{A}(t)$ and $\bar{B}(t)$ are respectively real-value functions $A(t)$ and $B(t)$, given in Theorem 2.1, with $\tilde{\lambda}, \bar{\lambda}$ and ρ_{sz} replaced by $\bar{\mu}, \bar{\mu}$ and $\bar{\rho}_{sz}$, respectively. Moreover, the first correction term $\bar{V}^{(1)}$ is given explicitly as

$$\bar{V}^{(1)}(t,x,z) = (\bar{g}_1(t)+\bar{g}_2(t)z)\,\bar{V}^{(0)}, \tag{43}$$

where the functions $\bar{g}_1(t)$ and $\bar{g}_2(t)$ are expressed in similar forms as $g_1(t)$ and $g_2(t)$ given in Theorem 2.2.

Proof. Substituting the expansions (33) and (34) in (10) and collecting the ϵ^{-1} order terms yields

$$z\mathcal{L}_0\bar{V}^{(0)} = 0. \tag{44}$$

Viewing (44) as a Poisson equation in y, we choose $\bar{V}^{(0)}$ to be independent of y.

Next, comparing the $\epsilon^{-\frac{1}{2}}$ order terms yields

$$z\mathcal{L}_0\bar{V}^{(1)} = 0, \tag{45}$$

due to the y–independence of $\bar{V}^{(0)}$. Likewise, $\bar{V}^{(1)}$ is also chosen to be independent of y. Then, collecting the order one terms gives

$$\begin{aligned}\sup_{\bar{\pi}\in\mathcal{B}}\Bigg[& z\mathcal{L}_0\bar{V}^{(2)}+\bar{V}_t^{(0)}+rx\bar{V}_x^{(0)}+\kappa(\theta-z)\bar{V}_z^{(0)} \\ & +\frac{1}{2}\sigma^2 z\bar{V}_{zz}^{(0)}+\frac{1}{2}\left(\bar{\pi}^{(0)}\right)^2 f^2(y)zx^2\bar{V}_{xx}^{(0)} \\ & +\bar{\pi}^{(0)}zx\Big(\mu(y)\bar{V}_x^{(0)}+\rho_{sz}\sigma f(y)\bar{V}_{xz}^{(0)}\Big)\Bigg] = 0.\end{aligned} \tag{46}$$

In order for the maximizer $\bar{\pi}^{(0)}$ to be independent of y, the argument being maximized must be y–independent, and then the only way to do so is to choose $\bar{V}^{(2)}$ as a solution of following Poisson equation

$$\begin{aligned} & z\mathcal{L}_0\bar{V}^{(2)}+\frac{1}{2}\left(\bar{\pi}^{(0)}\right)^2\left(f^2(y)-\tilde{f}^2\right)zx^2\bar{V}_{xx}^{(0)} \\ & \quad +\bar{\pi}^{(0)}zx\left(\left(\mu(y)-\bar{\mu}\right)\bar{V}_x^{(0)}+\rho_{sz}\sigma\left(f(y)-\bar{f}\right)\bar{V}_{xz}^{(0)}\right) = 0,\end{aligned} \tag{47}$$

where $\bar{f} = \langle f\rangle$ and $\tilde{f}^2 = \langle f^2\rangle$. With this choice of $\bar{V}^{(2)}$, (46) becomes

$$\sup_{\bar{\pi}\in\mathcal{B}}\Bigg[\bar{V}_t^{(0)}+rx\bar{V}_x^{(0)}+\kappa(\theta-z)\bar{V}_z^{(0)}+\frac{1}{2}\sigma^2 z\bar{V}_{zz}^{(0)} \\ +\frac{1}{2}\left(\bar{\pi}^{(0)}\right)^2\tilde{f}^2 zx^2\bar{V}_{xx}^{(0)}+\bar{\pi}^{(0)}zx\Big(\bar{\mu}\bar{V}_x^{(0)}+\rho_{sz}\sigma\bar{f}\bar{V}_{xz}^{(0)}\Big)\Bigg]=0. \tag{48}$$

Without loss of generality, we may assume that f satisfies $\langle f^2\rangle = 1$ and then we denote $\bar{\rho}_{sz} = \rho_{sz}\langle f\rangle$. Hence, we observe that the nonlinear PDE (48) is the HJB equation corresponding to the optimization problem under the pure Heston model with effective correlation $\bar{\rho}_{sz}$. It is easy to compute the maximizer in (48) as

$$\bar{\pi}^{*(0)} = -\frac{\bar{\mu}\bar{V}_x^{(0)}+\bar{\rho}_{sz}\sigma\bar{V}_{xz}^{(0)}}{x\bar{V}_{xx}^{(0)}}, \tag{49}$$

and with this maximizer, (48) is equivalent to

$$\bar{V}_t^{(0)}+rx\bar{V}_x^{(0)}+\kappa(\theta-z)\bar{V}_z^{(0)}+\frac{1}{2}\sigma^2 z\bar{V}_{zz}^{(0)} \\ -\frac{z\left(\bar{\mu}\bar{V}_x^{(0)}+\bar{\rho}_{sz}\sigma\bar{f}\bar{V}_{xz}^{(0)}\right)^2}{2\bar{V}_{xx}^{(0)}}=0. \tag{50}$$

Referring to the expansion (34), the terminal condition of (50) is given by

$$\bar{V}^{(0)}(T,x,z) = U(x). \tag{51}$$

The PDE (50) is exactly the PDE (17) with $\tilde{\lambda},\bar{\lambda}$ and ρ_{sz} replaced by $\bar{\mu},\bar{\mu}$ and $\bar{\rho}_{sz}$, respectively. Then from Theorem 2.1 a solution of the PDE (50) with terminal condition (51) is given by (42). Using the definition (21) of R, it follows from (49) that $\bar{\pi}^{*(0)}$ is given by (35).

