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# Invariants of Local Rings under Completion

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Abstract

We study the behavior of some invariants under completion. We also give a counterexample, which is the same as a counterexample to Ding's Conjecture, to Koh-Lee's Conjecture.

**Keywords:** Column-Invariant, Row-Invariant, Completion, Gorenstein local rings, Ding's Conjecturen

## 1. Introduction

It is known[1] that there are certain restrictions on the entries of the maps in the minimal free resolutions of finitely generated modules of infinite projective dimension over Noetherian local rings $A$. Using these restrictions, some new invariants were introduced[2]: They are (see Definition 2.1): $col\,(A)$ for a number associated with the columns of the maps, $fpd(A)$ for a number of socle of module of finite projective dimension, and $crs(A)$, which is associated with the cyclic modules determined by regular sequences.

The purpose of this paper is to study of stability of these invariants under completion.

In section 2, we make some remarks on the behavior of these invariants under completion. From the equalities we established, and the facts[3], it follows that $crs(A)=crs(\widehat{A})$ and $col\,(A) = col\,(\widehat{A})$ for Gorenstein local rings $A$. We show that $fpd(A)=fpd(\widehat{A})$ for local rings $A$ with approximation property using a standard argument in such situations utilizing Artin approximation theorem. It follows that if $A$ is excellent, then $fpd(\widehat{A})=fpd(B)$, where $B$ is a pointed etale extension of $A$.

In section 3, we introduce a counterexample to the conjecture posed by Koh and Lee[3], which states $col\,(A) = crs\,(A)$ for a Noetherian local ring $A$. That counterexample is given by Stefani[4] to show that Dings's Conjecture may fail in general. We here note that Koh-Lee's Conjecture and Ding's Conjecture are equivalent for a Gorenstein local ring. We also add a remark on the applications of row-invariant.

Although all rings we consider in this paper are commutative, Noetherian with identity, and all modules are unital, we emphasize the Noetherian property in our statements.

## 2. Invariants under completion

In this section we first recall the invariants defined[2,5-7], and the Auslander $index(A)$ and the generalized Loewy length $\ell\ell(A)$. We also state the basic properties of these invariants. At the end, we prove that the invariant $fpd(-)$ is stable under completion for a ring $A$ with the approximation property.

**Definition 2.1.** Let $(A,\boldsymbol{m})$ be a Noetherian local ring. We denote by $\varphi_i(M)$ the $i$-th map in a minimal resolution of a finitely generated $A$-module $M$. We also use the usual notation $Soc(M) := \mathrm{Hom}_A(A/\mathrm{m}, M)$ to denote the socle of $M$.

i) $col(A)$ = inf { $t \geq 1$: for each finitely generated $A$-module $M$ of infinite projective dimension, each column of $\varphi_i(M)$ contains an element outside $\boldsymbol{m}^t$, for all $i > 1 + depth\ A$ }.

When $A$ is regular local, we interpret the above definition as $col(A) = 1$.

ii) $fpd(A)$ = inf { $t \geq 1$: $Soc(N) \not\subseteq \boldsymbol{m}^t N$ for some finitely generated
$A$-module $N$ of finite projective dimension }.

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iii) $crs(A)$ = inf { $t \geq 1$ : $Soc(A/\boldsymbol{x}) \not\subseteq \boldsymbol{m}^t(A/\boldsymbol{x})$ for some maximal regular sequence $\boldsymbol{x} = x_1, \cdots, x_d$ }.

We conjectured[2] that the invariants in the above would not change under completion, which is a behavior one naturally expects from any invariant. We established the inequalities as follows:

**Proposition 2.2.** ([Proposition 2.6, 2.7][2]) Let $f:(A,\boldsymbol{m}) \to (B,\boldsymbol{n})$ be a flat local homomorphism of Noetherian local rings.

i) $col(A) \leq col(B)$.

ii) If $B/\boldsymbol{n}$ is regular, then $fpd(B) \leq fpd(A)$ and $crs(B) \leq crs(A)$.

Therefore, if $\hat{A}$ is its completion, then we have

$$col(A) \leq col(\hat{A}) \leq fpd(\hat{A}) \leq fpd(A).$$

Secondly, we recall the definition of the generalized Loewy length of $A$:

$\ell\ell(A)$ = inf {$t \geq 1$: $\boldsymbol{m}^t \subseteq (\boldsymbol{x})$ for some system of parameters $\boldsymbol{x}$}.

