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THE DETERMINANT MAP FROM THE AUTOMORPHISM GROUP OF A PROJECTIVE R-MODULE TO THE UNIT GROUP OF R

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Abstract. Let P be a finitely generated projective module over a commutative ring R with identity. If P has finite rank, then it will be shown that the map $\varphi : Aut_R(P) \to U(R)$ defined by $\varphi(\alpha) = det(\alpha)$ is locally surjective and $Ker(\varphi) = SL_R(P)$.

1. Introduction

Throughout this paper every *ring* R will be a commutative ring with identity and every *module* will be a finitely generated unitary R-module.

In section 2, we deal with properties of endomorphism rings of free modules which will be used later.

In section 3, if L is a projective R-module of rank 1, then we prove that $End_R(L) = R$. Let P be a projective R-module with rank $n < \infty$. Then $det(P) = \wedge^n P$ has rank 1 so that $End_R(det(P)) = R$. Hence since $\alpha \in Aut_R(P)$ if and only if $det(\alpha) = \wedge^n \alpha \in Aut_R(det(P)) = U(End_R(det(P)))$, we have

$$Aut_R(P) = U(End_R(P)) = \{\alpha \in End_R(P) \mid det(\alpha) \in U(R)\}.$$

If P is a projective R-module with finite rank, then we show in Theorem 3.5 that the map $\varphi : Aut_R(P) \to U(R)$ defined by $\varphi(\alpha) = det(\alpha)$ is locally surjective and $Ker(\varphi) = SL_R(P)$.

The study of this paper was motivated from the papers of [1] - [7].

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2. Endomorphism Rings of Free Modules

In this section, we deal with the exterior powers of free modules. For a given free module, we will discuss how the wedge product of its endomorphism, the determinant of its endomorphism, and the adjoint of its endomorphism are related. The results in this section will be used in the following sections.

Lemma 2.1. *Let R be a ring and let F be a free R-module of rank n. The r-th exterior power $\wedge^r F$ of F is a free R-module of rank ${}_nC_r$.*

Proof. Let F be a free R-module with a basis $x_1, \cdots, x_n$. Then $F = \oplus_{i=1}^n Rx_i$. So, $\wedge^r F = \oplus_{1\le i_1<i_2\cdots<i_r\le n} Rx_{i_1} \wedge Rx_{i_2} \wedge \cdots \wedge Rx_{i_r}$ and hence $\wedge^r F$ has an R-free basis $\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\}$. Hence $rank(\wedge^r F) = {}_nC_r$. □

Let $A \in Mat_{m\times n}(R)$. Define a map $L_A : R^n \to R^m$ by $L_A(x) = Ax$, where $x \in R^n$. Then L_A is an R-homomorphism. This is called the *left multiplication* by A. In R^n, let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then $\{e_1, \cdots, e_n\}$ is an R-free basis for the R-free module R^n. Define a map $\Phi : End_R(R^n) \to Mat_{n\times n}(R)$ by $\Phi(\sigma) = [\sigma]_{\{e_1,\cdots,e_n\}}$, where $\sigma \in End_R(R^n)$. Then Φ is an R-isomorphism. For $\alpha \in End_R(R^n)$, we define the *determinant*, denoted by $det\ (\alpha)$, of α to be the determinant of the matrix $[\alpha]_{\{e_1,\cdots,e_n\}}$. Also, we define the *adjoint*, $adj\ (\alpha)$, of α to be $\Phi^{-1}(adj([\alpha]_{\{e_1,\cdots,e_n\}}))$. We define a map $\wedge^n\alpha : \wedge^n R^n \to \wedge^n R^n$ by

$$(\wedge^n\alpha)(p_1 \wedge \cdots \wedge p_n) = \alpha(p_1) \wedge \alpha(p_2) \wedge \cdots \wedge \alpha(p_n),$$

where $p_1, \cdots, p_n \in R^n$. Then it is easy to prove that $\wedge^n\alpha \in End_R(\wedge^n R^n)$.

