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CONVERGENCE RESULTS FOR THE COOPERATIVE CROSS-DIFFUSION SYSTEM WITH WEAK COOPERATIONS

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Abstract. We prove convergence properties of the global solutions to the cooperative cross-diffusion system with the intra-specific cooperative pressures dominated by the inter-specific competition pressures and the inter-specific cooperative pressures dominated by intra-specific competition pressures. Under these conditions the W_2^1-bound and the time global existence of the solution for the cooperative cross-diffusion system have been obtained in [10]. In the present paper the convergence of the global solution is established for the cooperative cross-diffusion system with large diffusion coefficients.

1. Introduction

The cooperative cross-diffusion system refers the following quasilinear parabolic system in population dynamics :

$$(1.1)\quad \begin{cases} u_t = (d_1\, u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx} + u(a_1 - b_1 u + c_1 v) & \text{in } [0,1]\times(0,\infty), \\ v_t = (d_2\, v + \alpha_{21}uv + \alpha_{22}v^2)_{xx} + v(a_2 + b_2 u - c_2 v) & \text{in } [0,1]\times(0,\infty), \\ u_x(x,t) = v_x(x,t) = 0 & \text{at } x = 0,1, \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0 & \text{in } [0,1], \end{cases}$$

where α_{12}, α_{21}, d, a_i, b_i, c_i are positive constants for $i = 1,2$. Here we assume that the initial functions u_0, v_0 are positive functions on the domain $[0,1]$. In the system (1.1) u and v are nonnegative functions which represent the population densities of two competing species. d_1 and d_2 are the *diffusion* rates of the two species, respectively. a_1 and a_2 denote the intrinsic growth rates, b_2 and c_1 account for inter-specific cooperative pressures, b_1 and c_2 account for intra-specific competition pressures. Intra-specific competition pressures result in a reduction of population

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growth rate as population density increases. On the other hand, inter-specific cooperative pressure b_2 helps the population growth rate of u increase as the population density of v increases, and c_2 acts similarly. When $\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = 0$, (1.1) reduces to the well-known Lotka-Voltera cooperative-diffusion system. α_{11} and α_{22} are usually referred as *self-diffusion*, and α_{12}, α_{21} are *cross-diffusion* pressures. By adopting the coefficients α_{ij} $(i, j = 1, 2)$ the system (1.1) takes into account the pressures created by mutually competing species. For more details on the backgrounds of this model, we refer the reader to [6] and [8].

In [10] the existence of global solutions of the cooperative cross-diffusion system (1.1) is obtained under the following conditions.

$$\alpha_{12}^2 < 8\alpha_{11}\alpha_{21} \quad \text{and} \quad \alpha_{21}^2 < 8\alpha_{12}\alpha_{22} \tag{1.2}$$

$$b_1c_2 > b_2c_1. \tag{1.3}$$

The inequalities in (1.2) are reduced to

$$\alpha_{12}\alpha_{21} < 64\alpha_{11}\alpha_{22},$$

and this means that cross-diffusion pressures are controlled in low level compare to the self-diffusion pressures. The inequality (1.3) means the product of inter-specific cooperative pressures b_2c_1 is less than the product of intra-specific competition pressures b_1c_2. In this sense the inequality (1.3) may be called as *the weak cooperative condition* for the system (1.1).

Now in the present paper we are interested in the convergence of the global solutions of the cooperative cross-diffusion system (1.1). Throughout this this paper we assume condition (1.2) and (1.3) and use the following notations.

Notation 1. Let Ω be a domain(i.e., a bounded, connected open set) in $\mathbb{R}^n$. The norm in $L_p(\Omega)$ is denoted by $|\cdot|_{L_p(\Omega)}$, $1 \le p \le \infty$. The usual Sobolev spaces of real valued functions in Ω with exponent $k \ge 0$ are denoted by $W_p^k(\Omega)$, $1 \le p < \infty$. And $\|\cdot\|_{W_p^k(\Omega)}$ represents the norm in the Sobolev space $W_p^k(\Omega)$. For $\Omega = [0, 1] \subset \mathbb{R}^1$ we shall use the simplified notation $\|\cdot\|_{k,p}$ for $\|\cdot\|_{W_p^k(\Omega)}$ and $|\cdot|_p$ for $|\cdot|_{L_p(\Omega)}$.