Next, to define the index of a ring $A$, we recall the Auslander $\delta$-invariant[5,8,9]: Let $(A,\boldsymbol{m})$ be a Cohen-Macaulay local ring with a canonical module $\omega$. For a finitely generated $A$-module $X$, define

$f-rank(X) = r$ if $X = A^r \oplus U$, where $U$ has no free summands. Then $\delta(M)$ is defined as follows:

$\delta(M)$= inf {$f-rank(X)$: $X$ is a maximal Cohen-Macaulay module and $M$ is a homomorphic image of $X$}.

**Definition 2.3.** Let $(A,\boldsymbol{m})$ be a Cohen-Macaulay local ring with a canonical module. The index is define by

$$index(A) = \inf \{ t \geq 1 : \delta(A/\boldsymbol{m}^t) > 0 \}.$$

The generalized Loewy length $\ell\ell(A)$ can be described as $drs(A)$ when $A$ is Cohen-Macaulay, where

$drs(A)$= inf {$t \geq 1$: $Soc((A/(\boldsymbol{x}))^\vee) \not\subseteq \boldsymbol{m}^t((A/(\boldsymbol{x}))^\vee)$ for some system of parameters $\boldsymbol{x} = x_1, \cdots, x_d$ }.

It is known[2] that $crs(A) = drs(A)$, and thus $\ell\ell(A) = drs(A)$ if $A$ is Gorenstein. Also, $index(A)$ can be described as one of our invariants as follows:

**Theorem 2.4.** ([Corollary 1.7.][10]) Let $(A,\boldsymbol{m})$ be a Gorenstein local ring. Then

$$col(A) = index(A).$$

For a Gorenstein local ring $A$, it is known[3] (that $index(A)$ and $ll(A)$ are stable under completion, i.e., $index(\hat{A}) = index(A)$ and $\ell\ell(\hat{A}) = \ell\ell(A)$. Therefore, $col(A)$ and $crs(A)$ are also stable under completion provided that $A$ is Gorenstein by Theorem 2.4.

In the following, we apply a standard argument using Artin approximation theorem to show that for a local ring $A$ with the approximation property (see definition below), $fpd(\hat{A}) = fpd(A)$. We first recall the definition of the approximation property:

**Definition 2.5.** A Noetherian local ring $A$ is said to have the approximation property if for each $t \geq 1$ and for each system of polynomial equations $F_j(X_1, \cdots, X_q) : 1 \leq j \leq s$ with coefficients in $A$ having a solution $(\hat{a}_1, \cdots, \hat{a}_q) \in \hat{A}^q$, there is a solution $(a_1, \cdots, a_q) \in A^q$ such that $a_j \equiv \hat{a}_j \bmod \hat{\boldsymbol{m}}^t$ for all $1 \leq j \leq q$.

We remark that an excellent henselian local ring has the approximation property[11].

**Theorem 2.6.** Let $(A,\boldsymbol{m})$ be a Noetherian ring of dimension $d$ with the approximation property. Then $fpd(\hat{A}) = fpd(A)$.

**Proof.** Since we have $fpd(A) \geq fpd(\hat{A})$ by Proposition 2.2, we only need to show that $fpd(A) \leq fpd(\hat{A})$. Write $f = fpd(\hat{A})$ and let $Q$ be a finitely generated $\hat{A}$-module of finite projective dimension such that $Soc(Q) \not\subseteq \hat{\boldsymbol{m}}^f Q$.

Let

$$(G_\bullet, \psi_\bullet) : \cdots \to \hat{A}^{n_i} \xrightarrow{\psi_i} \hat{A}^{n_{i-1}} \to \cdots \to \hat{A}^{n_1} \xrightarrow{\psi_1} \hat{A}^{n_0} \to 0$$

be a minimal resolution of $Q$. For each $1 \leq i \leq d$, let $\Psi_i$ be a matrix of indeterminates of same size as $\psi_i$. Suppose that the image of $(\hat{a}_1, \cdots, \hat{a}_{n_0})$ is in $Soc(Q) - \hat{\boldsymbol{m}}^f Q$. Let $\boldsymbol{m} = (x_1, \cdots, x_n)$. Since the image of $(\hat{a}_1, \cdots, \hat{a}_{n_0})$ is in $Soc(Q)$, there are elements $\{\hat{y}_{ij}\}$ such that for each $1 \leq k \leq n$, $x_k(\hat{a}_1, \cdots, \hat{a}_{n_0}) = \sum_j \hat{y}_{kj} R_j$, where $R_j$ denotes the $j$-th row of $\psi_1$.