Lemma 2.2. *Let R be a ring. Let F be an R-free module of rank 1. If we define a map $\Phi : R \to End_R(F)$ by $\Phi(r) =$ the left multiplication by r, where $r \in R$, then Φ is an R-isomorphism.*

Corollary 2.3. *For every ring R, $End_R(\wedge^n R^n) \cong R$.*

Proof. $\wedge^n R^n$ is a rank 1 R-free module with an R-free basis $\{e_1 \wedge \cdots \wedge e_n\}$. Hence the result follows from Lemma 2.2. □

We will identify each element $r \in R$ with the left multiplication by r, so that we may identify R with $End_R(\wedge^n R^n)$.

Lemma 2.4. *Let R be a ring. Then the following statements are true.*

1. *For every $\alpha \in End_R(R^n)$, $\wedge^n\alpha = det(\alpha)$.*
2. *For every α, $\beta \in End_R(R^n)$,*

$$(\wedge^n\alpha) \circ (\wedge^n\beta) = det(\alpha)\, det(\beta) = det(\alpha \circ \beta) = \wedge^n(\alpha \circ \beta).$$

Proof. (1) Consider the standard ordered R-free basis $\{e_1, \cdots, e_n\}$ for R^n and write

$$[\alpha]_{\{e_1,e_2,\cdots,e_n\}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Then

$$\begin{aligned} \wedge^n\alpha(e_1 \wedge e_2 \wedge \cdots \wedge e_n) &= \alpha\,(e_1) \wedge \alpha\,(e_2) \wedge \cdots \wedge \alpha\,(e_n) \\ &= (a_{11}e_1 + a_{21}e_2 + \cdots + a_{n1}e_n) \\ &\quad \wedge(a_{12}e_1 + a_{22}e_2 + \cdots + a_{n2}e_n) \\ &\quad \wedge \quad \cdots\cdots \\ &\quad \wedge(a_{1n}e_1 + a_{2n}e_2 + \cdots + a_{nn}e_n) \\ &= det\,([\alpha]_{\{e_1,\,e_2,\,\cdots,\,e_n\}})(e_1 \wedge e_2 \wedge \cdots \wedge e_n) \\ &= det(\alpha)(e_1 \wedge e_2 \wedge \cdots \wedge e_n). \end{aligned}$$

By the proof of Lemma 2.1, $\{e_1 \wedge e_2 \wedge \cdots \wedge e_n\}$ is an R-free basis for the rank 1 free R-module $\wedge^n(R^n)$. Hence $\wedge^n\alpha$ is the left multiplication by $det(\alpha)$, so that $\wedge^n\alpha = det(\alpha)$.

(2) Note that

$$\begin{aligned} ((\wedge^n\alpha) \circ (\wedge^n\beta))(e_1 \wedge \cdots \wedge e_n) &= (\wedge^n\alpha)((\wedge^n\beta)(e_1 \wedge \cdots \wedge e_n)) \\ &= (\wedge^n\alpha)(det(\beta)(e_1 \wedge \cdots \wedge e_n)) \\ &= det(\beta)((\wedge^n\alpha)(e_1 \wedge \cdots \wedge e_n)) \\ &= det(\beta)(det(\alpha)(e_1 \wedge \cdots \wedge e_n)) \\ &= det(\beta)\, det(\alpha)(e_1 \wedge \cdots \wedge e_n). \end{aligned}$$

Then $(\wedge^n\alpha) \circ (\wedge^n\beta)$ is the left multiplication by $det(\beta)\, det(\alpha)$, so that $(\wedge^n\alpha) \circ (\wedge^n\beta) = det(\beta)\, det(\alpha) = det(\alpha)\, det(\beta)$, since R is commutative. The remainder of the proof is routine. □

$U(R)$ denotes the group of units of a ring R.

Lemma 2.5. *The following statements are true.*

1. *If* $\alpha \in End_R(R^n)$, *then* $\alpha(adj(\alpha)) = (det(\alpha))id_{R^n}$ *and* $(adj(\alpha))\alpha = (det(\alpha))id_{R^n}$.
2. *If* $\alpha \in Aut_R(R^n)$, *then* $det(\alpha) \in U(R)$ *and* $\alpha^{-1} = (det(\alpha))^{-1}adj(\alpha)$.

