

SASAKIAN TWISTOR SPINORS AND THE FIRST DIRAC EIGENVALUE

EUI CHUL KIM

ABSTRACT. On a closed eta-Einstein Sasakian spin manifold of dimension $2m + 1 \geq 5$, $m \equiv 0 \pmod{2}$, we prove a new eigenvalue estimate for the Dirac operator. In dimension 5, the estimate is valid without the eta-Einstein condition. Moreover, we show that the limiting case of the estimate is attained if and only if there exists such a pair $(\varphi_{\frac{m}{2}-1}, \varphi_{\frac{m}{2}})$ of spinor fields (called *Sasakian duo*, see Definition 2.1) that solves a special system of two differential equations.

1. Introduction

Friedrich proved [6] that, on a closed Riemannian spin manifold (M^n, g) with positive scalar curvature $S > 0$, any eigenvalue λ of the Riemann Dirac operator D satisfies

$$(1.1) \quad |\lambda| \geq \sqrt{\frac{n S_{\min}}{4(n-1)}},$$

where S_{\min} denotes the minimum of the scalar curvature. Equality in (1.1) occurs if and only if there exists a non-trivial solution ψ to the *Killing spinor equation*

$$\nabla_X \psi = -\frac{\lambda}{n} X \cdot \psi, \quad 0 \neq \lambda \in \mathbb{R}, \quad X \in \Gamma(T(M)),$$

where the dot “ \cdot ” indicates the Clifford multiplication [7]. Improvement of (1.1) depends on the geometric structure that the manifold in question may possess and is closely related to a generalization of the Killing spinor equation. For Kähler manifolds [14, 15, 18] and quaternionic Kähler manifolds [16, 17], Friedrich’s estimate has been improved to

$$(1.2) \quad |\lambda| \geq \begin{cases} \sqrt{\frac{n+2}{4n}} S_{\min} & \text{if } \frac{n}{2} \text{ is odd,} \\ \sqrt{\frac{n}{4(n-2)}} S_{\min} & \text{if } \frac{n}{2} \text{ is even} \end{cases}$$

Received August 29, 2015.

2010 *Mathematics Subject Classification.* 53C25, 53C27.

Key words and phrases. Dirac operator, first eigenvalue, Sasakian twistor spinor.

and

$$(1.3) \quad |\lambda| \geq \sqrt{\frac{(n+12)S}{4(n+8)}},$$

respectively. Moreover, one can explicitly write down the spinor field equation (a variant of the Killing spinor equation) characterizing the limiting case of (1.2) (resp. (1.3)). For more information about the first Dirac eigenvalue and generalizations of the Killing spinor equation, we refer the readers to references [4, 5, 8, 9, 10, 11, 12].

We now turn to the first Dirac eigenvalue problem on Sasakian spin manifolds. A Sasakian spin manifold is an odd-dimensional Riemannian spin manifold (M^{2m+1}, g) , $m \geq 1$, equipped with a tensor field ϕ of type $(1, 1)$, a unit Killing vector field ξ and a 1-form η such that

$$\begin{aligned} \eta(\xi) &= 1, \quad \phi^2(X) = -X + \eta(X)\xi, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X \end{aligned}$$

hold for all vector fields X, Y . Denote by $\nabla\psi$ the Levi-Civita connection acting on sections $\psi \in \Gamma(\Sigma(M))$ of the spinor bundle $\Sigma(M)$ over M^{2m+1} . The Riemann Dirac operator D is defined by

$$D\psi = \sum_{u=1}^{2m+1} E_u \cdot \nabla_{E_u} \psi,$$

where $(E_1, \dots, E_{2m}, E_{2m+1} = \xi)$ is a local orthonormal frame. Let ξ^\perp denote the orthogonal complement of the Killing vector field ξ in the tangent bundle $T(M)$ and consider a natural deformation [1, 13] of the Levi-Civita connection ∇ in the subbundle $\xi^\perp \subset T(M)$

$$\bar{\nabla}_V \psi = \nabla_V \psi - \frac{1}{2} \phi(V) \cdot \xi \cdot \psi, \quad V \in \Gamma(\xi^\perp).$$

Define non-elliptic, first-order operators $\bar{C}, \bar{Q}, D_+, D_-$ by

$$\begin{aligned} \bar{C}\psi &= \sum_{u=1}^{2m} E_u \cdot \bar{\nabla}_{E_u} \psi, & \bar{Q}\psi &= \sum_{u=1}^{2m} \phi(E_u) \cdot \bar{\nabla}_{E_u} \psi, \\ D_+\psi &= \frac{1}{2}\bar{C}\psi + \frac{\sqrt{-1}}{2}\bar{Q}\psi, & D_-\psi &= \frac{1}{2}\bar{C}\psi - \frac{\sqrt{-1}}{2}\bar{Q}\psi. \end{aligned}$$