Finally, collecting the $\sqrt{\epsilon}$ order terms gives

$$\sup_{\bar{\pi}\in\mathcal{B}}\Bigg[z\mathcal{L}_0\bar{V}^{(3)}+\bar{V}_t^{(1)}+rx\bar{V}_x^{(1)}+\kappa(\theta-z)\bar{V}_z^{(1)} \\ +\frac{1}{2}\sigma^2 z\bar{V}_{zz}^{(1)}+\rho_{yz}\sigma\alpha(y)zV_{yz}^{(2)} \\ +f^2(y)zx^2\left(\frac{1}{2}\left(\bar{\pi}^{(0)}\right)^2\bar{V}_{xx}^{(1)}+\bar{\pi}^{(0)}\bar{\pi}^{(1)}\bar{V}_{xx}^{(0)}\right) \\ +\bar{\pi}^{(0)}zx\left(\mu(y)\bar{V}_x^{(1)}+\rho_{sz}\sigma f(y)\bar{V}_{xz}^{(1)}+\rho_{sy}\alpha(y)f(y)\bar{V}_{xy}^{(2)}\right) \\ +\bar{\pi}^{(1)}zx\left(\mu(y)\bar{V}_x^{(0)}+\rho_{sz}\sigma f(y)\bar{V}_{xz}^{(0)}\right)\Bigg]=0. \tag{52}$$

In order for the maximizers $\bar{\pi}^{(0)}$ and $\bar{\pi}^{(1)}$ to be independent of y, the argument being maximized must be independent of y, and then the only way to do so is to choose $\bar{V}^{(3)}$ as a solution of following Poisson equation

$$\begin{aligned}
&z\mathcal{L}_0\bar{V}^{(3)} + \rho_{yz}\sigma z\left(\alpha(y)V_{yz}^{(2)} - \left\langle \alpha V_{yz}^{(2)}\right\rangle\right)\\
&+\left(f^2(y)-1\right)zx^2\left(\frac{1}{2}\left(\bar{\pi}^{(0)}\right)^2\bar{V}_{xx}^{(1)} + \bar{\pi}^{(0)}\bar{\pi}^{(1)}\bar{V}_{xx}^{(0)}\right)r\\
&+\bar{\pi}^{(0)}zx\Bigg[\big(\mu(y)-\bar{\mu}\big)\bar{V}_x^{(1)} + \rho_{sz}\sigma\big(f(y)-\bar{f}\big)\bar{V}_{xz}^{(1)}\\
&\qquad\qquad + \rho_{sy}\left(\alpha(y)f(y)\bar{V}_{xy}^{(2)} - \left\langle \alpha f\bar{V}_{xy}^{(2)}\right\rangle\right)\Bigg]\\
&+\bar{\pi}^{(1)}zx\left(\big(\mu(y)-\bar{\mu}\big)\bar{V}_x^{(0)} + \rho_{sz}\sigma\big(f(y)-\bar{f}\big)\bar{V}_{xz}^{(0)}\right) = 0.
\end{aligned}$$

Then, with this choice, (52) becomes

$$\begin{aligned}
\sup_{\bar{\pi}\in\mathcal{B}}\Bigg[&\bar{V}_t^{(1)} + rx\bar{V}_x^{(1)} + \kappa(\theta-z)\bar{V}_z^{(1)} + \frac{1}{2}\sigma^2 z\bar{V}_{zz}^{(1)} + \rho_{yz}\sigma z\left\langle \alpha V_{yz}^{(2)}\right\rangle\\
&+zx^2\left(\frac{1}{2}\left(\bar{\pi}^{(0)}\right)^2\bar{V}_{xx}^{(1)} + \bar{\pi}^{(0)}\bar{\pi}^{(1)}\bar{V}_{xx}^{(0)}\right)\\
&+\bar{\pi}^{(0)}zx\left(\bar{\mu}\bar{V}_x^{(1)} + \bar{\rho}_{sz}\sigma\bar{V}_{xz}^{(1)} + \rho_{sy}\left\langle \alpha f\bar{V}_{xy}^{(2)}\right\rangle\right)\\
&\qquad\qquad + \bar{\pi}^{(1)}zx\left(\bar{\mu}\bar{V}_x^{(0)} + \bar{\rho}_{sz}\sigma\bar{V}_{xz}^{(0)}\right)\Bigg] = 0.
\end{aligned} \tag{53}$$