For readers reference we state the global existence result that have obtained in [10].

Theorem 1.1 ([10, Theorem 1.4])**.** *Suppose that the initial functions u_0, v_0 are in $W_2^2([0, 1])$. Also assume the conditions* (1.2) *and* (1.3)*. Let $(u(x, t), v(x, t))$ be the maximal solution to the system* (1.1) *as in the result of Amann*([1])*. Then* *there exist positive constants* $M' = M'(\|u_0\|_1, \|v_0\|_1, d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$ *and* $M = M(d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$ *such that*

$$\sup\{\|u(\cdot,t)\|_{1,2}, \|v(\cdot,t)\|_{1,2} : t \in [0,T)\} \leq M',$$
$$\sup\{u(x,t),\ v(x,t) : (x,t) \in [0,1] \times [0,T)\} \leq M.$$

Also it is concluded that $T = +\infty$*, and thus the maximal solution* $(u(x,t), v(x,t))$ *is a global solution.*

The main result of the present paper is the convergence of the solution to the system (1.1) as stated in the following theorem.

Theorem 1.2. *Assume the conditions* (1.2)*,* (1.3)*, and that* u_0*,* v_0 *are in* $W_2^2([0,1])$ *for the system* (1.1)*. If* $d_1, d_2 \geq 1$ *satisfy that*

$$(b_2^2\alpha_{12}^2\overline{u}^2 + c_1^2\alpha_{21}^2\overline{v}^2)M^2 < 4b_2c_1\overline{u}\,\overline{v}d_1d_2, \tag{1.4}$$

where M *is the positive constant in* Theorem 1.1 *and* $(\overline{u}, \overline{v}) = \left(\frac{a_1c_2+a_2c_1}{b_1c_2-b_2c_1}, \frac{a_1b_2+a_2b_1}{b_1c_2-b_2c_1}\right)$*, then the solution* $(u(t), v(t))$ *converges to* $(\overline{u}, \overline{v})$ *uniformly in* $[0,1]$ *as* $t \to \infty$*, and the constant steady-state* $(\overline{u}, \overline{v})$ *is globally asymptotically stable.*

Remark 1. The W_2^1-bound of u, v for the system (1.1) has been obtained under the conditions (1.2), (1.3) in [10]. In the case that condition (1.2) fails, and condition (1.3) holds, we only have the boundedness result of the L_1-norms of u, v for the system (1.1) from Theorem 1.2 in [10]. In that case the solution u, v may not exists globally in time. If the cross-diffusion pressures are in high level compare to the self-diffusion pressures, or the intra-specific cooperative pressures exceed the inter-specific competition pressures, then we may expect that u, v blow-up in finite time for the system (1.1).

In Section 2 we collect calculus inequalities and comparison results which are necessary for the proof of Theorem 1.2 in Section 3.

2. Calculus Inequalities and Comparison Results

Theorem 2.1 (A Sobolev type embedding Theorem by Rellich and Kondrachov)**.** *Let* Ω *be a bounded domain with with smooth boundary in* R^n *and* $1 \leq p \leq \infty$*. Then*

$$W_p^1(\Omega) \subset C(\overline{\Omega}) \qquad \textit{for } p > n$$

Proof. The proof may found in [3] or [7]. □

Lemma 2.1. *For every function u in $W_2^2([0,1])$ with $u_x(0) = u_x(1) = 0$*

$$|u_x|_2 \le |u_{xx}|_2^{\frac{1}{2}} |u|_2^{\frac{1}{2}}. \tag{2.1}$$

Proof. Using the given boundary conditions and Hölder's inequality

$$\int_0^1 u_x^2\, dx = -\int_0^1 u u_{xx}\, dx \le |u_{xx}|_2 |u|_2,$$

and thus the inequality (2.1) holds. □

Lemma 2.2 (positivity of the maximal solution to (1.1)). *Suppose that the initial functions u_0, v_0 are in $W_2^2([0,1])$. Let $(u(x,t), v(x,t))$ be the maximal solution to the system* (1.1) *for $x \in [0,1]$, $t \in [0,T]$. Then*