Let $Z = (Z_1, \cdots, Z_{n_0})$ and $Y = [Y_{kj}]$ be matrices of indeterminates of appropriate sizes. Let (※) denote the following system of polynomial equations with coefficients in $A$:

1) for each $1 \le i \le d-1$, each entry of $\Psi_{i+1} \cdot \Psi_i = 0$,

2) each entry of $(x_1, \cdots, x_n)^t \cdot Z - Y \cdot \Psi_1 = 0$.

Since (※) has a solution in $\hat{A}$ and $A$ has the approximation property, we

may find a solution in $A$ which agrees with the solution in $\hat{A}$ (mod $\hat{m}^f$). Let $\varphi_i$ be a solution corresponding to $\Psi_i$. We then have a complex

$$(F_\bullet, \varphi_\bullet) : \cdots \to A^{n_i} \xrightarrow{\varphi_i} A^{n_{i-1}} \to \cdots \to A^{n_1} \xrightarrow{\varphi_1} A^{n_0} \to 0.$$

By Corollary 6.3[12], $F_\bullet$ is acyclic. Let $a := (a_1, \cdots, a_{n_0})$ be a solution corresponding to $Z$. By definition of the system (※), $a$ is an element of $Soc(M)$, where $M := H_0(F_\bullet)$. Since the image of $a$ in $\hat{M}/\hat{m}^f\hat{M} \cong Q/\hat{m}^f Q$ is not 0, $Soc(M) \not\subseteq m^f M$. Hence $fpd(\hat{A}) = fpd(A)$.

**Corollary 2.7.** Let $(A, m)$ be an excellent Noetherian local ring. Then there is a pointed etale extension $B$ of $A$ such that $fpd(B) = fpd(\hat{A})$. ■

**Proof.** The conclusion follows from Theorem 2.7 and the fact that the henselization of $A$ is a directed union of pointed etale extensions of $A$. ■

## 3. Some Remarks

In this section, we introduce a counterexample to the conjecture posed by Koh and Lee[2]. We first recall their conjecture and Ding's Conjecture:

**Conjecture** (Koh-Lee[2,13]) Let $A$ be a Noetherian local ring with the minimal reduction. Then $col(A) = crs(A)$, and if $A$ is Cohen-Macaulay then $row(A) = drs(A)$.

**Ding's Conjecture**[5-7,14] Let $(R, m, k)$ be a Gorenstein local ring, where $k$ is infinite. Then $index(R) = \ell\ell(R)$.

Since $col(R) = index(R)$ and $crs(R) = \ell\ell(R)$ if $R$ is Gorenstein by Theorem 2.7, we can say that Koh-Lee's Conjecture is equivalent to Ding's conjecture when $R$ is a Gorenstein local ring. Koh and Lee prove[2] that their conjecture is in the affirmative in a few cases. For instances, if a non-regular Cohen-Macaulay local ring $A$ has a minimal multiplicity, i.e., $mult(A) = 1 + edim(A) - \dim(A)$, then these invariants are all equal to 2 (see Corollary 3.7[2]), and if $A$ is hypersurface, these invariants are the same as the multiplicity of $A$ (see Theorem 4.3[2]).

Also, they prove that if a ring $A$ is of depth 0 (and thus if $A$ is of dimension 0), then the conjecture holds. However, it is recently shown[4] that the conjecture may not be true in general if $A$ has a dimension greater than 0. Stefani[4] gives a counterexample to Ding's conjecture. In fact, M. Hashimoto and A. Shida[4] showed that Ding's conjecture cannot hold if the residue field of a ring is finite; their example is $A = F[[x,y]]/(xy(x+y))$, where $F$ is a field with two elements. Stefani proves[4] that Ding's Conjecture may fail even though the residue field of a ring is infinite:

**Counterexample to Conjecture**[4] Let $S = k[x,y,z]_{(x,y,z)}$, where $k$ is any field, and let $n = (x,y,z)S$ its maximal ideal. Let $R = S/I$, where $I = (x^{2-}y^5, xy^2 + yz^3 - z^5)S$. Then $R$ is one dimensional local complete intersection domain. It can be shown by using CoCoA and Macaulay2 that $index(R) = 5$ and $\ell\ell(R) = 6$, which says Ding's Conjecture fails for $R$. Since Ding's Conjecture is equivalent to Koh-Lee's Conjecture in the case of Gorenstein ring, Koh-Lee's Conjecture also fails; $col(R) = 5$, and $crs(R) = 6$.

