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SURFACES FOLIATED BY ELLIPSES WITH CONSTANT GAUSSIAN CURVATURE IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we study the surfaces foliated by ellipses in three dimensional Euclidean space $\mathbf{E}^3$. We prove the following results: **(1)** The surface foliated by an ellipse have constant Gaussian curvature K if and only if the surface is flat, i.e. $K = 0$. **(2)** The surface foliated by an ellipse is a flat if and only if it is a part of generalized cylinder or part of generalized cone.

1. Introduction

A cyclic surface in Euclidean space $\mathbf{E}^3$ is a surface determined by a smooth uni-parametric family of pieces of circles [4, 5]. The first example of cyclic surface is a surface of revolution, that is, a surface which is stable under a group of rotations that leave a straight-line point wise fixed. The Riemann examples play a major role in the theory of minimal surfaces [10]. Enneper [4,5] proved that for a cyclic minimal surface, the planes containing the circles must be parallel and it is one of the examples obtained by Riemann. Nitsche [9] Proved that the surface must be an open set of a sphere or, in non-spherical case, the circles must be lie in parallel planes. In the latter case, the only possibilities are the surfaces of revolution discovered by Delaunay [3].

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Let $\Gamma=\Gamma(u)$ be an orthogonal smooth curve to each u-plane of the foliation and denote by u its arc-length parameter. We assume that the planes of the foliation are not parallel and let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ be a moving frame of the curve Γ, where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ denote the tangent, principal normal and binormal vectors respectively. Locally we parameterize the surface $M(u,v)$ by

(1)

$$\mathbf{X}(u,v)=\mathbf{C}(u)+r(u)\Big(a\,\cos[v]\,\mathbf{n}+b\,\sin[v]\,\mathbf{b}\Big),\quad v\in[0,2\pi],\quad a\neq b\in R$$

where $r=r(u)>0$ and $\mathbf{C}=\mathbf{C}(u)$ denote the center of each ellipse of the foliation, the Frenet equations of the curve Γ are

$$\begin{cases}\mathbf{t}'=\kappa\,\mathbf{n}.\\ \mathbf{n}'=-\kappa\,\mathbf{t}+\tau\,\mathbf{b}.\\ \mathbf{b}'=-\tau\,\mathbf{n}.\end{cases}\tag{2}$$

where the prime $'$ denotes the derivative with respect to the u-parameter, κ and τ are the curvature and torsion of Γ respectively. We assume that $\kappa\neq 0$ because Γ is not straight line and let

$$\mathbf{C}'(u)=\alpha\,\mathbf{t}+\beta\,\mathbf{n}+\gamma\,\mathbf{b}\tag{3}$$

where α, β and γ are smooth functions on u.

Without loss of generality, we can assume that $a=1$. It is worth noting that when $b=1$ the surface (1) is a cyclic surface foliated by a smooth one-parameter family of circles in three dimensional Euclidean space $\mathbf{E}^3$ which studied in Lopez [6–8]. In [6] he studied a surface in Euclidean three space $\mathbf{E}^3$ with constant Gauss curvature foliated by circles. In [7, 8] he studied surfaces in Euclidean 3-space that satisfy a special Linear Weingarten (LW) condition of linear type as $\kappa_1=c_1\kappa_2+c_2$ and $c_3H+c_4K=c_5$, where $c_i\,i=1,2,...,5$ are real numbers and $\kappa_1$and k_2 denote the principal curvatures while H and K denotes the the mean and Gaussian curvatures at each point of the surface. Also, he proved that:

(1): *All cyclic surface with non-zero constant Gauss curvature must be a surface of revolution* [6].

(2): *All cyclic LW-surface with* $\kappa_1=c_1\kappa_2$ *must be of Riemann type* [7].

(3): *All cyclic LW-surface with* $c_3H + c_4K = c_5$ *must be a surface of revolution, a Riemann minimal surface or a generalized cone* [8].

In this paper, we will study the surface foliated by an ellipse. In this case, the surface takes the form (1) such that $b \neq 1$. We will prove the following main results for constant Gaussian curvature:

THEOREM 1.1. *The surface (1) foliated by an ellipse in* $\mathbf{E}^3$ *is a flat if and only if it is a part of generalized cylinder or part of generalized cone.*

THEOREM 1.2. *The surface (1) foliated by an ellipse has constant Gaussian curvature* K *in* $\mathbf{E}^3$ *if and only if* $K = 0$.

On the other hand, Let M be a surface in $\mathbf{R}^3$ foliated by a pieces of ellipses in parallel planes. Without loss of generality, we assume that the planes of the foliation are parallel to x_1x_2-plane. Let

$$\text{(4)} \quad \mathbf{X}(u,v) = \Big(f(u) + r(u)\cos[v], g(u) + b\,r(u)\sin[v], u\Big),\ u \in I,\ v \in J,$$

be a local parametrization of M. Then, we have the following theorem:

THEOREM 1.3. *Let* M *be a surface (4) in* $\mathbf{E}^3$ *with constant Gaussian curvature* $K = K_0$ *and foliated by pieces of ellipses in parallel planes. Then*

(1): $K_0 = 0$.

(2): M *must be parameterized, up a rigid motion of* $\mathbf{E}^3$, *as*

$$\text{(5)} \quad \mathbf{X}(u,v) = \Big(f_1\,u + f_0, g_1\,u + g_0, u\Big) + \Big(r_1\,u + r_0\Big)\Big(\cos[v],\ b\,\sin[v],\ 0\Big),$$

where f_0, f_1, g_0, g_1, r_0, r_1, $b \in \mathbf{R}$.