Proof. (1) Note that

$$[\alpha]_{\{e_1, e_2, \cdots, e_n\}}(adj([\alpha]_{\{e_1, e_2, \cdots, e_n\}})) = (det([\alpha]_{\{e_1, e_2, \cdots, e_n\}}))I_n = det(\alpha)I_n.$$

As in previous paragraph of Lemma 2.2, let $\Phi : End_R(R^n) \to Mat_{n\times n}(R)$ be defined by $\Phi(\sigma) = [\sigma]_{\{e_1, \cdots, e_n\}}$.

$$\begin{aligned}
\alpha(adj(\alpha)) &= \Phi^{-1}([\alpha]_{\{e_1,e_2,\cdots,e_n\}})\Phi^{-1}(adj([\alpha]_{\{e_1,e_2,\cdots,e_n\}}))\\
&= \Phi^{-1}([\alpha]_{\{e_1,e_2,\cdots,e_n\}}adj([\alpha]_{\{e_1,e_2,\cdots,e_n\}}))\\
&= \Phi^{-1}((det(\alpha))I_n)\\
&= (det(\alpha))\Phi^{-1}(I_n)\\
&= (det(\alpha))id_{R^n}.
\end{aligned}$$

By a similar proof, we can show that $(adj(\alpha))\alpha = (det(\alpha))id_{R^n}$.

(2) Assume that $\alpha \in Aut_R(R^n)$. Then there exists $\beta \in End_R(R^n)$ such that $\alpha \circ \beta = id_{R^n}$ and $\beta \circ \alpha = id_{R^n}$. Then by Lemma 2.4,

$$det(\alpha)\, det(\beta) = \wedge^n(\beta \circ \alpha) = \wedge^n(id_{R^n}) = 1.$$

By a similar proof, we can show that $det(\beta)\, det(\alpha) = 1$. It follows that $det(\alpha)$ is a unit in R, so that $det(\alpha) \in U(R)$. Now, use (1) to prove the remainder. □

Corollary 2.6. $Aut_R(R^n) = \{\alpha \in End_R(R^n) \,|\, det(\alpha) \in U(R)\}$.

Let a be any element of $U(R)$. Write

$$diag\{a, 1, \cdots, 1\} = \begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Consider the left multiplication $L_{diag\{a, 1, \cdots, 1\}}$ by $diag\{a, 1, \cdots, 1\}$. Then $L_{diag\{a, 1, \cdots, 1\}}$ is an R-automorphism on R^n. Define a map $s : U(R) \to Aut_R(R^n)$ by $s(a) = L_{diag\{a, 1, \cdots, 1\}}$. Then we can see that s is a group homomorphism. Moreover,

$$\begin{aligned}(det \circ s)(a) &= det(L_{diag\{a, 1, \cdots, 1\}}) \\ &= det([L_{diag\{a, 1, \cdots, 1\}}]_{\{e_1, e_2, \cdots, e_n\}}) \\ &= det(diag\{a, 1, \cdots, 1\}) \\ &= a,\end{aligned}$$

so that $det \circ s = id_{U(R)}$ (s is called a *section* of det). This shows that the determinant map $det : Aut_R(R^n) \to U(R)$ is a surjective group homomorphism and the exact sequence of groups

$$0 \longrightarrow SL_R(R^n) \xrightarrow{inc} Aut_R(R^n) \xrightarrow{det} U(R) \longrightarrow 0$$

splits. Here, $SL_R(R^n)$ denotes $Ker(det)$. Therefore, $Aut_R(R^n)$ is the semidirect product, denoted by $SL_R(R^n) \rtimes s(U(R))$, of the normal subgroup $SL_R(R^n)$ and a subgroup isomorphic to the unit group $U(R)$ of R. In fact, we have the following result:

Theorem 2.7. *$Aut_R(R^n) = SL_R(R^n) \rtimes s(U(R))$ and $s(U(R)) \cong U(R)$.*