The maximizing conditions for (53) are given by

$$\begin{aligned}
\bar{\pi}^{*(0)} &= -\frac{\bar{\mu}\bar{V}_x^{(0)} + \bar{\rho}_{sz}\sigma\bar{V}_{xz}^{(0)}}{x\bar{V}_{xx}^{(0)}},\\
\bar{\pi}^{*(1)} &= \frac{1}{x\bar{V}_x^{(0)}}\Bigg(\bar{\mu}D_1\bar{V}^{(1)} + \bar{\rho}_{sz}\sigma D_1\bar{V}_z^{(1)}\\
&\qquad + \left(\bar{\mu} + \bar{\rho}_{sz}\sigma\bar{B}(t)\right)D_2\bar{V}^{(1)} + \rho_{sy}D_1\left\langle \alpha f V_y^{(2)}\right\rangle\Bigg).
\end{aligned} \tag{54}$$

Substituting these maximizers into (53) and expressing in terms of the operator (23) gives

$$\mathcal{L}_{t,x,z}\left(\bar{\mu},\bar{\mu}\right)\bar{V}^{(1)} + z\left\langle \mathcal{L}_2\bar{V}^{(2)}\right\rangle = 0, \tag{55}$$

where $\mathcal{L}_{t,x,z}$ was given in (23) with ρ_{sz} replaced by $\bar{\rho}_{sz} = \rho_{sz}\bar{f}$ and $\mathcal{L}_2$ is defined by

$$\mathcal{L}_2 = \rho_{yz}\sigma\alpha(y)\frac{\partial^2}{\partial y\partial z} + \rho_{sy}\left(\bar{\mu} + \bar{\rho}_{sz}\sigma\bar{B}(t)\right)\alpha(y)f(y)R\frac{\partial^2}{\partial x\partial y} \tag{56}$$

From (47), we choose $\bar{\pi}^{(0)}$ to be $\bar{\pi}^{*(0)}$ defined in (35) and then up to a constant in y, we have

$$\bar{V}^{(2)} = \left\{\gamma_1(y)C(t)^2 - \gamma_2(y)C(t) - \rho_{sz}\sigma\gamma_3(y)C(t)\bar{B}(t)\right\} D_1\bar{V}^{(0)}, \tag{57}$$

where the function $C(t)$ is defined by (37), and the functions $\gamma_i(y)$ are given in (39)-(41). Then from (56) and (57) we can compute

$$\begin{aligned}\left\langle \mathcal{L}_2\bar{V}^{(2)}\right\rangle = &\rho_{yz}\sigma P\left\{\tilde{F}_1C^2(t) - \tilde{F}_2C(t) - \rho_{sz}\sigma\tilde{F}_3C(t)\bar{B}(t)\right\}B(t)V^{(0)}\\ &+ \rho_{sy}P^2C(t)\left\{\tilde{F}_4C^2(t) - \tilde{F}_5C(t) - \rho_{sz}\sigma\tilde{F}_6C(t)\bar{B}(t)\right\}V^{(0)},\end{aligned}$$

where the group parameters $\tilde{F}_i$ are given by (38). Then it follows from (55) that $\bar{V}^{(1)}$ satisfies the following PDE

$$\mathcal{L}_{t,x,z}\left(\bar{\mu},\bar{\mu}\right)\bar{V}^{(1)} = \bar{\mathcal{A}}\bar{V}^{(0)}, \tag{58}$$

where

$$\begin{aligned}\bar{\mathcal{A}} = &z\rho_{yz}\sigma P\left\{-\tilde{F}_1C^2(t) + \tilde{F}_2C(t) + \rho_{sz}\sigma\tilde{F}_3C(t)\bar{B}(t)\right\}\bar{B}(t)\\ &+ z\rho_{sy}P^2C(t)\left\{-\tilde{F}_4C^2(t) + \tilde{F}_5C(t) + \rho_{sz}\sigma\tilde{F}_6C(t)\bar{B}(t)\right\}.\end{aligned} \tag{59}$$

From the expansion (34), we have the terminal condition

$$\bar{V}^{(1)}(T,x,z) = 0. \tag{60}$$

We observe that the PDE (58) has the same form of the PDE (28) satisfied by $V^{(1)}$. Therefore, $\bar{V}^{(1)}$ can be derived in an explicit form similar to $V^{(1)}$ given in Theorem 2.2. With similar proof, we can show that $\bar{V}^{(1)}$ is given by (43).

Finally, using the fact that $D_1\bar{V}^{(0)} = -D_2\bar{V}^{(0)}$ and (43), $\bar{\pi}^{*(1)}$ in (54) can be simplified as (36). The proof is complete. □

Remark 3. The principal terms $\bar{\pi}^{*(0)}$ and $\bar{\pi}^{*(1)}$ of the asymptotic approximation for the practical suboptimal strategy $\bar{\pi}^*$ are of similar forms of $\pi^{*(0)}$ and $\pi^{*(1)}$ respectively given in (31) and (32), without that the expressions for $\bar{\pi}^{*(0)}$ and $\bar{\pi}^{*(1)}$ do not depend on y.

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