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A FINITE ELEMENT METHOD USING SIF FOR CORNER SINGULARITIES WITH AN NEUMANN BOUNDARY CONDITION

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Abstract. In [8] they introduced a new finite element method for accurate numerical solutions of Poisson equations with corner singularities, which is useful for the problem with known stress intensity factor. They consider the Poisson equations with homogeneous Dirichlet boundary condition, compute the finite element solution using standard FEM and use the extraction formula to compute the stress intensity factor, then they pose a PDE with a regular solution by imposing the nonhomogeneous boundary condition using the computed stress intensity factor, which converges with optimal speed. From the solution they could get accurate solution just by adding the singular part. This approach works for the case when we have the reasonably accurate stress intensity factor. In this paper we consider Poisson equations defined on a domain with a concave corner with Neumann boundary conditions. First we compute the stress intensity factor using the extraction formular, then find the regular part of the solution and the solution.

1. Introduction

Let Ω be an open, bounded polygonal domain in $\mathbb{R}^2$ and let Γ_D and Γ_N be a partition of the boundary of Ω such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, assume that Γ_D is not empty (i.e., $\mathrm{meas}(\Gamma_D) \neq 0$). Let ν denote the outward unit vector normal to the boundary.

As a model problem, we consider the following Poisson equation with mixed boundary conditions:

$$\left\{ \begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} &=& 0 & \text{on } \Gamma_N, \end{array} \right. \tag{1}$$

where $f \in L^2(\Omega)$ and Δ stands for the Laplacian operator. Moreover we assume the Neumann boundary condition along two line segements adjacent to the

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concave corner as in **Figure 1**.(i.e., Two line segments adjacent to the concave corner are contained in Γ_N.)

For simplicity, we assume there is only one concave corner with the inner angle w : $\pi < \omega < 2\pi$. We also assume $\Gamma_D \neq \emptyset$ for simplicity. In this case the singular function s and its dual singular function s_- can be expressed by

$$s = s(r,\theta) = r^{\frac{\pi}{\omega}} \cos\frac{\pi\theta}{\omega}, \quad s_- = s_-(r,\theta) = r^{-\frac{\pi}{\omega}} \cos\frac{\pi\theta}{\omega} \tag{2}$$

for the model problem (1) and the unique solution $u \in H_D^1(\Omega)$ has the representation (see [4, 5]):

$$u = w + \lambda\eta s, \tag{3}$$

where $w \in H^2(\Omega) \cap H_D^1(\Omega)$, and η is a smooth cut-off function which equals one identically in a neighborhood of the origin and the support of η is small enough so that the function ηs vanishes identically on Γ_D. (Here, (r,θ) is the polar coordinate.)

The coefficient, λ, is called 'stress intensity factor' and can be computed by the following extraction formula (see [4]):

$$\lambda = \frac{1}{\pi}\int_\Omega f\eta s_- dx + \frac{1}{\pi}\int_\Omega u\Delta(\eta s_-)dx. \tag{4}$$

Note that both s and s_- are harmonic functions in Ω.

As observed in [8], some numerical approaches (e.g. [1, 2, 3]) use this extraction formula for λ and seek the regular part $w \in H^2(\Omega)$ from a new partial differential equation, for example,

$$-\Delta w = f + \lambda\Delta(\eta s) \qquad \text{in } \Omega. \tag{5}$$

Unfortunately, the results were not good enough because the input function f was replaced by $f + \lambda\Delta(\eta s)$, etc., whose L^2- norms are quite large compared to that of f (see Lemma 2.2 in [8]).

In [8] they introduced a new partial differential equation, whose solution is in $H^2(\Omega)$ with the same input function by simple changing of the boundary condition. Using this partial differential equation, they suggested an efficient algorithm to compute the numerical solution for Poisson equation with Dirichlet boundary condition containing domain singularity.

In this paper we consider a Poisson problem with a concave corner with the Neumann boundary condition and suggest a proper algorithm similar to that in [8] together with some theorems. We give three results of numerical experiments, including the standard finite element method, the dual singular function method using the equation (5), and the one similar to that in [8]. An example will be given in Section 4 with computational results using FreeFEM++ code ([6]).

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot,\cdot)_{t,\Omega}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively, although we will omit Ω if there is no chance of misunderstanding. $H^1_D(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$.