We close this article with an additional remark. Using the theory developed from column-row invariants, some previously known results in commutative ring theory are slightly improved; for examples, Herzog's extension of Kunz's result to a characterization of modules of finite projective and injective dimensions in characteristic $p > 0$ (Corollary 2.8[1]), and Eisenbud's and Dutta's results on the existence of free summands in syzygy modules (Proposition 2.2[1]). We give one more application of row-invariant regarding Bass Conjecture. We first recall two old conjectures, which are theorems now since 'Intersection Theorem' was proved by P. Roberts.

**Two Theorems.** Let $(A, m)$ be a Noetherian local ring.

(1) (Bass Conjecture) If there is a finite $A$-module $M$ of finite injective dimension, then a Noetherian local ring $A$ is Cohen-Macaulay.

(2) (Zero Divisor Conjecture) if $x \in A$ is $M$-regular, where $M$ is a finite $A$-module of finite projective dimension, then $x$ is $A$-regular.

We may pose the following questions:

**Question 3.1.** Let $(A, m)$ be a Noetherian local ring.

(1) Suppose that there is a finite $A$-module $M$ of finite injective dimension, and $x$ is $A$-regular. Then is there a finite $A/xA$-module $M$ of finite injective dimension?

(2) Let $M$ be a finite $A$-module such that $M$ is of finite injective dimension. When $x \in A$ is $M$-regular, is $x$ also $A$-regular?

We can show that Question 3.1 (1) is equivalent to Bass Conjecture in the theorem below: we may find a proof[15], but since it is not stated explicitly anywhere, we include a proof for reader's convenience. However, we don't know the relation between Zero Divisor Conjecture and Question 3.1 (2).

Before we state a theorem, let's recall the definition of row-invariant; $row(A)$ = inf { $t \geq 1$: for each finitely generated $A$-module $M$ of infinite projective dimension, each row of $\varphi_i(M)$ contains an element outside $m^t$, for all $i > depth(A)$ }.

**Theorem 3.2.** If Question 3.1(2) is true, then it is equivalent to Bass Conjecture.

**Proof.** Suppose that there is a finite $A$-module $M$ of finite injective dimension, and a non zero divisor $x$ in $A$ such that $A/xA$ has a finite $A/xA$-module $M$ of finite injective dimension. If $A$ is of dimension 0, then clearly, $A$ is Cohen-Macaulay. We claim that if $\dim(A) > 0$, then $depth(A) > 0$. (In particular, if $\dim(A) = 1$, then $depth(A) = 1$, and so $A$ is Cohen-Macaulay.) Suppose to the contrary that $\dim(A) > 0$, but $depth(A) = 0$. Since there is a finite $A$-module of finite injective dimension, $row(A)$ is a finite value[1]. Since $\dim(A) > 0$, there is an $x \in m$ such that $x^* = x^{row(A)} \neq 0$ (if not, $m^{row(A)} = 0$). Then $A/x^*A$ has an infinite minimal resolution:

$$\cdots \to A \xrightarrow{x^*} A \to A/x^*A \to 0,$$

which contradicts to the definition of $row(A)$. Thus $depth(A) > 0$. Now if $\dim(A) > 0$, and $depth(A) > 0$, then by assumption there is a non zero divisor $x$ in $A$ such that $A/xA$ has a finite $A/xA$-module $M$ of finite injective dimension. By induction, $A/xA$ is Cohen-Macaulay, and so is $A$, i.e., Bass Conjecture holds.

Conversely, suppose that Bass Conjecture holds. Then $\overline{A} = A/xA$ is also Cohen-Macaulay for a nonzero divisor $x$ in $A$. We know that $(\overline{A}/(y)\overline{A})^{\vee}$ is a finite $\overline{A}$-module of finite injective dimension, where $(y)$ is a system of parameters of $\overline{A}$, and $(-)^{\vee}$ denotes the Matlis dual. ■

**Corollary 3.3.** Let $M$ be a finite $A$-module of finite injective dimension. Suppose that $M$ has a positive depth, and that Question 3.1(2) is true. Then $A$ is Cohen-Macaulay, i.e., Bass Conjecture holds.

**Proof.** Since the depth of $M$ is positive, there is $x \in A$ which is $M$-regular, and so $A$-regular. Now, we can complete the proof by Theorem 3.2, and the fact: For an $A$ and $M$-regular element $x$, $injdim_{A/xA} M/xM = injdim_A M - 1$. ■

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