Proof. Let α be any element of $Aut_R(R^n)$. Then with the same notations as in the statements prior to this result,

$$\alpha = \alpha \circ ((s \circ det)(\alpha))^{-1} \circ s(det(\alpha)).$$

Furthermore, since det is a group homomorphism, it follows from Lemma 2.4 that

$$\begin{aligned}det(\alpha \circ ((s \circ det)(\alpha))^{-1}) &= det((s \circ det(\alpha))^{-1})det(\alpha) \\ &= (det(s \circ det(\alpha)))^{-1}det(\alpha) \\ &= (det(\alpha))^{-1}det(\alpha) \\ &= 1,\end{aligned}$$

so that $\alpha \circ ((s \circ det)(\alpha))^{-1} \in SL_R(R^n)$. Hence $\alpha \in SL_R(R^n) \circ s(U(R))$. This shows that $Aut_R(R^n) = SL_R(R^n) \circ s(U(R))$.

Let $\alpha \in SL_R(R^n) \cap s(U(R))$. Then $\alpha = s(a)$ for some $a \in U(R)$ and hence $a = id_{U(R)}(a) = (det \circ s)(a) = det(\alpha) = 1$. This implies that $\alpha = s(a) = s(1) = id_{R^n}$. Hence $SL_R(R^n) \cap s(U(R)) = \{id_{R^n}\}$.

Since $det \circ s = id_{U(R)}$, we can see that s is injective. This shows that $s(U(R)) \cong U(R)$. Therefore the proof is completed. □

For any $\alpha \in s(U(R))$, define $\varphi_\alpha : SL_R(R^n) \to SL_R(R^n)$ by $\varphi_\alpha(\beta) = \alpha\beta\alpha^{-1}$, where $\beta \in SL_R(R^n)$. Then $\varphi_\alpha \in Inn(SL_R(R^n)) \subseteq Aut_R(SL_R(R^n))$. Define a map $\varphi : s(U(R)) \to Aut_R(SL_R(R^n))$ by $\varphi(\alpha) = \varphi_\alpha$, where

$\alpha \in s(U(R))$. Then φ is a group homomorphism. Consider the cartesian product $SL_R(R^n) \times s(U(R))$. Define the multiplication on this cartesian product as follows: $(\alpha,\, s(a))(\beta,\, s(b)) = (\alpha \circ \varphi_{s(a)}(\beta),\, s(a) \circ s(b))$. Then the cartesian product with this multiplication forms a group. This group is called the *external semidirect product*, denoted by $SL_R(R^n) \rtimes_\varphi s(U(R))$, of $SL_R(R^n)$ and $s(U(R))$.

Theorem 2.8. *$Aut_R(R^n) \cong SL_R(R^n) \rtimes_\varphi s(U(R))$.*

Proof. Define a map $\Delta : Aut_R(R^n) \to SL_R(R^n) \rtimes_\varphi s(U(R))$ by $\Delta(\alpha) = (\alpha \circ (s(det(\alpha)))^{-1},\, s(det(\alpha)))$, where $\alpha \in Aut_R(R^n)$. For any $\alpha,\, \beta \in Aut_R(R^n)$,

$$\begin{aligned}
\Delta(\alpha)\Delta(\beta) &= (\alpha \circ (s(det(\alpha)))^{-1},\, s(det(\alpha)))(\beta \circ (s(det(\beta)))^{-1},\, s(det(\beta))) \\
&= (\alpha \circ (s(det(\alpha)))^{-1} \circ \varphi_{s(det(\alpha))}(\beta \circ (s(det(\beta)))^{-1}), \\
&\qquad s(det(\alpha)) \circ s(det(\beta))) \\
&= (\alpha \circ \beta \circ (s(det(\beta))^{-1}) \circ (s(det(\alpha))^{-1}),\, s(det(\alpha)det(\beta))) \\
&= (\alpha \circ \beta \circ (s(det(\alpha)) \circ s(det(\beta)))^{-1},\, s(det(\alpha)det(\beta))) \\
&= (\alpha \circ \beta \circ (s(det(\alpha \circ \beta)))^{-1},\, s(det(\alpha \circ \beta))) \\
&= \Delta(\alpha \circ \beta)
\end{aligned}$$