$$u(x,t) > 0, \quad v(x,t) > 0 \quad \textit{for } x \in [0,1],\ t \in [0,T]$$

Proof. Each of the first two equations in the system (1.1) is expressed as

$$u_t = d_1(1+2\alpha_{11}u+\alpha_{12}v)u_{xx}+2(\alpha_{11}u_x+\alpha_{12}v_x)u_x+(\alpha_{12}v_{xx}+a_1-b_1u+c_1v)u \tag{2.2}$$

$$v_t = d_2(1+\alpha_{12}u+2\alpha_{22}v)v_{xx}+2(\alpha_{21}u_x+\alpha_{22}v_x)v_x+(\alpha_{21}u_{xx}+a_2+b_2u-c_2v)v \tag{2.3}$$

Here application of the parabolic maximum principles(may refer to [7], Theorem 5 on p. 173) for (2.2) and (2.3) yields that the maximum values of $u(x,t)$ and $v(x,t)$ do not occur on $(0,1) \times (0,T]$. Then by using the Neumann boundary condition of the system (1.1) and the Hopf-type boundary point lemma (may refer to [7], Theorem 6 on p. 174) we see that the maximum values of $u(x,t)$ and $v(x,t)$ occurs at $t = 0$. Now from the positivity of the initial functions $u_0(x)$, $v_0(x)$ of the system (1.1), it is concluded that $u(x,t) > 0$, $v(x,t) > 0$ for all $x \in [0,1]$, $t \in [0,T]$. □

Lemma 2.3 (Positivity of the global solution to (1.1)). *Suppose that the initial functions u_0, v_0 are in $W_2^2([0,1])$. Also assume the conditions* (1.2) *and* (1.3). *Then the solution $(u(x,t), v(x,t))$ to the system* (1.1) *satisfies*

$$u(x,t) > 0, \quad v(x,t) > 0 \quad \textit{for } x \in [0,1],\ t \in [0,\infty)$$

Proof. Under the conditions (1.2) and (1.3) the maximal solution $(u(x,t), v(x,t))$ to the system (1.1) is a global solution by Theorem 1.1. And from Lemma 2.2 we obtain the positivity result on $u(x,t)$ and $v(x,t)$ for $x \in [0,1]$, $t \in [0,\infty)$. □

3. Convergence Results(Proof of Theorem 1.2)

In [10] the proof of Theorem (1.1) deals with the constant M depending on the parameters d_i, α_{ij}, a_i, b_i, c_i, $(i = 1, 2)$. Here following similar arguments as in [9] it is possible to conclude the independence of the constant M in the proof of Theorem (1.1) on d_1, d_2 in the case that $d_1 > 1$, $d_2 > 1$ are are sufficiently large. Using these results we prove the convergence result Theorem 1.2 for the global solution $(u(x,t), v(x,t))$ to the system (1.1) as $t \to \infty$ in this section.

By Lemma 2.3 we have that $u(x,t) > 0$ and $v(t,x) > 0$ in $[0,1] \times [0,\infty)$. Under the weak cooperative condition (1.3), that is $b_1c_2 > b_2c_1$, the system (1.1) has the unique constant steady-state $(\overline{u}, \overline{v}) = \left(\frac{a_1c_2+a_2c_1}{b_1c_2-b_2c_1}, \frac{a_1b_2+a_2b_1}{b_1c_2-b_2c_1}\right)$ in the first quadrant of the phase plane of (u,v).