As a corollary of both theorem 1.2 and 1.3, we obtain:

COROLLARY 1.4. *All surfaces foliated by an ellipses with constant Gauss curvatures must be a surfaces of revolution.*

2. Gaussian curvatures

Consider M a surface in $\mathbf{E}^3$ parameterized by $\mathbf{X} = \mathbf{X}(u,v)$ and let $\mathbf{U}$ denote the unit normal vector field on M . The tangent vectors to the parametric curves of the surface $\mathbf{X}(u, v)$ are

$$\mathbf{X}_u = \frac{\partial\, \mathbf{X}}{\partial\, u}, \qquad \mathbf{X}_v = \frac{\partial\, \mathbf{X}}{\partial\, v},$$

and the unit normal on the surface is given by

$$\mathbf{U} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}$$

where $\times$ denotes the cross product of $\mathbf{E}^3$. The first fundamental quadratic form on the surface is

$$I = \langle\, d\mathbf{X}, d\mathbf{X}\,\rangle = E\, du^2 + 2F\, du\, dv + G\, dv^2$$

with the first fundamental coefficients

$$E = \langle\, \mathbf{X}_u, \mathbf{X}_u\,\rangle, \qquad F = \langle\, \mathbf{X}_u, \mathbf{X}_v\,\rangle, \qquad G = \langle\, \mathbf{X}_v, \mathbf{X}_v\,\rangle.$$

The second fundamental quadratic form is given by

$$II = \langle\, \mathbf{U}, d^2\mathbf{X}\,\rangle = e\, du^2 + 2f\, du\, dv + g\, dv^2,$$

with second fundamental coefficients

$$e = \langle\, \mathbf{U}, \mathbf{X}_{uu}\,\rangle, \qquad f = \langle\, \mathbf{U}, \mathbf{X}_{uv}\,\rangle, \qquad g = \langle\, \mathbf{U}, \mathbf{X}_{vv}\,\rangle.$$

Under this parameterization of the surface $\mathbf{X} = \mathbf{X}(u, v)$, the Gaussian curvature K is

$$K = \frac{e\, g - f^2}{E\, G - F^2}. \tag{6}$$

The proof of our results depend on we can reduce the equation $K =$ constant to an expression as a linear combination of the trigonometric function $\{\cos[i\, v], \sin[i\, v]\}$, $i \in \mathbb{N}$, namely,

$$\sum_{i=0}^{8} \Big(E_i \cos[i\, v] + F_i \sin[i\, v]\Big) = 0 \tag{7}$$

and E_i and F_i are functions on the variable u. In particular, the coefficients must vanish. The work then is to compute explicitly these coefficients E_i and F_i by successive manipulations. The author were able to obtain the results using the symbolic program Mathematica to check their work. The computer was used in each calculation several times, giving understandable expressions of the coefficients E_i and F_i.

Although the explicit computation of the Gaussian curvature K can be obtained, for example, by using the Mathematica programme, its expression is some cumbersome. However, the key in our proofs lies that one can write K as

$$K = \frac{\mathcal{P}\big(\cos[i\,v], \sin[i\,v]\big)}{\mathcal{Q}\big(\cos[i\,v], \sin[i\,v]\big)} = \frac{\sum_{i=0}^{4}\big(A_i\cos[i\,v] + B_i\sin[i\,v]\big)}{\sum_{i=0}^{8}\big(C_i\cos[i\,v] + D_i\sin[i\,v]\big)}. \tag{8}$$

The assumption of the constancy of the scalar curvature K implies that (8) converts into

$$K\ \mathcal{Q}(\cos[i\,v], \sin[i\,v]) -\ \mathcal{P}(\cos[i\,v], \sin[i\,v]) = 0. \tag{9}$$

Equation (9) means that if we write it as a linear combination of the functions $\{\cos[i\,v], \sin[i\,v]\}$ namely, (7), the corresponding coefficients must vanish. From here, we will be able to describe all surfaces foliated by an ellipses with constant Gaussian curvature. As we will see, it is not necessary to give the (long) expression of K but only the coefficients of higher order for the trigonometric functions.

We distinguish the cases $K = 0$ and $K \neq 0$. Without loss of generality, we can put $\tau(u) = \lambda(u)\,\kappa(u)$ and $\beta(u) = \mu(u)\,\kappa(u)$, where λ and μ are functions of u.

3. Proof Theorem 1.1

In this section we assume that $K = 0$ on the surface $\mathbf{X}(u, v)$. From (9), we have

$$\mathcal{P}(\cos[i\,v], \sin[i\,v]) = \sum_{i=0}^{4}(A_i\cos[i\,v] + B_i\sin[i\,v]) = 0.$$

Then the work consists in the explicit computations of the coefficients A_i and B_i. We distinguish different cases that fill all possible cases. Since

$$B_4 = \frac{b\,r^2\,\kappa^3}{2}\Big[2\,\mu\,\gamma - (1-b^2)\big(r^2\,\lambda' - 2\,\lambda\,\mu\,\alpha\big)\Big] \tag{10}$$

must be vanished, then we have two possibilities for μ as the following:

3.1. When $\mu \neq 0$ then $\gamma = \dfrac{(b^2-1)\Big(2\,\lambda\,\mu\,\alpha - r^2\,\lambda'\Big)}{2\,\mu}$. A straightforward computation shows that:

$$A_3 = -b^2\,\kappa^3\Big[3\,f\,\alpha - \Big(4\,\mu\,r' - r\,\mu'\Big)\,r\Big], \tag{11}$$

where $f = \Big[(b^2-1)\,\lambda^2-1\Big]\,r^2$. According to the condition $A_3 = 0$, let us distinguish two possibilities of f as as the following:

3.1.1. *When* $f \neq 0$ *then* $\alpha = \dfrac{\Big(4\,\mu\,r' - r\,\mu'\Big)\,r}{3\,f}$. From $B_2 = 0$ and $B_3 = 0$, their imply

(12)
$$\left\{\begin{array}{l} (b^2-1)\,\kappa\,f\,r^2\,\big(2\,f\,r' - r\,f'\big)\,\mu'' = 4\,f\,\kappa^3\,\mu\,\Big[b^2\,f\,\big(3\,r^2\,r' - 4\,\mu^2\,r' + r\,\mu\,\mu'\big) \\ \qquad +b^2\,r^2\,\mu\,\big(4\,\mu\,r' - r\,\mu'\big) - 3\,f^2\,r'\Big] + (1-b^2)\,\Bigg[6\,f^3\,k^3\,\mu\,r' \\ \qquad +4\,\kappa\,f'\,\big(r\,\mu' - 4\,\mu\,r'\big)\,\Big[8\,\mu^2\,r' + r\big(f' - 2\,\mu\,\mu'\big)\Big] + f^2\Bigg(40\,\kappa\,\mu\,r6\prime 3 \\ \qquad -2\,r^2\,r'\,\kappa'\,\mu' + r\,\Big[3\,\mu\,\kappa^3\,f' + 8\,\mu\,\kappa'\,r'^2 - 8\,\kappa\,r'\big(2\,r'\,\mu' + \mu\,r''\big)\Big]\Bigg) \\ \qquad +f\,\Bigg(64\,\kappa\,\mu^3\,r'^3 + r^3\,\kappa'\,\mu'\,f' - 4\,\kappa\,\mu\,r\,r'^2\,\big[3\,f' + 8\,\mu\,\mu'\big] \\ \qquad +r^2\,\Big[6\,\kappa\,\mu'\,r'\,f' + 4\,\mu\,\Big(\kappa\,r'\,\mu'^2 + f'\,\big[\kappa\,r' - r'\,\kappa'\big]\Big)\Big]\Bigg)\Bigg], \end{array}\right.$$

(13)
$$\left\{\begin{array}{l} 2\,\kappa\,\mu\,f\,r^2\,\big(f - r^2\big)\,f'' = 4\,\kappa\,\mu^2\,r^3\,f'\,\big(r\,\mu' - 4\,\mu\,r'\big) \\ \qquad +r\,f\,\Bigg[\mu\,\Bigg(\kappa\Big[r\,f'^2 + 32\,r\,\mu^2\,r'^2 + 4\,f'\,r'\,\big(4\,\mu^2 - 3\,r^2\big)\Big] - 2\,r^3\,f'\,\kappa'\Bigg) \\ \qquad -2\,\kappa\,r\,\mu'\Big[f'\,\big(r^2 + 2\,\mu^2\big) + 4\,\mu^2\,r\,r'\Big]\Bigg] \\ \qquad +4\,f^3\,\Big(r\,\kappa\,\mu\,r'' - r\,\Big[r\,\mu\,\kappa' + \kappa\,\big(4\,\mu\,r' + r\,\mu'\big)\Big]\Big) \\ \qquad +2\,f^2\,\Bigg[4\,\kappa\,\mu\,r\,r'\,\big(f' + \mu\,\mu'\big) - 16\,\kappa\,\mu^3\,r'^2 + r^2\big(10\,\kappa\,\mu\,r'^2 + \mu\,\kappa\,f' \\ \qquad +\kappa\,\mu'\,f'\big) + 2\,r^3\,\Big[r'\,\big(\mu\,\kappa' + \kappa\,\mu'\big) - \kappa\,\mu\,r''\Big]\Bigg]. \end{array}\right.$$

From the above equation, we will consider two cases:

3.1.1.1 $f \neq f_0\, r^2$, for all arbitrary constant f_0. If we substitute μ'' and f'' from (12) and (13), in the coefficient B_1 we obtain the following condition:
(14)
$$\Big(2\,f\,r' - r\,f'\Big)\,\Big[\Big(r\,\mu' - 4\,\mu\,r'\Big)\mu - 3\,f\,r'\Big]\,\Big[\Big(r\,\mu' - 4\,\mu\,r'\Big)^2 - 9\,f^2\,\kappa^2\Big] = 0.$$

The above condition reduces to two solutions: $\mu' = \dfrac{4\,\mu\,r' \pm 3\,\kappa\,f}{r}$ and $\mu' = \dfrac{\left(3\,f + 4\,\mu^2\right)r'}{\mu\,r}$.