Then the operator identities

$$\bar{C} \circ \bar{Q} + \bar{Q} \circ \bar{C} = 0, \quad \bar{C}^2 = \bar{Q}^2$$

and

$$D_+ \circ D_+ = D_- \circ D_- = 0$$

hold [13]. Now consider a two-parameter generalization of the Riemann Dirac operator

$$D_{ab} := \bar{C} + aD_0 + \left(b - \frac{1}{2}\right) \xi \circ \Phi,$$

where $a, b \in \mathbb{R}$ are real numbers, $\Phi := g(\cdot, \phi(\cdot))$ is the fundamental form and $D_0 := \xi \circ \nabla_\xi - \frac{1}{2}\xi \circ \Phi$. The generalized Dirac operator D_{ab} coincides with the Riemann Dirac operator D if $(a = 1, b = 0)$ and with the so-called *cubic Dirac operator* in [1, 2, 3] if $(a = 1, b = \frac{1}{2})$. Recall that a Sasakian manifold is said to be *eta-Einstein* if the scalar curvature S is constant and the Ricci tensor Ric is given by

$$\text{Ric} = \left(\frac{S}{2m} - 1\right)g + \left(2m + 1 - \frac{S}{2m}\right)\eta \otimes \eta.$$

Recently, the author proved:

Theorem A (see Theorem 5.2 in [13]). *Let $(M^5, \phi, \xi, \eta, g)$ be a 5-dimensional closed Sasakian spin manifold. Assume that the parameters a, b of D_{ab} and the minimum S_{\min} of the scalar curvature satisfy*

$$0 < a \leq 1, \quad 0 \leq b \leq \frac{1}{2}$$

and

$$a^2(S_{\min} + 4) > \frac{1}{2} [8 - a(1 - 2b)]^2,$$

respectively. Then, for any eigenvalue λ of D_{ab} , the inequality

$$(1.4) \quad |\lambda| \geq \min \left\{ \omega, \frac{a(S_{\min} + 4)}{16} + 1 - 2b \right\}$$

holds, where ω denotes

$$\omega := \max \left\{ b - \frac{1}{2} + \sqrt{\frac{S_{\min} + 4}{3}}, b - \frac{1}{2} + \sqrt{\frac{3(S_{\min} + 4)}{8} - \frac{9}{4a^2} + \frac{3(1 - 2b)}{2a}}, \right. \\ \left. b - \frac{1}{2} + \sqrt{\frac{S_{\min} + 4}{2} - \frac{16}{a^2} + \frac{4(1 - 2b)}{a}} \right\}.$$

Theorem B (see Theorem 6.3 in [13]). *Let $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 3$, be a closed eta-Einstein Sasakian spin manifold. Let the parameters a, b of D_{ab} and the scalar curvature S satisfy*

$$0 < a \leq 1, \quad 0 \leq b < \frac{1}{2}$$

and

$$(2b - 1)^2 \cdot \frac{m}{m + 1} < S + 2m \leq \frac{(2b - 1)^2}{2 - a(1 - 2b)} \cdot 2m(m + 1),$$

respectively. Then, for any eigenvalue λ of D_{ab} , the inequality

$$(1.5) \quad |\lambda| \geq \begin{cases} \min \left\{ \frac{1}{a} \rho_{ab}(0), b - \frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)} \right\} & \text{for odd } m \geq 3, \\ b - \frac{1}{2} + \sqrt{\frac{2m(m+2)+a(1-2b)}{8m(m+1)}(S+2m) + (b - \frac{1}{2})^2} & \text{for even } m \geq 4, \end{cases}$$

holds, where $\frac{1}{a} \rho_{ab}(0) := \frac{a(S+2m)}{8m} + \frac{m(1-2b)}{2}$.

Equality $|\lambda| = \frac{a(S_{\min}+4)}{16} + 1 - 2b$ in (1.4) can be attained and characterized by the existence of a *phi-Killing spinor* (see Definition 3.3 in [13]). Equality for odd m in (1.5) can be attained and characterized by the existence of a *phi-Killing spinor* if $|\lambda| = \frac{1}{a} \rho_{ab}(0)$ (resp. *Sasakian Killing spinor* if $|\lambda| = b - \frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)}$, see Definition 3.4 in [13]). However, we don't know yet whether other equality cases in (1.4)-(1.5) may occur.

The aim of the paper is to prove that (1.4) (resp. the latter part of (1.5) for even m) can be improved to (3.5) (resp. (3.13)). In the next section we will consider a special class of spinors called *Sasakian duos* (see Definition 2.1) whose existence characterizes the equality case in (3.5) and (3.13). It turns out that the Sasakian duos generalize the Sasakian Killing spinors in a natural way (see Remark 2.1).