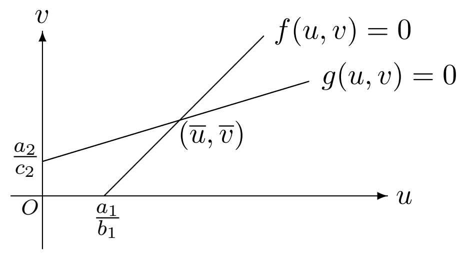

Figure 1. The zero sets of the functions $f(u,v) = a_1 - b_1u + c_1v$, $g(u,v) = a_2 + b_2u - c_2v$ in the phase plane of (u,v) for the system (1.1) with the weak cooperative condition $b_1c_2 > b_2c_1$

In order to observe the convergence of global solutions of the system (1.1) in the weak cooperative case, we use the functional $K(u,v)$ defined as :

$$K(u,v) = \int_0^1 \left\{ b_2 \left(u - \overline{u} - \overline{u} \log \frac{u}{\overline{u}} \right) + c_1 \left(v - \overline{v} - \overline{v} \log \frac{v}{\overline{v}} \right) \right\} dx.$$

Here using the natural logarithmic function $y = \log x$ and its tangent line $y = x - 1$ at $x = 1$, we notice that $K(u,v) \geq 0$ for all (u,v) in the first quadrant of the phase plane, and $K(u,v) = 0$ only at $(\overline{u}, \overline{v})$. Now let us compute the time derivative of $K(u(t), v(t))$ for the solution of the system (1.1).

$$\frac{dK(u(t),v(t))}{dt} = \int_0^1 \left\{ b_2 \left(1 - \frac{\overline{u}}{u} \right) u_t + c_1 \left(1 - \frac{\overline{v}}{v} \right) v_t \right\} dx$$

$$= \int_0^1 \left\{ b_2\left(1-\frac{\overline{u}}{u}\right)\left(d_1u+\alpha_{11}u^2+\alpha_{12}uv\right)_{xx} + c_1\left(1-\frac{\overline{v}}{v}\right)\left(d_2v+\alpha_{21}uv+\alpha_{22}v^2\right)_{xx}\right\} dx$$
$$+ \int_0^1 \{b_2(u-\overline{u})(a_1-b_1u+c_1v)+c_1(v-\overline{v})(a_2+b_2u-c_2v)\}\, dx$$

From the Neumann boundary conditions $u_x(x,t)=v_x(x,t)=0$ at $x=0,1$ as in the third line of the system (1.1) it is reduced as

$$\begin{aligned}
&\int_0^1 b_2\left(1-\frac{\overline{u}}{u}\right)\left(d_1u+\alpha_{11}u^2+\alpha_{12}uv\right)_{xx}\, dx \\
&= \left[b_2\left(1-\frac{\overline{u}}{u}\right)\left(d_1u+\alpha_{11}u^2+\alpha_{12}uv\right)_x\right]_0^1 \\
&\quad - \int_0^1 b_2\left(1-\frac{\overline{u}}{u}\right)_x\left(d_1u+\alpha_{11}u^2+\alpha_{12}uv\right)_x\, dx \\
&= 0 - \int_0^1 b_2\left(\frac{\overline{u}}{u^2}\right)u_x\left(d_1u+\alpha_{11}u^2+\alpha_{12}uv\right)_x\, dx \\
&= -\int_0^1 \left\{\frac{b_2\overline{u}}{u^2}(d_1+2\alpha_{11}u+\alpha_{12}v)u_x^2+\frac{b_2\alpha_{12}\overline{u}}{u}u_xv_x\right\} dx,
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^1 c_1\left(1-\frac{\overline{v}}{v}\right)\left(d_2v+\alpha_{21}uv+\alpha_{22}v^2\right)_{xx}\, dx \\
&\qquad = -\int_0^1 \left\{\frac{c_1\overline{v}}{v^2}(d_2+\alpha_{21}u+2\alpha_{22}v)v_x^2+\frac{c_1\alpha_{21}\overline{v}}{v}u_xv_x\right\} dx.
\end{aligned}$$