(I): $\mu' = \dfrac{4\,\mu\,r' + 3\,\kappa\,f}{r}$. If we substitute μ' from this equation and μ'' from (12), in the condition $\mu'' - \dfrac{d\mu'}{du} = 0$, then we can obtain the following condition:

$$(15)\qquad \Big(\kappa\,\mu - r'\Big)\,\Big[5\,(b^2-1)\,(r\,f' - 2\,f\,r') + 4\,b^2\,\kappa\,\mu\,(f - r^2)\Big] = 0.$$

The above condition yields two solutions:

(I-A): $f' = 2\left[\dfrac{f\,r'}{r} - \dfrac{2\,b^2\,\kappa\,\mu\,(f - r^2)}{5\,(b^2-1)\,r}\right]$. Again the condition $f'' - \dfrac{df'}{du} = 0$ reduced that:

$$(16)\qquad \Big(f - r^2\Big)\,\Big[2\,(14\,b^2 - 15)\,\kappa\,\mu - 15\,(b^2-1)\,r'\Big] = 0.$$

Because $f \neq r^2$, the above condition leads to $r' = \dfrac{2\,(14\,b^2 - 15)\,\kappa\,\mu}{15\,(b^2-1)}$. Now, a straightforward computation shows that, the remain coefficients are:

(17)

$$\begin{cases} A_0 = \dfrac{b^2\,r\,\kappa^4}{50\,(b^2-1)}\Big[57\,b^2\,r^2 - 9\left(218\,b^2-225\right)f - \dfrac{\left(51\,b^2-55\right)\left(71\,b^2-75\right)\mu^2}{b^2-1}\Big],\\[2ex] A_1 = \dfrac{4\,b^2\,r\,\kappa^4}{45\,(b^2-1)}\Big[18\left(b^2\,r^2 - \left(34\,b^2-35\right)f\right) - \dfrac{\left(25\,b^2-27\right)\left(43\,b^2-45\right)\mu^2}{b^2-1}\Big],\\[2ex] A_2 = \dfrac{2\,b^2\,r\,\kappa^4}{225\,(b^2-1)}\Big[18\left(3\,b^2\,r^2 - 2\left(49\,b^2-50\right)f\right) - \dfrac{\left(2925-5580\,b^2+2659\,b^4\right)\mu^2}{b^2-1}\Big],\\[2ex] A_4 = b^2\,r^2\,\kappa^4\left[\mu^2 - f - \dfrac{b^2\left(r^2-f\right)}{25\,(b^2-1)}\right]. \end{cases}$$

The system $\{A_0 = 0,\ A_1 = 0,\ A_2 = 0,\ A_4 = 0\}$ is an algebraic system of four unknown r, μ, f and b. Solving this system by Mathematica Program, we have $r = \pm\sqrt{6}\,f$, $\mu = 0$ and $b = \pm\dfrac{\sqrt{5}}{2}$ which is contradiction with $\mu \neq 0$.

(I-B): $\mu = \dfrac{r'}{\kappa}$. From this equation and equation of μ' above, the condition $\mu' - \dfrac{d\mu}{du} = 0$, gives

$$r'' = \frac{3\,f\,\kappa^3 + 4\,\kappa\,r'^2 + r\,r'\,\kappa'}{r\,\kappa}. \tag{18}$$

From $A_1 = 0$, it implies

$$f' = \frac{18\,f\,\kappa^2\,r' + 16\,r'^3}{r\,\kappa^2}. \tag{19}$$

The condition $f'' - \dfrac{df'}{du} = 0$ gives

$$r'^2\left(f\,\kappa^2 - r'^2\right)\Big[\left(17\,f - 25\,r^2\right)\kappa^2 - 8\,r'^2\Big] = 0. \tag{20}$$

If $r' = 0$, then $\mu = 0$ contradiction. Therefore the above condition reduce two solutions:

(I-B1): $f = \frac{1}{17}\left[25\, r^2 + \frac{8\, r'^2}{\kappa^2}\right]$. Again the condition $f' - \frac{df}{du} = 0$ leads $r'^2\left(\kappa^2\, r^2 + r'^2\right) = 0$ which is contradiction.

(I-B2): $f = -\frac{r'^2}{\kappa^2}$. Again the condition $f' - \frac{df}{du} = 0$ gives

$$r\, r'\, \kappa' + \kappa\left(r'^2 - r\, r''\right) = 0. \tag{21}$$

The general solution of this equation is

$$r(s) = r_0 \exp\Big[r_1 \int \kappa(s)\, ds\Big], \quad r_0,\, r_1 \in R. \tag{22}$$

Now, all coefficients A_i and A_j are equal zero. In this case we have:

(23)

$$\begin{cases} \lambda(u) = m, & m = \sqrt{\dfrac{1 + r_1^2}{b^2 - 1}}, \qquad \tau(u) = m\, \kappa(u), \\ \alpha(u) = -\kappa(u)\, r(s), & \beta(u) = r_1\, \kappa(u)\, r(u), \quad \gamma(u) = -\Big(\dfrac{1 + r_1^2}{m}\Big)\, \kappa(u)\, r(u). \end{cases}$$

Therefore, the parametrization of the tangent of the curve $\mathbf{C}(u)$ is given by:

$$\mathbf{C}'(u) = -r'(u)\left[\frac{(1 + r_1^2)\, \mathbf{b} + m\, \mathbf{t}}{m\, r_1}\right] + r_1\, \kappa(u)\, r(u)\, \mathbf{n}.$$

Since $\lambda(u) = m$, then $\Gamma(u)$ is a general helix, we can prove that:

$$-\frac{d}{du}\left[\frac{(1 + r_1^2)\, \mathbf{b} + m\, \mathbf{t}}{m\, r_1}\right] = r_1\, \kappa(u)\, \mathbf{n}.$$