If $(\alpha \circ (s(det(\alpha)))^{-1},\, s(det(\alpha))) = (id, id)$, where $\alpha \in s(U(R))$, then $\alpha = id$. For any $(\alpha,\, s(a)) \in SL_R(R^n) \rtimes s(U(R))$, take $\beta = \alpha \circ s(a)$. Then $\beta \in Aut_R(R^n)$ and $det(\beta) = det(\alpha \circ s(a)) = det(\alpha)det(s(a)) = a$, so that

$$\begin{aligned}
\Delta(\beta) &= (\beta \circ (s(det(\beta)))^{-1},\, s(det(\beta))) \\
&= (\beta \circ (s(a))^{-1},\, s(a)) \\
&= (\alpha,\, s(a)).
\end{aligned}$$

Therefore, Δ is a group isomorphism. □

We have constructed the following commutative diagram of groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & SL_R(R^n) & \xrightarrow{inc} & Aut_R(R^n) & \xrightarrow{det} & U(R) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi|_{SL_R(R^n)}} & & \downarrow{\scriptstyle \Phi} & & \| & & \\
0 & \longrightarrow & SL_n(R) & \xrightarrow{inc} & GL_n(R) & \xrightarrow{det} & U(R) & \longrightarrow & 0
\end{array}$$

Here, the two row exact sequences split and the first two vertical arrows are isomorphisms.

Corollary 2.9. *$GL_n(R) = SL_n(R) \rtimes s(U(R))$ and $s(U(R)) \cong U(R)$, and $GL_n(R) \cong SL_n(R) \rtimes_\varphi s(U(R))$.*

Let us summarize the results:

$$\begin{array}{ccccccc}
Aut_R(R^n) & = & SL_R(R^n) & \rtimes & s(U(R)) & \text{and} & s(U(R)) \cong U(R) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
GL_n(R) & = & SL_n(R) & \rtimes & s(U(R)) & \text{and} & s(U(R)) \cong U(R),
\end{array}$$

and

$$\begin{array}{ccccc}
Aut_R(R^n) & \xrightarrow{\sim} & SL_R(R^n) & \rtimes_\varphi & s(U(R)) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
GL_n(R) & \xrightarrow{\sim} & SL_n(R) & \rtimes_\varphi & s(U(R)).
\end{array}$$

3. Endomorphism Rings of Projective Modules

We adopt the definition of the *rank* of a projective module in [5, Definition 2.3.1] and [3, Definitions 2.2.11]. We will discuss various properties of free modules and more generally of projective modules and their endomorphism rings which are available for later discussion. The main objective of this section is to show that if P is a projective R-module with finite rank, then the map $\varphi : Aut_R(P) \to U(R)$ defined by $\varphi(\alpha) = det(\alpha)$ is locally surjective. However, φ is not (globally) surjective in general. We will give an example of this.

If L is a projective R-module of rank 1, then every R-homomorphism $L \to L$ is scalar. We prove this as follows:

Lemma 3.1. *If L is a rank 1 projective R-module, then $End_R(L) \cong R$.*

Proof. Define a map $\varphi : R \to End_R(L)$ by

$$\varphi(a) = \text{the left multiplication by } a.$$

Then φ is an R-homomorphism. To show that φ is bijective, it suffices to show that for every $\mathfrak{p} \in Spec(R)$, the $R_\mathfrak{p}$-homomorphism $\varphi_\mathfrak{p} : R_\mathfrak{p} \to (End_R(L))_\mathfrak{p}$ is bijective. Assume $\varphi_\mathfrak{p}(a/s) = 0$, where $a/s \in R_\mathfrak{p}$. Then $\varphi(a)/s = 0$. There exists an element $t \in R\backslash\mathfrak{p}$ such that $t\varphi(a) = 0$. $\varphi(ta) = 0$. $taL = 0$. $(ta/1)L_\mathfrak{p} = 0$. Since $L_\mathfrak{p} \cong R_\mathfrak{p}$, we have $(ta/1)R_\mathfrak{p} = 0$. It follows that $a/s = (ta/1)(1/ts) = 0$. This shows that $\varphi_\mathfrak{p}$ is injective.