Also using that $(\overline{u},\overline{v})$ satisfies both equations $a_1-b_1\overline{u}+c_1\overline{v}=0$ and $a_2+b_2\overline{u}-c_2\overline{v}=0$ it is reduced as

$$\begin{aligned}
a_1-b_1u+c_1v &= -b_1(u-\overline{u})+c_1(v-\overline{v}),\\
a_2+b_2u-c_2v &= b_2(u-\overline{u})-c_2(v-\overline{v}).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{dK(u(t),v(t))}{dt} = &-\int_0^1\left\{\frac{b_2\overline{u}}{u^2}(d_1+2\alpha_{11}u+\alpha_{12}v)u_x^2+\left(\frac{b_2\alpha_{12}\overline{u}}{u}+\frac{c_1\alpha_{21}\overline{v}}{v}\right)u_xv_x\right.\\
&\left.\qquad +\frac{c_1\overline{v}}{v^2}(d_2+\alpha_{21}u+2\alpha_{22}v)v_x^2\right\} dx\\
&-\int_0^1\{b_1b_2(u-\overline{u})^2-2b_2c_1(u-\overline{u})(v-\overline{v})+c_1c_2(v-\overline{v})^2\}\, dx.
\end{aligned}$$

From the weak cooperative condition (1.3) we have a positive constant

$$\delta = \frac{1}{2}\min\left\{b_1b_2,\ c_1c_2,\ \frac{b_2c_1(b_1c_2-b_2c_1)}{b_1b_2+c_1c_2}\right\}.$$

Using this constant we show that

$$b_1b_2(u-\overline{u})^2 - 2b_2c_1(u-\overline{u})(v-\overline{v}) + c_1c_2(v-\overline{v})^2 \ \geq\ \delta\{(u-\overline{u})^2+(v-\overline{v})^2\} \tag{3.1}$$

by noticing the determinant of the quadratic expression

$$(b_1b_2-\delta)(u-\overline{u})^2 - 2b_2c_1(u-\overline{u})(v-\overline{v}) + (c_1c_2-\delta)(v-\overline{v})^2$$

is negative as

$$\begin{aligned} D &= (b_2c_1)^2 - (b_1b_2-\delta)(c_1c_2-\delta) \\ &= -\delta^2 + (b_1b_2+c_1c_2)\delta - b_2c_1(b_1c_2-b_2c_1) \\ &< (b_1b_2+c_1c_2)\delta - b_2c_1(b_1c_2-b_2c_1) \\ &< 0 \end{aligned}$$

Now we remind that the uniform bound M in Theorem 1.1 for the solution of the system (1.1) in the case $d_1, d_2 \geq 1$ is independent of d_1, d_2. Thus from the condition $d_1, d_2 \geq 1$ we have a constant $M = M(\alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$ such that

$$0 \leq u(x,t),\ v(x,t) \leq M \qquad \text{for every } (x,t) \in [0,1]\times[0,\infty). \tag{3.2}$$

Thus for the constant M in (3.2) we may choose d_1, d_2 sufficiently large to satisfy that

$$d_1d_2 > \frac{\left(b_2^2\alpha_{12}^2\overline{u}^2 + c_1^2\alpha_{21}^2\overline{v}^2\right)M^2}{4b_2c_1\overline{u}\,\overline{v}}$$

as given in the condition (1.4). Hence by taking the positive constant

$$\gamma = \frac{4b_2c_1\overline{u}\,\overline{v}d_1d_2 - \left(b_2^2\alpha_{12}^2\overline{u}^2 + c_1^2\alpha_{21}^2\overline{v}^2\right)M^2}{8M^2\left[b_2\overline{u}\left\{d_1 + (2\alpha_{11}+\alpha_{12})M\right\} + c_1\overline{v}\left\{d_2 + (\alpha_{21}+2\alpha_{22})M\right\}\right]}$$

we aim to show that the following inequality holds :

$$\begin{aligned} &\frac{b_2\overline{u}}{u^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 \\ &\qquad + \left(\frac{b_2\alpha_{12}\overline{u}}{u} + \frac{c_1\alpha_{21}\overline{v}}{v}\right)u_xv_x + \frac{c_1\overline{v}}{v^2}(d_2 + \alpha_{21}u + 2\alpha_{22}v)v_x^2 \\ &\geq \gamma\left\{u_x^2 + v_x^2\right\}. \end{aligned} \tag{3.3}$$