Hence, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\mathbf{C}(u) = \mathbf{C}_0 - r(u)\left[\frac{(1 + r_1^2)\, \mathbf{b} + m\, \mathbf{t}}{m\, r_1}\right].$$

The parametrization of this surface is given by

$$X(u, v) = \mathbf{C}_0 - r(u)\left[\frac{(1 + r_1^2)\, \mathbf{b} + m\, \mathbf{t}}{m\, r_1} - \cos[v]\, \mathbf{n} - b\, \sin[v]\, \mathbf{b}\right], \tag{24}$$

where $b = \frac{\sqrt{1 + m^2 + r_1^2}}{m}$, $r(u) = r_0 \exp\Big[r_1 \int \kappa(u)\, du\Big]$ such that the curvature of the base curve $\kappa(u)$ is arbitrary function of u and r_0, r_1 and the ratio $\dfrac{\tau(u)}{\kappa(u)} = m$ are arbitrary constants.

(II): $\mu' = \dfrac{\left(3\,f + 4\,\mu^2\right) r'}{\mu\, r}$. If, we compute the μ'' and compare it with (12) we can obtain the following condition:

$$r'' = \frac{\kappa^2\,\mu^2}{r} + \frac{3\,r'^2}{r} + \frac{3\,f^2\,r'^2}{r\,\mu^2} + \frac{r'\,\kappa'}{\kappa}. \tag{25}$$

If we substitute r'' from the equation $A_1 = 0$ and solve it we get:

$$f' = \frac{2\,f\,\kappa^2\,\mu^2\,r' + 2\left(17\,f + 16\,\mu^2\right) r'^3}{2\,r\left(\kappa^2\,\mu^2 + r'^2\right)}. \tag{26}$$

Again, by computing f'' from (26) and equating with f'' from (13), yields the following condition:

$$r'^2\left(f+\mu^2\right)\left[\kappa^2\,\mu^2\left(f-r^2\right)\left(3\,\kappa^2\,\mu^2+7\,r'^2\right)+8\left(3\,f-5\,r^2-2\,\mu^2\right) r'^4\right] = 0.$$

The above condition implies three cases:

(II-A): $f = \dfrac{8\left(2\,\mu^2 + 5\,r^2\right) r'^4 + \kappa^2\,\mu^2\,r^2\left(3\,\kappa^2\,\mu^2 + 7\,r'^2\right)}{3\,\kappa^4\,\mu^4 + 7\,\kappa^2\,\mu^2\,r'^2 + 24\,r'^4}$. In this case we have

$$A_1 = -\frac{32\,b^2\,r\,\kappa\,r'^3\left(r^2+\mu^2\right)}{\mu^3\left(3\,\kappa^4\,\mu^4 + 7\,\kappa^2\,\mu^2\,r'^2 + 24\,r'^4\right)^2}\Big[3\,\kappa^8\,\mu^8 + 26\,\kappa^6\,\mu^6\,r'^2 + 217\,\kappa^4\,\mu^4\,r'^4 + 314\,\kappa^2\,\mu^2\,r'^6 + 840\,r'^8\Big]. \tag{27}$$

It is easy to see that $A_1 = 0$ if $r = r_0$, where r_0 is an arbitrary constant. The condition $r'' - \dfrac{d^2r}{du^2}$ gives $\kappa^2\,\mu^2 = 0$ contradiction.

(II-B): When $f = -\mu^2$, then $\mu' = \dfrac{\mu\, r'}{r}$ which means $\mu(u) = \mu_1\, r(u)$, where μ_1 is an arbitrary constant. From (25) we get $r'' = \mu_1^2\, r\,\kappa^2 + \dfrac{\kappa'\,r'}{r}$. By solving this equation, one can obtain $r(s) = R_0 \cosh\Big[R_1 + \mu_0 \int \kappa(s)\, ds\Big]$. Now, all coefficients A_i and A_j are zero. Therefore, we have:

(28)

$$\begin{cases} \lambda(s) = m, & m = \sqrt{\dfrac{1+\mu_1^2}{b^2-1}}, & \tau(s) = m\,\kappa(u), \\ \alpha(u) = -\dfrac{r'(u)}{\mu_1}, & \beta(u) = \mu_1\,\kappa(u)\,r(u), & \gamma(u) = -\dfrac{(1+\mu_1^2)\,r'(u)}{m\,\mu_1}. \end{cases}$$

The parametrization of the tangent of the curve $\mathbf{C}(u)$ can be written in the following form:

$$\mathbf{C}'(u) = -r'(u)\left[\frac{(1+\mu_1^2)\,\mathbf{b} + m\,\mathbf{t}}{m\,r_1}\right] + \mu_1\,\kappa(u)\,r(u)\,\mathbf{n}.$$

Because $\lambda(u) = m$, then $\Gamma(u)$ is a general helix and then:

$$-\frac{d}{du}\left[\frac{(1+\mu_1^2)\,\mathbf{b} + m\,\mathbf{t}}{m\,\mu_1}\right] = \mu_1\,\kappa(u)\,\mathbf{n}.$$

So that, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\mathbf{C}(u) = \mathbf{C}_0 - r(u)\left[\frac{(1+\mu_1^2)\,\mathbf{b} + m\,\mathbf{t}}{m\,\mu_1}\right].$$