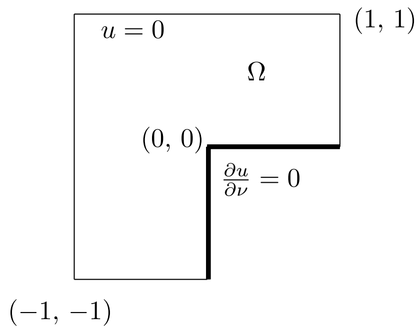

FIGURE 1. L-shape domain with a corner with a Neumann boundary condition

2. Extraction formula and algorithms

We need a cut-off function to derive the singular behavior of the problem. We set

$$B(r_1;r_2) = \{(r,\theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega$$

and

$$B(r_1) = B(0;r_1),$$

and define a smooth enough cut-off function of r as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in} \quad B(\frac{1}{2}\rho), \\ \frac{1}{16}\{8 - 15p(r) + 10p(r)^3 - 3p(r)^5\} & \text{in} \quad \overline{B}(\frac{1}{2}\rho;\rho), \\ 0 & \text{in} \quad \Omega\backslash\overline{B}(\rho), \end{cases} \tag{6}$$

with $p(r) = 4r/\rho - 3$. Here, ρ is a parameter which will be determined so that the singular part $\eta_\rho s$ has the same boundary condition as the solution u of the model problem, where s is the singular function which is given in (2). Note $\eta_\rho(r)$ is C^2.

The solution of the Poisson equation on the polygonal domain is well known ([1, 2, 5]). Given $f \in L^2(\Omega)$, if we assume there is only one reentrant corner with inner angle $\pi < \omega < 2\pi$, then there exists a unique solution u and in addition there exists a unique number λ such that

$$u - \lambda s \in H^2(\Omega). \tag{7}$$

By using the cut-off function $\eta = \eta_\rho$, we may write

$$u = w + \lambda \eta s, \tag{8}$$

with $w \in H^2(\Omega) \cap H_0^1(\Omega)$.

2.1. Extraction formula and theorems

The constant λ is referred as stress intensity factor and computed by the following formula ([4]);

Lemma 2.1. *The stress intensity factor λ can be expressed in terms of u and f by the following extraction formula:*

$$\lambda = \frac{1}{\pi}\int_\Omega f\eta s_- dx + \frac{1}{\pi}\int_\Omega u\Delta(\eta s_-)dx. \tag{9}$$

Assume that (1) has a solution u as in (8) and the stress intensity factor λ is known, then we introduce the following boundary value problem:

$$\begin{cases} -\Delta w &= f & \text{in } \Omega, \\ w &= -\lambda s & \text{on } \Gamma_D, \\ \frac{\partial w}{\partial \nu} &= 0 & \text{on } \Gamma_N. \end{cases} \tag{10}$$

Note the input function f is the same as in (1) and $s = s|_{\Gamma_D}$ is the restriction of the singular function s to the boundary Γ_D.

The following theorems show (10) has a regular solution. The proofs of the following two theorems are very similar to those in [7], although the singular function s is different. We just state them for the completeness.

Theorem 2.2. *If* (1) *has a solution u as in* (8) *with the stress intensity factor λ, then* (10) *has a unique solution w in $H^2(\Omega)$.*

Proof. First, we note (1) has a unique solution and its stress intensity factor is λ. The uniqueness of the solution of Poisson problem also implies the following equation has a unique solution with the stress intensity factor $-\lambda$:

$$\begin{cases} -\Delta p &= 0 & \text{in } \Omega, \\ p &= -\lambda s & \text{on } \Gamma_D, \\ \frac{\partial p}{\partial \nu} &= 0 & \text{on } \Gamma_N. \end{cases} \tag{11}$$

(Note $p = -\lambda s$ is the unique solution and the coefficient of the singular function s is the stress intensity factor.) By adding two equations, (1) and (11), we have the following equation:

$$\begin{cases} -\Delta w &= f & \text{in } \Omega, \\ w &= -\lambda s & \text{on } \Gamma_D, \\ \frac{\partial w}{\partial \nu} &= 0 & \text{on } \Gamma_N, \end{cases} \tag{12}$$

whose solution $w = u + p$ belongs to $H^2(\Omega)$. □

Theorem 2.3. *If λ is the stress intensity factor given by* (9) *with the solution u in* (1) *and w is the solution of* (10)*, then $u = w + \lambda s$ is the unique solution of* (1)*.*