Let $f/s \in (End_R(L))_{\mathfrak{p}}$, where $s \in R\backslash\mathfrak{p}$ and $f \in End_R(L)$. Then $f_{\mathfrak{p}} : L_{\mathfrak{p}} \to L_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-homomorphism. Let $\alpha : L_{\mathfrak{p}} \to R_{\mathfrak{p}}$ be an $R_{\mathfrak{p}}$-isomorphism. Consider the following commutative diagram

$$\begin{array}{ccc} L_{\mathfrak{p}} & \xrightarrow{f_{\mathfrak{p}}} & L_{\mathfrak{p}} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ R_{\mathfrak{p}} & \xrightarrow[\alpha\circ f_{\mathfrak{p}}\circ\alpha^{-1}]{} & R_{\mathfrak{p}} \end{array}$$

Write $(\alpha \circ f_{\mathfrak{p}} \circ \alpha^{-1})(1/1) = b/t$, where $b \in R$, $t \in R\backslash\mathfrak{p}$. Let x be any element of L. Write $\alpha(x/1) = c/u$, where $c \in R$, $u \in R\backslash\mathfrak{p}$. Then

$$\begin{aligned} f(x)/1 &= f_{\mathfrak{p}}(x/1) \\ &= (f_{\mathfrak{p}} \circ \alpha^{-1})(c/u) \\ &= (c/u)(f_{\mathfrak{p}} \circ \alpha^{-1})(1/1) \\ &= (c/u)\,(\alpha^{-1} \circ \alpha \circ f_{\mathfrak{p}} \circ \alpha^{-1})(1/1) \\ &= (c/u)\alpha^{-1}(b/t) \\ &= \alpha^{-1}((c/u)(b/t)) \\ &= (b/t)\alpha^{-1}(c/u) \\ &= (b/t)(x/1) \\ &= (bx)/t \\ &= (\varphi(b)(x))/t. \end{aligned}$$

There exists $v \in R\backslash\mathfrak{p}$ such that $v(tf(x) - \varphi(b)(x)) = 0$. So, $(vtf)(x) = (v\varphi(b))(x)$. Hence $vtf = v\varphi(b)$ in $Ent_R(L)$, so that $vstf = vs\varphi(b)$. It follows that $f/s = \varphi(b)/st = \varphi_{\mathfrak{p}}(b/st)$ and $b/st \in R_{\mathfrak{p}}$. Therefore $\varphi_{\mathfrak{p}}$ is surjective.

This shows that for every $\mathfrak{p} \in Spec(R)$, $\varphi_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-isomorphism, so that φ is an R-isomorphism. □

Of course, Lemma 3.1 is a generalization of Corollary 2.3. Let L be a projective R-module of rank 1. Regarding each element a of R as the left multiplication by a, we may have $End_R(L) = R$.

Lemma 3.2. *Let R be a Noetherian ring and let P be a projective R-module of rank n. The r-th exterior power $\wedge^r P$ of P is a projective R-module of rank ${}_nC_r$.*

Proof. Assume that P is a projective R-module. Then there exists an R-module Q and a positive integer N such that $P \oplus Q = R^N$. Hence

$$\begin{aligned} \wedge^r(R^N) &= \wedge^r(P \oplus Q) \\ &\cong (\wedge^r P \otimes \wedge^0 Q) \oplus (\wedge^{r-1} P \otimes \wedge^1 Q) \oplus \cdots \\ &\quad \oplus (\wedge^1 P \otimes \wedge^{r-1} Q) \oplus (\wedge^0 P \otimes \wedge^r Q) \\ &\cong \wedge^r P \oplus (\wedge^{r-1} P \otimes \wedge^1 Q) \oplus \cdots \oplus (\wedge^1 P \otimes \wedge^{r-1} Q) \oplus \wedge^r Q \end{aligned}$$