Now, the parametrization of this surface is given by

$$X(u,v) = \mathbf{C}_0 - r(u)\left[\frac{(1+\mu_1^2)\,\mathbf{b} + m\,\mathbf{t}}{m\,\mu_1} - \cos[v]\,\mathbf{n} - b\,\sin[v]\,\mathbf{b}\right], \tag{29}$$

where $b = \dfrac{\sqrt{1+m^2+\mu_1^2}}{m}$, $r(u) = R_0 \cosh\left[R_1 + \mu_0 \int \kappa(u)\,du\right]$ such that the curvature of the base curve $\kappa(s)$ is arbitrary function of u and R_0, R_1, μ_1 and the ratio $\dfrac{\tau(u)}{\kappa(u)} = m$ are arbitrary constants.

3.1.1.2 $f = f_0\,r^2$, where f_0 is an arbitrary constant. In this case, the solution of the equation $A_4 = 0$ is $\mu(u) = \mu_2\,r(u)$, where $\mu_2 = \sqrt{-f_0}$ is an arbitrary constant. Here, all coefficients A_i and A_j are zero.

Therefore, we can obtain the following:

(30)
$$\begin{cases} \lambda(s) = m, & m = \sqrt{\dfrac{1+\mu_2^2}{b^2-1}}, & \tau(s) = m\,\kappa(u), \\ \alpha(u) = -\dfrac{r'(u)}{\mu_2}, & \beta(u) = \mu_2\,\kappa(u)\,r(u), & \gamma(u) = -\dfrac{(1+\mu_2^2)\,r'(u)}{m\,\mu_2}. \end{cases}$$

Hence, the parametrization of the tangent of the curve $\mathbf{C}(u)$ is given by:

$$\mathbf{C}'(u) = -r'(u)\left[\frac{(1+\mu_2^2)\,\mathbf{b} + m\,\mathbf{t}}{m\,r_2}\right] + \mu_2\,\kappa(u)\,r(u)\,\mathbf{n}.$$

Since $\lambda(u) = m$, then $\Gamma(u)$ is a general helix, we can prove that:

$$-\frac{d}{du}\left[\frac{(1+\mu_2^2)\,\mathbf{b} + m\,\mathbf{t}}{m\,\mu_2}\right] = \mu_2\,\kappa(u)\,\mathbf{n}.$$

Therefore, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\mathbf{C}(u) = \mathbf{C}_0 - r(u)\left[\frac{(1+\mu_2^2)\,\mathbf{b} + m\,\mathbf{t}}{m\,\mu_2}\right].$$

Now, the parametrization of this surface is given by

$$X(u,v) = \mathbf{C}_0 - r(u)\left[\frac{(1+\mu_2^2)\,\mathbf{b} + m\,\mathbf{t}}{m\,\mu_2} - \cos[v]\,\mathbf{n} - b\,\sin[v]\,\mathbf{b}\right], \tag{31}$$

where $b = \dfrac{\sqrt{1+m^2+\mu_2^2}}{m}$, while $r(u)$ is an arbitrary function of u. Also, μ_1 and the ratio $\dfrac{\tau(u)}{\kappa(u)} = m$ are an arbitrary constants while the curvature and $\kappa(s)$ of the base curve is arbitrary function of u.

3.1.2. *When* $f = 0$, *then* $\lambda = \pm\dfrac{1}{\sqrt{b^2-1}}$. For computation, $A_4 = \dfrac{1}{2}\,b^2\,r\,\kappa^4\,\mu$ leads contradiction with $\mu \neq 0$.

3.2. $\mu = 0$**.** Then $B_4 = \dfrac{1}{2}\, b\,(b^2-1)\,\kappa^3\, r^3\, \lambda'$ is vanished when $\lambda(u) = \lambda_0$, where λ_0 is arbitrary constant. Then $A_4 = 3\, b^2 \big[\lambda_0^1 \big(b^2-1\big) - 1\big]\, r^2\, \kappa^3\, \alpha$, that is $\alpha = 0$ or $\lambda = \dfrac{1}{\sqrt{b^2-1}}$. If $\alpha = 0$, then $\Big(\alpha,\, \beta, \gamma\Big) = (0,\, 0,\, 0)$ contradiction. Now, $\lambda = \dfrac{1}{\sqrt{b^2-1}}$ and the equation $A_4 = 0$, lead $\gamma(u) = \sqrt{b^2-1}\,\alpha(u)$. Therefore $A_2 = \dfrac{2\, b^3\, r^2\, \kappa^3\, r'}{\sqrt{b^2-1}} = 0$, gives $r(u) = r_0$, where r_0 is arbitrary constant. Now all coefficients A_i, B_i are trivially zero and the parametrization of the tangent of the curve $\mathbf{C}(u)$ is given by

$$\mathbf{C}'(u) = \frac{\alpha(s)}{m}\,\Big(m\,\mathbf{t} + \mathbf{b}\Big),$$

where $m = \dfrac{1}{\sqrt{b^2-1}}$. Since $\Gamma(u)$ is a general helix, then $\dfrac{d\big(m\,\mathbf{t} + \mathbf{b}\big)}{du} = 0$ and hence, $m\,\mathbf{t} + \mathbf{b}$ is a fixed vector. Therefore, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\mathbf{C}(u) = \mathbf{C}_0 + \big(m\,\mathbf{t} + \mathbf{b}\big)\,\Omega(u),$$

where $\Omega(u) = \dfrac{1}{m}\int \alpha(u)\, du$ is arbitrary function of u. The parameterization of this surface is given by

$$X(u,v) = \mathbf{C}_0 + \big(m\,\mathbf{t} + \mathbf{b}\big)\,\Omega(u) + c\,\Big(m\,\cos[v]\,\mathbf{n} + \sqrt{1+m^2}\,\sin[v]\,\mathbf{b}\Big), \quad (32)$$

where $c = \dfrac{r_0}{m}$ and m are arbitrary constants.