2. A special class of Sasakian twistor spinors

We start with recalling some general properties in Sasakian spin geometry [13]. Throughout this paper we will denote by $(M^{2m+1}, \phi, \xi, \eta, g)$ a Sasakian spin manifold of dimension $2m + 1 \geq 3$ and by $\Sigma = \Sigma(M)$ the spinor bundle over M^{2m+1} . Denote $P_+(V), P_-(V)$ the complex vector fields

$$P_+(V) = \frac{1}{2} [V + \sqrt{-1}\phi(V)], \quad P_-(V) = \frac{1}{2} [V - \sqrt{-1}\phi(V)], \quad V \in \Gamma(\xi^\perp).$$

Then it holds that

$$\begin{aligned} (P_+ \circ P_+)(V) &= P_+(V), & (P_- \circ P_-)(V) &= P_-(V), \\ (P_+ \circ P_-)(V) &= (P_- \circ P_+)(V) = 0, \\ P_+(V) \cdot P_+(V) &= P_-(V) \cdot P_-(V) = 0, \\ \sum_{u=1}^{2m} P_+(E_u) \cdot P_-(E_u) &= -m + \sqrt{-1}\Phi, \\ \sum_{u=1}^{2m} P_-(E_u) \cdot P_+(E_u) &= -m - \sqrt{-1}\Phi \end{aligned}$$

and

$$D_+ = \frac{1}{2}\bar{C} + \frac{\sqrt{-1}}{2}\bar{Q} = \sum_{u=1}^{2m} P_+(E_u) \cdot \bar{\nabla}_{E_u} = \sum_{u=1}^{2m} E_u \cdot \bar{\nabla}_{P_-(E_u)},$$

$$D_- = \frac{1}{2}\overline{C} - \frac{\sqrt{-1}}{2}\overline{Q} = \sum_{u=1}^{2m} P_-(E_u) \cdot \overline{\nabla}_{E_u} = \sum_{u=1}^{2m} E_u \cdot \overline{\nabla}_{P_+(E_u)},$$

where $\Phi := g(\cdot, \phi(\cdot))$ is the fundamental form.

Under the action of the fundamental form Φ , the spinor bundle Σ splits into the orthogonal direct sum $\Sigma = \Sigma_0 \oplus \Sigma_1 \oplus \dots \oplus \Sigma_m$ with

$$\Phi|_{\Sigma_r} = \sqrt{-1}(2r - m)I, \quad \dim(\Sigma_r) = \binom{m}{r}, \quad r \in \{0, 1, \dots, m\},$$

where I stands for the identity map. Since $\overline{\nabla}_V$ commutes with Φ , we have

$$\overline{\nabla}_V(\Gamma(\Sigma_r)) \subset \Gamma(\Sigma_r), \quad D_+(\Gamma(\Sigma_r)) \subset \Gamma(\Sigma_{r+1}), \quad D_-(\Gamma(\Sigma_r)) \subset \Gamma(\Sigma_{r-1}),$$

where we use the convention that Σ_s is the zero subbundle of Σ if $s \notin \{0, 1, \dots, m\}$.

Let $\varphi_r \in \Gamma(\Sigma_r)$ and $\varphi_{r+1} \in \Gamma(\Sigma_{r+1})$, $r \in \{0, 1, \dots, m - 1\}$, be non-trivial spinor fields on $(M^{2m+1}, \phi, \xi, \eta, g)$. Then φ_r and φ_{r+1} are called a *left* and a *right Sasakian twistor spinor* if, for all vector fields $V \in \Gamma(\xi^\perp)$,

$$\overline{\nabla}_V \varphi_r = -\frac{1}{2(r+1)}P_-(V) \cdot D_+ \varphi_r$$

and

$$\overline{\nabla}_V \varphi_{r+1} = -\frac{1}{2(m-r)}P_+(V) \cdot D_- \varphi_{r+1}$$

hold, respectively. Via direct computations one verifies the formulas in the following two propositions.

Proposition 2.1. *Let $\varphi_r \in \Gamma(\Sigma_r)$, $r \in \{0, 1, \dots, m - 1\}$, be a left Sasakian twistor spinor on $(M^{2m+1}, \phi, \xi, \eta, g)$. Then $D_- \varphi_r = 0$ vanishes identically and*

$$\begin{aligned} \overline{C}^2(\varphi_r) &= \frac{r+1}{2(2r+1)}(S+2m)\varphi_r + (-1)^{m+r} \frac{4(r+1)(m-2r)}{2r+1} D_0 \varphi_r \\ &= -\frac{(r+1)\sqrt{-1}}{2} \sum_{u=1}^{2m} E_u \cdot \text{Ric}(\phi E_u) \cdot \varphi_r + 2(r+1)(m-2r)\varphi_r \\ (2.1) \quad &+ (-1)^{m+r} 4m(r+1)D_0 \varphi_r. \end{aligned}$$