It follows from Lemma 2.1 that $\wedge^r(R^N)$ is an R-free module of rank ${}_N C_r$, so that $\wedge^r P$ is projective. Let $\mathfrak{p} \in Spec(R)$. Then

$$(\wedge^r P)_{\mathfrak{p}} \cong \wedge^r(P_{\mathfrak{p}}).$$

Since P has rank n, $P_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-free module with rank n. Hence by Lemma 2.1 again, $\wedge^r(P_{\mathfrak{p}})$ is free with rank ${}_n C_r$. Therefore $\wedge^r P$ has rank ${}_n C_r$. □

Let P be a rank n projective R-module over a Noetherian ring R. Then it follows from Lemma 3.2 that $\wedge^n P$ is a projective R-module and $rank(\wedge^n P) = {}_n C_n = 1$. Let $\alpha \in End_R(P)$. Like the free case, we define a map $\wedge^n \alpha : \wedge^n P \to \wedge^n P$ by

$$(\wedge^n \alpha)(p_1 \wedge \cdots \wedge p_n) = \alpha(p_1) \wedge \alpha(p_2) \wedge \cdots \wedge \alpha(p_n),$$

where $p_1, \cdots, p_n \in P$. Then $\wedge^n \alpha \in End_R(\wedge^n P)$. By the proof of Lemma 3.1, there exists a unique element $a \in R$ such that $\wedge^n \alpha$ is the left multiplication by a. That is, for any elements $p_1, p_2, \cdots, p_n \in P$,

$$(\wedge^n \alpha)(p_1 \wedge p_2 \wedge \cdots \wedge p_n) = a(p_1 \wedge p_2 \wedge \cdots \wedge p_n).$$

Such the unique element a is called the *determinant* of α and is denoted by $det(\alpha)$.

Theorem 3.3. *Let R be a Noetherian ring. If P is a projective R-module with rank n, then $Aut_R(P) = \{\alpha \in End_R(P) \mid det(\alpha) \in U(R)\}$.*

Proof. Let P be a projective R-module of rank n. Let $\alpha \in Aut_R(P) \subseteq End_R(P)$. Then there exists $\beta \in End_R(P)$ such that $\alpha \circ \beta = id_P$. $det(\alpha)det(\beta) = det(\alpha \circ \beta) = det(id_P) = 1$, so $det(\alpha) \in U(R)$. Conversely, let $\alpha \in End_R(P)$ such that $det(\alpha) \in U(R)$. To show that $\alpha \in Aut_R(P)$, we need to prove that for all $\mathfrak{p} \in Spec(R)$, $\alpha_{\mathfrak{p}} \in Aut_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$. $\alpha_{\mathfrak{p}} \in End_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$ and $det(\alpha_{\mathfrak{p}}) = (det(\alpha))_{\mathfrak{p}} \in U(R_{\mathfrak{p}})$, so by Corollary 2.6, $\alpha_{\mathfrak{p}} \in Aut_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$, as claimed. □

If P is a projective R-module with finite rank, then write

$$SL_R(P) = \{\alpha \in Aut_R(P) \mid det(\alpha) = 1\}.$$

Then $SL_R(P)$ is a normal subgroup of $Aut_R(P)$.

Let $\mathfrak{p}$ be a prime ideal of R and let P be a projective R-module. If we define a map $\lambda : (End_R(P))_{\mathfrak{p}} \to End_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$ by defining $\lambda(\alpha/s) : P_{\mathfrak{p}} \to P_{\mathfrak{p}}$ as $(\lambda(\alpha/s))(p/t) = \alpha(p)/st$, then λ is an isomorphism. Let P be a projective R-module with rank n. If we define $\mu : (End_R(\wedge^n P))_{\mathfrak{p}} \to End_{R_{\mathfrak{p}}}(\wedge^n(P_{\mathfrak{p}}))$ by $\mu((\wedge^n \alpha)/s) = \wedge^n(\alpha/s)$, then μ is an isomorphism.