Remark 3.1. The developable surface (24) foliated by an ellipse is a special surface of (31) when $r(u) = r_0 \exp\Big[r_1 \int \kappa(u)\, du\Big]$.

Remark 3.2. The developable surface (29) foliated by an ellipse is a special surface of (31) when $r(u) = R_0\, \cosh\Big[R_1 + \mu_0 \int \kappa(u)\, du\Big]$.

Lemma 3.3. *The surface (1) foliated by an ellipse is developable if and only if it takes the general forms (31) or (32).*

Lemma 3.4. *If the base curve* $\Gamma(u)$ *is not a general helix, then the surface (1) foliated by an ellipse is non-developable.*

Now, we will introduce the position vector of two obtained developable surfaces foliated by an ellipse in Euclidean 3-space. Firstly, we write the following Theorem

THEOREM 3.5. *The position vector* **C** *of a general helix whose tangent vector makes a constant angle with a fixed straight line in the space, is expressed in the natural representation form as follows:*

(33)
$$\mathbf{C}(u) = \sqrt{1-n^2}\int\Big(\cos\Big[\sqrt{1+m^2}\int\kappa(u)du\Big],\sin\Big[\sqrt{1+m^2}\int\kappa(u)du\Big],m\Big)du,$$

where $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ *and* ϕ *is the angle between the fixed straight line* $\mathbf{e}_3$ *(axis of a general helix) and the tangent vector of the curve* **C**.

(1): The position vector $\mathbf{X}(u,v) = \Big(x_1,\, x_2,\, x_3\Big)$ of the surface (31) takes the following form:

$$(34)\quad \begin{cases} x_1 = r(u)\Bigg[\Big(Q-\sqrt{1+Q^2}\,\sin[v]\Big)\,\cos\big[\Phi\big]-\cos[v]\,\sin\big[\Phi\big]\Bigg], \\[2ex] x_2 = r(u)\Bigg[\Big(Q-\sqrt{1+Q^2}\,\sin[v]\Big)\,\sin\big[\Phi\big]+\cos[v]\,\cos\big[\Phi\big]\Bigg], \\[2ex] x_3 = \dfrac{\sqrt{1+Q^2}\,r(u)}{m\,Q}\Bigg[Q\,\sin[v]-\sqrt{1+Q^2}\Bigg], \end{cases}$$

where $Q = \dfrac{\mu_2}{\sqrt{1+m^2}}$, and $\Phi = \sqrt{1+m^2}\int\kappa(u)du$. Note that in this surface, $r(s)$ and $\kappa(s)$ are arbitrary function of the variable u while Q and m are arbitrary constants. It is worth noting that for the above surface we have

$$x_1^2 + x_2^2 = \Big(\frac{m^2\,Q^2}{1+Q^2}\Big)\,x_3^2.$$

So that, we can write the following Lemma:

LEMMA 3.6. *The developable surface (31) foliated by an ellipse is represented a generalized cone in Euclidean 3-space.*

(2): The position vector of the surface (32) reduces to the following form:

$$\mathbf{X}(u,v) = \Big(X_1,\, X_2,\, X_3\Big)) = \Bigg(- M_0 \,\sin\big[\Theta\big], M_0 \,\cos\big[\Theta\big], \Omega\Bigg) \tag{35}$$

where $M_0 = m\,c$, $\Theta = v - \sqrt{1+m^2}\,\int \kappa(u)du$ and $\Psi = \Omega(u) + c\,\sin\Big[\Theta + \sqrt{1+m^2}\,\int \kappa(u)du\Big]$. Note that, $\Omega(u)$ and $\kappa(u)$ are arbitrary functions of the variable u while m and c are an arbitrary constant. It is easy to see that, for the above surface the following condition is satisfied:

$$X_1^2 + X_2^2 = M^2.$$

The above condition gives the following Lemma:

Lemma 3.7. *The developable surface (32) foliated by an ellipse is represents a generalized cylinder in Euclidean 3-space.*

From the above, the main Theorem (1.1) is proved.

4. Proof Theorem 1.2

In this section, we assume that the the surface (1) has a non-zero constant Gaussian curvature K. In this case equation (9) can be written in the form

$$\sum_{i=0}^{8} E_i(u)\cos[i\,v] + F_i(u)\sin[i\,v] = 0$$

One begin to compute the coefficients E_i and F_i. The first coefficient

$$E_8 = \frac{1}{16}\,(b^2-1)\,K\,\kappa^4\,r^6\,\big[1-(b^2-1)\,\lambda^2\big] = 0,$$

leads to

$$\lambda = \pm\frac{1}{\sqrt{b^2-1}}.$$

The coefficient F_6 becomes

$$F_6 = 2\,b\,(b^2-1)\,K\,\kappa^3\,r^4\,\mu\,\big[\gamma - \alpha\,\sqrt{b^2-1}\big].$$

For vanishes the above coefficient, we have two possibilities:

4.1. When $\mu(u) = 0$, the coefficient E_6 becomes:

$$E_6 = (1-b^2)\,K\,\kappa^2\,r^4\left(\alpha\sqrt{b^2-1}-\gamma\right)^2.$$

The vanishing of E_6 gives $\gamma = \alpha\sqrt{b^2-1}$. Now, $F_4 = 4\,b^3\sqrt{b^2-1}\,K\,\kappa^3\,r^5\,r' = 0$, which means $r = r_0$ where r_0 is arbitrary constant. Finally, the coefficient $E_4 = -b^4\,K\,\kappa^4\,r^6$ leads to contradiction.

