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ADDITIVE ρ-FUNCTIONAL EQUATIONS IN β-HOMOGENEOUS F-SPACES

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Abstract. In this paper, we solve the additive ρ-functional equations

(0.1) $f(x+y)+f(x-y)-2f(x)=\rho\left(2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right)$,

and

(0.2) $2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)=\rho(f(x+y)+f(x-y)-2f(x))$,

where ρ is a fixed (complex) number with $\rho \neq 1$,

Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional equations (0.1) and (0.2) in β-homogeneous (complex) F-spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [22] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional

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equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25]).

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN_1) $\|x\| = 0$ if and only if $x = 0$;

(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;

(FN_3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;

(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;

(FN_5) $\|\lambda x_n\| \to 0$ provided $\|x_n\| \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space.

A sequence $\{x_n\}$ is called a Cauchy sequence if, for a given $\epsilon > 0$, there is a nautral number N such that $\|x_n - x_m\| \leq \epsilon$ for all $n, m \geq N$. A sequence $\{x_n\}$ is called a convergernt sequence if, for a given $\epsilon > 0$, there are a nautral number N and $x_0 \in X$ such that $\|x_n - x_0\| \leq \epsilon$ for all $n \geq N$. If every Cauchy sequence converges, then the space is called *complete*. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [16]).

In Section 2, we solve the additive ρ-functional equation (0.1) and prove the Hyers-Ulam stability of the additive ρ-functional equation (0.1) in β_2-homogeneous (complex) F-spaces.

In Section 3, we solve the additive ρ-functional equation (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional equation (0.2) in β_2-homogeneous (complex) F-spaces.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that X is a β_1-homogeneous (complex) normed space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous (complex) F-space with norm $\|\cdot\|$. Assume that ρ is a (complex) number with $\rho \neq 1$.

2. Additive ρ-functional Equation (0.1) in β-homogeneous (Complex) F-spaces

We solve and investigate the additive ρ-functional equation (0.1) in (complex) normed spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$f(x+y)+f(x-y)-2f(x)=\rho\left(2f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)\right) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y)+f(x-y)-2f(x) &= \rho\left(2f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)\right) \\ &= \rho(f(x+y)+f(x-y)-2f(x)) \end{aligned}$$

and so $f(x + y) + f(x - y) = 2f(x)$ for all $x, y \in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (2.1) in β-homogeneous (complex) F-spaces.

Theorem 2.2. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\left\|f(x+y)+f(x-y)-2f(x)-\rho\left(2f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)\right)\right\| \\ &\qquad \le \theta(\|x\|^r+\|y\|^r) \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\|\le\frac{2\theta}{2^{\beta_1 r}-2^{\beta_2}}\|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.3), we get

$$\|f(2x)-2f(x)\|\le 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|\le\frac{2}{2^{\beta_1 r}}\theta\|x\|^r$$

for all $x \in X$. Hence

$$
\begin{aligned}
\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\leq \frac{2}{2^{\beta_1 r}} \sum_{j=l}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \qquad (2.6)
\end{aligned}
$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$
\begin{aligned}
&\left\|A(x+y) + A(x-y) - 2A(x) - \rho\left(2A\left(\frac{x+y}{2}\right) + A(x-y) - 2A(x)\right)\right\| \\
&= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right)\right.\right. \\
&\left.\left. - \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right)\right)\right)\right\| \leq \lim_{n\to\infty} \frac{2^{\beta_2 n}}{2^{\beta_1 r n}} \theta(\|x\|^r + \|y\|^r) = 0
\end{aligned}
$$

for all $x, y \in X$. So

$$A(x+y) + A(x-y) - 2A(x) = \rho\left(2A\left(\frac{x+y}{2}\right) + A(x-y) - 2A(x)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then we have

$$
\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\leq \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\leq \frac{4\theta}{2^{\beta_1 r} - 2^{\beta_2}} \frac{2^{\beta_2 q}}{2^{\beta_1 q r}} \|x\|^r,
\end{aligned}
$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A, as desired. □

Theorem 2.3. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive* *mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{2.7}$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{2}{2^{\beta_2}}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \frac{2}{2^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{2^{\beta_2 j}}\theta\|x\|^r \end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.7).

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Additive ρ-functional Equation (0.2) in β-homogeneous (Complex) F-spaces

We solve and investigate the additive ρ-functional equation (0.2) in β-homogeneous (complex) normed spaces.

Lemma 3.1. *If a mapping* $f : X \to Y$ *satisfies*

$$2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) = \rho(f(x+y) + f(x-y) - 2f(x)) \tag{3.1}$$

for all $x, y \in X$, *then* $f : X \to Y$ *is additive.*

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0$ and so

$$2f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) &= 2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) \\ &= \rho(f(x+y) + f(x-y) - 2f(x)) \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (3.1) in β-homogeneous (complex) F-spaces.

Theorem 3.2. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - \rho(f(x+y) + f(x-y) - 2f(x))\right\| \\ &\leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^{\beta_1 r}\theta}{2^{\beta_1 r} - 2^{\beta_2}}\|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be an odd mapping satisfying* (3.3). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^{\beta_1 r}\theta}{2^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{2^{\beta_1 r}}{2^{\beta_2}}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l+1}^{m}\frac{2^{\beta_1 rj}}{2^{\beta_2 j}}\theta\|x\|^r \qquad (3.8)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 2.2. □

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