Proof. We only need to show $u = w + \lambda s$ is the solution to (1) when w is the solution of (10). Since $\Delta s = 0$, we have

$$-\Delta u = -\Delta w - \lambda \Delta s = \Delta w = f.$$

Moreover, we have

$$u|_{\Gamma_D} = w|_{\Gamma_D} + \lambda s|_{\Gamma_D} = -\lambda s + \lambda s = 0,$$

and

$$\frac{\partial u}{\partial \nu}|_{\Gamma_N} = \frac{\partial w}{\partial \nu}|_{\Gamma_N} + \lambda \frac{\partial s}{\partial \nu}|_{\Gamma_N} = 0 + \lambda \cdot 0 = 0.$$

□

2.2. Two algorithms

Now we suggest two algorithms in variational form for the solution u of the model problem (1), say DSFM method and KL method. We use the DSFM method for comparision with our KL method. The KL method is the modifed algorithm for the mixed boundary problem form the one introduced in [8].

For the first algorithm we use the approximated stress intensity factor λ_{BD} form the formula in (9) with the approximated solution obtained by standard finite element method. Then we use input function $f + \lambda_{BD}\Delta(\eta s)$ instead of f. For the second algorithm we use the same input function f with changed boundary condition so that the solution has good regularity as in (10). Here we state two algorithms;

The first algorithm : DSFM

DSFM-1: To find $u \in H^1_D(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall\ v \in H^1_D(\Omega). \tag{13}$$

DSFM-2: Then compute $\lambda = \lambda_{BD}$ by (9) with u.

DSFM-3: To find $w \in H^1_D(\Omega)$ and

$$(\nabla w, \nabla v) = (f + \lambda_{BD}\Delta(\eta s), v), \quad \forall\ v \in H^1_D(\Omega). \tag{14}$$

DSFM-4: Finally set $u = w + \lambda_{BD}\eta s$.

The existence and uniqueness of the solution u and w in DSFM-1 and DSFM-3 is clear. Now we state the second algorithm:

The second algorithm : KL

KL-1: To find $u \in H^1_D(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall\ v \in H^1_D(\Omega). \tag{15}$$

KL-2: Then compute $\lambda = \lambda_{BD}$ by (9) with u.

KL-3: To find w such that $w + \lambda_{BD} s \in H^1_D(\Omega)$ and

$$(\nabla w, \nabla v) = (f, v), \quad \forall\, v \in H^1_D(\Omega). \tag{16}$$

KL-4: Finally set $u = w + \lambda_{BD} s$.

By Theorem 2.2 and Theorem 2.3 we have the solution $w \in H^2(\Omega)$ in **KL-3** and u, in **KL-4**, is the solution of (1).

3. Finite Element Approximation

In this section we present the standard finite element approximation for the algorithms considered in the previous section. Let T_h be a partition of the domain Ω into triangular finite elements; i.e., $\Omega = \cup_{K \in T_h} K$ with $h = \max\{\mathrm{diam} K : K \in T_h\}$. Let V_h be continuous piecewise linear finite element space; i.e.,

$$V_h = \{\phi_h \in C^0(\Omega) : \phi_h|_K \in P_1(K)\ \forall K \in T_h, \phi_h = 0 \text{ on } \Gamma_{\mathrm{D}}\} \subset \mathrm{H}^1_{\mathrm{D}}(\Omega),$$

where $P_1(K)$ is the space of linear functions on K.

Then the standard error analysis of the method in the standard norms, $\|\cdot\|$ and $|\cdot|_1$, can be carried out with a regular triangulation and continuous piecewise linear finite element space V_h (see [8]).

Note we can find approximated solution u_h using the following Algorithm.

Algorithm 1 (**A1 : DSFM**)

A1-1: To find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v) = (f, v) \quad \forall\, v \in V_h. \tag{17}$$

A1-2: Then compute $\lambda_{BD,h}$ by

$$\lambda_{BD,h} = \frac{1}{\pi} \int_\Omega f \eta s_- dx + \frac{1}{\pi} \int_\Omega u_h \Delta(\eta s_-) dx. \tag{18}$$

A1-3: To find $w_h \in V_h$ such that

$$(\nabla w_h, \nabla v) = (f + \lambda_{BD,h} \Delta(\eta s), v) \quad \forall\, v \in V_h. \tag{19}$$

A1-4: Then $u_h = w_h + \lambda_{BD,h} \eta s$.

The second approximation motivated from [8] is the following.