4.2. $\gamma = \alpha\sqrt{b^2-1}$**.** Now, $F_6 = b^2\,(b^2-1)\,K\,\kappa^4\,r^4\,\mu^2 = 0$ gives $\mu = 0$ and this case has been discussed previously. Therefore the proof of the Theorem 1.2 is completed.

5. Proof Theorem 1.3

Let M be a surface (4)in $\mathbf{E}^3$ with constant Gauss curvature $K = K_0$ and foliated by a pieces of ellipses in parallel planes. If we put $K_0 = \dfrac{P}{W}$ in the computations of the Gauss curvature K, yields

(36)
$$W = r\Big[\left(1+g'^2\right)\left(1-\cos[2v]\right)+b^2\left(1+f'^2\right)\left(1+\cos[2v]\right) \\ +2\,b\left(b\,r'^2+2\,r'\left(g'\sin[v]+b\,f'\cos[v]\right)+2\,f'\,g'\,\sin[2v]\right)\Big]^2,$$

(37)
$$P = -4\,b\left(b\,r''+g''\sin[v]+b\,f''\cos[v]\right).$$

5.1. $K_0 \neq 0$**.** The identity (9) writes as:

$$\sum_{i=0}^{4}\left(A_i\cos[i\,v]+B_i\sin[i\,v]\right)=0.$$

A computations yields

(38)
$$\begin{cases} A_4 = -\dfrac{K_0\,r}{2}\left(b^4\,f'^4-6\,b^2\,f'^2\,g'^2+g'^4+(b^2-1)\left[b^2\left(1+2\,f'^2\right)-\left(1+2\,g'^2\right)\right]\right), \\ B_4 = 2\,b\,K_0\,r\,f'\,g'\left[1+g'^2-b^2\left(1+f'^2\right)\right]. \end{cases}$$

According to the condition $B_4 = 0$, we have distinguish three possibilities as the following:

5.1.1. $f' = 0$ *and* $f(u) = f_0$, *where* f_0 *is arbitrary constant.* In this case $A_4 = \dfrac{K_0\, r}{2}\left(b^2 - 1 - g'^2\right)^2 = 0$ which implies $g(u) = \pm\sqrt{b^2-1}\,u + g_0$, where g_0 is an arbitrary constant and $|b| > 1$. Therefore the coefficient A_2 implies $8\,(b^2-1)\,K_0\,r\,r'^2 = 0$ and thus $r(u) = r_0$, where r_0 is arbitrary constant. In this case, $A_0 = 0$, yields $4\,b^4\,K_0\,r_0 = 0$ contradiction.

5.1.2. $g' = 0$ *and* $g(u) = g_0$, *where* g_0 *is arbitrary constant.* In this case $A_4 = \dfrac{K_0\, r}{2}\left[1 - b^2\left(1 + f'^2\right)\right]^2 = 0$ which implies $f(u) = \pm\dfrac{\sqrt{1-b^2}\,u}{b} + f_0$, where f_0 is an arbitrary constant and $|b| < 1$. Therefore the coefficient A_2 implies $8\,(b^2-1)\,K_0\,r\,r'^2 = 0$ and thus $r(u) = r_0$, where r_0 is arbitrary constant. With these conditions $A_0 = 4\,K_0\,r_0 = 0$ contradiction.

5.1.3. $f' \neq 0$ *and* $g' \neq 0$ *at some* u*-interval.* Since $B_4 = 0$, then $g' = \pm\sqrt{b^2\left(1+f'^2\right) - 1}$. Taking in account $A_4 = 0$, it follows

$$2\,b^2\,K_0\,r\,f'^2\left[b^2\left(1+f'^2\right) - 1\right] = 0,$$

which leads to $f(u) = \left(\sqrt{\dfrac{1}{b^2} - 1}\right)u + f_0$. Then, the coefficient $A_2 = 8\,b^2\,K_0\left(b^2-1\right)r\,r'$ must be vanishes. Therefore $r(u) = r_0$ and $A_0 = -4\,K_0\,r_0 = 0$ contradiction.

5.2. $K_0 = 0$**.** Identity (9) leads to $P = 0$. In view of the above expression of P, it follows that $r'' = f'' = g'' = 0$. As consequence, there are constants r_0, r_1, f_0, f_1, g_0 and g_1 such that

$$\left\{\begin{array}{ll} r(u) & = r_1\,u + r_0, \\ f(u) & = f_1\,u + f_0, \\ g(u) & = g_1\,u + g_0, \end{array}\right. \tag{39}$$

that is, the functions f, g and r are linear on u and so, the surface is a generalized cone. Therefore the proof of Theorem 1.3 is completed.

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