For this purpose we observe the quadratic expression

$$\begin{aligned} &\left\{\frac{b_2\overline{u}}{u^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v) - \gamma\right\}u_x^2 + \left(\frac{b_2\alpha_{12}\overline{u}}{u} + \frac{c_1\alpha_{21}\overline{v}}{v}\right)u_xv_x \\ &\qquad + \left\{\frac{c_1\overline{v}}{v^2}(d_2 + \alpha_{21}u + 2\alpha_{22}v) - \gamma\right\}v_x^2. \end{aligned} \tag{3.4}$$

Using the positive constant γ we see that the determinant of the quadratic expression (3.4) is negative as in the following :

$$\begin{aligned}
&\left(\frac{b_2\alpha_{12}\overline{u}}{u}+\frac{c_1\alpha_{21}\overline{v}}{v}\right)^2-4\left\{\frac{b_2\overline{u}}{u^2}(d_1+2\alpha_{11}u+\alpha_{12}v)-\gamma\right\}\\
&\cdot\left\{\frac{c_1\overline{v}}{v^2}(d_2+\alpha_{21}u+2\alpha_{22}v)-\gamma\right\}\\
&\qquad\le\frac{b_2^2\alpha_{12}^2\overline{u}^2}{u^2}+\frac{c_1^2\alpha_{21}^2\overline{v}^2}{v^2}-\frac{4b_2c_1\overline{u}\,\overline{v}d_1d_2}{u^2v^2}\\
&\qquad\quad+4\gamma\left\{\frac{b_2\overline{u}}{u^2}(d_1+(2\alpha_{11}+\alpha_{12})M)+\frac{c_1\overline{v}}{v^2}(d_2+(\alpha_{21}+2\alpha_{22})M)\right\}\\
&\qquad\le\frac{1}{u^2v^2}\left[(b_2^2\alpha_{12}^2\overline{u}^2+c_1^2\alpha_{21}^2\overline{v}^2)M^2-4b_2c_1\overline{u}\,\overline{v}d_1d_2\right.\\
&\qquad\quad\left.+4\gamma M^2\{b_2\overline{u}(d_1+(2\alpha_{11}+\alpha_{12})M)+c_1\overline{v}(d_2+(\alpha_{21}+2\alpha_{22})M)\}\right]\\
&\qquad<0.
\end{aligned}$$

Thus we have the inequality (3.3).

From (3.1) and (3.3) we have

(3.5)
$$\frac{dK(u(t),v(t))}{dt}\le-\gamma\int_0^1\left\{u_x^2+v_x^2\right\}\,dx-\delta\int_0^1\left\{(u-\overline{u})^2+(v-\overline{v})^2\right\}\,dx\le 0.$$

And in (3.5) we see that $\frac{dK(u(t),v(t))}{dt}=0$ only if $u(x,t)\equiv\overline{u}$ and $v(x,t)\equiv\overline{v}$. Since $K(u,v)\ge 0$ for all (u,v) in the first quadrant of the phase plane, it is concluded that the functional $K(u(x,t),v(x,t))$ is decreasing to zero as $t\to\infty$. Here by using the uniform boundedness of $(u(x,t),v(x,t))$ in $[0,1]$ we obtain the L_2 convergences, $|u(t)-\overline{u}|_2\to 0$, $|v(t)-\overline{v}|_2\to 0$ as $t\to\infty$.

Using the uniform boundedness results in Theorem 1.1 that

$$\sup_{0\le t<\infty}|u_{xx}(t)|_2<\infty,\qquad\sup_{0\le t<\infty}|v_{xx}(t)|_2<\infty,$$

and applying the calculus inequality in Lemma 2.1 to the functions $u(x,t)-\overline{u}$ and $v(x,t)-\overline{v}$, we obtain the convergence $(u(t),v(t))\to(\overline{u},\overline{v})$ as $t\to\infty$ in $W_2^1([0,1])$. By using the Sobolev embedding result in Theorem 2.1 we show that $(u(t),v(t))$ converges to $(\overline{u},\overline{v})$ uniformly in $[0,1]$ as $t\to\infty$. It is also shown that $(\overline{u},\overline{v})$ is locally asymptotically stable in $C([0,1])$ from the fact that $K(u(t),v(t))$ is decreasing for $t\ge 0$. Therefore we conclude that $(\overline{u},\overline{v})$ is globally asymptotically stable for the system 1.1.

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