Algorithm 2 (**A2 : KL**)

A2-1: To find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v) = (f, v) \quad \forall\, v \in V_h. \tag{20}$$

A2-2: Then compute $\lambda_{BD,h}$ by

$$\lambda_{BD,h} = \frac{1}{\pi} \int_\Omega f \eta s_- dx + \frac{1}{\pi} \int_\Omega u_h \Delta(\eta s_-) dx. \tag{21}$$

A3-3: To find w_h such that $w_h + \lambda_{BD,h} s \in V_h$ and

$$(\nabla w_h, \nabla v) = (f, v), \quad \forall\, v \in V_h. \tag{22}$$

A4-4: Finally set $u_h = w_h + \lambda_{BD,h} s$.

4. Example and Numerical results

In this section as an example we consider a Poisson problem with the mixed boundary condition, together with the Neumann boundary condition posed on a concave corner with an inner angle $\omega = \frac{3\pi}{2}$.

Example 1. Consider the Poisson equation in (1) with mixed boundary conditions on the L-shape domain $\Omega = (-1,1) \times (-1,1) \setminus ([0,1] \times [-1,0])$ with $\Gamma_N = \{(0,y) \in R^2 \,:\, -1 < y < 0\} \cup \{(x,0) \in R^2 \,:\, 0 \le x < 1\}$ and $\Gamma_D = \partial\Omega \backslash \Gamma_N$ (see **Figure 1**). This problem has a singularity at the origin (0, 0), where the Neumann boundary conditions is satisfied with the internal angle $\omega = \frac{3\pi}{2}$. More specifically, the corresponding singular function has the form

$$s = r^{\frac{2}{3}} \cos(\frac{2\theta}{3}).$$

Let $\eta_* = \eta_{3/4}$ be the cut-off function in (6) with $\rho = 3/4$ and choose the right-hand side function in (1) to be

$$f = -\Delta(\eta_* s).$$

Then the exact solution of the underlying problem is

$$u = \eta_* s.$$

Note the solution of this problem is singular and the stress intensity factor is exactly 1. First we compute this example by the standard finite element method as in Table 1. Then we give the errors and convergence rates of approximated solutions by two algorithms, (**A1**) and (**A2**), are presented in Table 2 and 3, respectively.

h	$\|E\|_{L^2}$		$\|E\|_{H^1}$	
1/4	6.15737E-02	ratio	8.56289E-01	ratio
1/8	1.71935E-02	1.84045E+00	4.51119E-01	0.92459
1/16	5.27518E-03	1.70457E+00	2.36558E-01	0.93131
1/32	1.56417E-03	1.75382E+00	1.22677E-01	0.94733
1/64	5.37861E-04	1.54010E+00	6.48211E-02	0.92033
1/128	1.96364E-04	1.45371E+00	3.47017E-02	0.90146
1/256	7.47953E-05	1.39251E+00	1.90075E-02	0.86844

TABLE 1. Standard FEM : The L^2-error and H^1-error

h	λ_{BD}	$\|E\|_{L^2}$		$\|E\|_{H^1}$	
1/4	0.666579	6.53186E-02	ratio	9.06505E-01	ratio
1/8	0.860668	2.77534E-02	1.23483	5.40082E-01	0.74714
1/16	0.981446	6.73769E-03	2.04234	2.53121E-01	1.09335
1/32	0.993592	1.77880E-03	1.92135	1.34511E-01	0.91210
1/64	0.998858	4.43244E-04	2.00473	6.51405E-02	1.04610
1/128	0.999662	1.13850E-04	1.96096	3.40128E-02	0.93748
1/256	0.999946	2.81404E-05	2.01642	1.67311E-02	1.02355

TABLE 2. DSFM : The L^2-error and H^1-error

h	λ_{BD}	$\|E\|_{L^2}$		$\|E\|_{H^1}$	
1/4	0.666579	6.15737E-02	ratio	8.56289E-01	ratio
1/8	0.857109	1.55875E-02	1.98193	4.38744E-01	0.96472
1/16	0.979867	4.17363E-03	1.90101	2.24354E-01	0.96760
1/32	0.992853	1.03516E-03	2.01145	1.11783E-01	1.00507
1/64	0.998563	2.66962E-04	1.95515	5.63480E-02	0.98827
1/128	0.999544	6.86290E-05	1.95974	2.81864E-02	0.99936
1/256	0.999900	1.77735E-05	1.94909	1.40902E-02	1.00031

TABLE 3. Our algorithm : The L^2-error and H^1-error

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