Lemma 3.4. *Let P be a projective R-module with rank n. Define a map $\varphi : End_R(P) \to End_R(\wedge^n P)$ by $\varphi(\alpha) = \wedge^n \alpha$. Then for every prime ideal $\mathfrak{p}$ of R the following statements are true:*

(1) *The following diagram is commutative:*

$$\begin{array}{ccc}
(End_R(P))_{\mathfrak{p}} & \xrightarrow{\ \lambda\ } & End_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}) \\
\downarrow{\scriptstyle \varphi_{\mathfrak{p}}} & & \downarrow{\scriptstyle det} \\
(End_R(\wedge^n P))_{\mathfrak{p}} & \xrightarrow[\ \mu\ ]{} & End_{R_{\mathfrak{p}}}(\wedge^n(P_{\mathfrak{p}}))
\end{array}$$

(2) *$\varphi_{\mathfrak{p}}$ is surjective.*

Proof. (1) Let α/s be any element of $(End(P))_{\mathfrak{p}}$ and let

$$\{p_1/s_1, p_2/s_2, \cdots, p_n/s_n\}$$

be an $R_{\mathfrak{p}}$-free basis for $P_{\mathfrak{p}}$. Then

$$\begin{array}{rl}
\wedge^n\lambda \quad (\alpha/s) & (p_1/s_1 \wedge p_2/s_2 \wedge \cdots \wedge p_n/s_n) \\
= & \lambda(\alpha/s)(p_1/s_1) \wedge \lambda(\alpha/s)(p_2/s_2) \wedge \cdots \wedge \lambda(\alpha/s)(p_n/s_n) \\
= & \alpha(p_1)/ss_1 \wedge \alpha(p_2)/ss_2 \wedge \cdots \wedge \alpha(p_n)/ss_n \\
= & \wedge^n(\alpha/s)(p_1/s_1 \wedge p_2/s_2 \wedge \cdots \wedge p_n/s_n)
\end{array}$$

This shows that $\wedge^n\lambda(\alpha/s) = \wedge^n(\alpha/s)$. Hence by Lemma 2.4(1),

$$(det \circ \lambda)(\alpha/s) = det(\lambda(\alpha/s)) = \wedge^n(\alpha/s) = \mu((\wedge^n\alpha)/s) = \mu \circ \varphi_{\mathfrak{p}}(\alpha/s).$$

(2) Let a be any element of $End_R(\wedge^n P)$ and let s be any element of $R\backslash\mathfrak{p}$. Then $\mu(a/s) \in End_{R_{\mathfrak{p}}}(\wedge^n(P_{\mathfrak{p}}))$. It is known that $det : End_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}) \to End_{R_{\mathfrak{p}}}(\wedge^n(P_{\mathfrak{p}}))$ is surjective. So, there exists $\beta \in End_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$ such that $det(\beta) = \mu(a/s)$. We have already known that λ is surjective. So, there exists an element $\alpha \in End_R(P)$ and an element $t \in R\backslash\mathfrak{p}$ such that

$\lambda(\alpha/t) = \beta$ (actually, take $t = s$.) By (1), we have that

$$\begin{aligned}\varphi_{\mathfrak{p}}(\alpha/t) &= \mu^{-1} \circ det \circ \lambda(\alpha/t) \\ &= \mu^{-1} \circ det(\lambda(\alpha/t)) \\ &= \mu^{-1} \circ det(\beta) \\ &= \mu^{-1}(\mu(a/s)) \\ &= a/s,\end{aligned}$$

as required.

Theorem 3.5. *Let R be a ring. If P is a projective R-module of finite rank, then the map $\varphi : Aut_R(P) \to U(R)$ defined by $\varphi(\alpha) = det(\alpha)$ is locally surjective and $Ker(\varphi) = SL_R(P)$.*

Proof. It is clear that φ is a group homomorphism. Moreover, it follows from Lemma 3.4(2) that for every $\mathfrak{p} \in Spec(R)$ $\varphi_{\mathfrak{p}}$ is surjective. Hence φ is an epimorphism. Also, $Ker(\varphi) = SL_R(P)$. □

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