In particular, if $(M^{2m+1}, \phi, \xi, \eta, g)$ is eta-Einstein and $r \neq 0$, then it holds that

$$(2.2) \quad D_0 \varphi_r = (-1)^{m+r} \frac{2r+1-m}{8m(m+1)}(S+2m)\varphi_r$$

and

$$(2.3) \quad \overline{C}^2(\varphi_r) = \frac{(r+1)(m-r)}{m(m+1)}(S+2m)\varphi_r.$$

Proposition 2.2. *Let $\varphi_{r+1} \in \Gamma(\Sigma_{r+1})$, $r \in \{0, 1, \dots, m - 1\}$, be a right Sasakian twistor spinor on $(M^{2m+1}, \phi, \xi, \eta, g)$. Then $D_+\varphi_{r+1} = 0$ vanishes identically and*

$$\begin{aligned} \overline{C}^2(\varphi_{r+1}) &= \frac{m-r}{2(2m-2r-1)}(S+2m)\varphi_{r+1} \\ &\quad + (-1)^{m+r} \frac{4(m-r)(2r+2-m)}{2m-2r-1} D_0\varphi_{r+1} \\ &= \frac{(m-r)\sqrt{-1}}{2} \sum_{u=1}^{2m} E_u \cdot \text{Ric}(\phi E_u) \cdot \varphi_{r+1} \\ (2.4) \quad &\quad + 2(m-r)(2r+2-m)\varphi_{r+1} + (-1)^{m+r} 4m(m-r)D_0\varphi_{r+1}. \end{aligned}$$

In particular, if $(M^{2m+1}, \phi, \xi, \eta, g)$ is eta-Einstein and $r \neq m - 1$, then it holds that

$$(2.5) \quad D_0\varphi_{r+1} = (-1)^{m+r+1} \frac{2r+1-m}{8m(m+1)} (S+2m)\varphi_{r+1}$$

and

$$(2.6) \quad \overline{C}^2(\varphi_{r+1}) = \frac{(r+1)(m-r)}{m(m+1)} (S+2m)\varphi_{r+1}.$$

We now consider a special class of Sasakian twistor spinors that characterizes the equality case in (3.5) and (3.13).

Definition 2.1. A pair $(\varphi_r, \varphi_{r+1})$ of non-trivial spinor fields φ_r, φ_{r+1} , $r \in \{0, 1, \dots, m - 1\}$, is called a *Sasakian duo* with characteristic numbers (c_+, c_-) if, for all vector fields $V \in \Gamma(\xi^\perp)$, the systems of two differential equations

$$(2.7) \quad \begin{aligned} \overline{\nabla}_V \varphi_r &= -\frac{c_+}{2(r+1)} P_-(V) \cdot \varphi_{r+1}, \\ \overline{\nabla}_V \varphi_{r+1} &= -\frac{c_-}{2(m-r)} P_+(V) \cdot \varphi_r \end{aligned}$$

hold, where $c_+, c_- \in \mathbb{C}$ are complex numbers such that the product $c_+c_- \in \mathbb{R}$ is a real number.

Obviously, if $(\varphi_r, \varphi_{r+1})$ is a Sasakian duo with characteristic numbers (c_+, c_-) , then φ_r (resp. φ_{r+1}) is a left (resp. right) Sasakian twistor spinor and it holds that

$$\begin{aligned} D_-\varphi_r &= D_+\varphi_{r+1} = 0, \\ D_+\varphi_r &= c_+\varphi_{r+1}, \quad D_-\varphi_{r+1} = c_-\varphi_r, \\ D_-\varphi_r &= c_+c_-\varphi_r, \quad D_+\varphi_{r+1} = c_+c_-\varphi_{r+1}. \end{aligned}$$

Remark 2.1. Let $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 3$, be a Sasakian spin manifold with $m \equiv 1 \pmod{2}$. Suppose that $(M^{2m+1}, \phi, \xi, \eta, g)$ admits a Sasakian duo $(\varphi_l, \varphi_{l+1})$, $l := \frac{m-1}{2}$, with characteristic numbers (c, c) , where $0 \neq c \in \mathbb{R}$ is a non-zero real number. Then the spinor field $\varphi := \varphi_l + \varphi_{l+1}$ is a Sasakian

Killing spinor with characteristic number c , i.e., φ is a solution to the differential equation

$$\bar{\nabla}_V \varphi = -\frac{c}{2(m+1)} V \cdot \varphi - (-1)^{l+1} \frac{c}{2(m+1)} \phi(V) \cdot \xi \cdot \varphi, \quad V \in \Gamma(\xi^\perp).$$

Using Propositions 2.1 and 2.2 we obtain:

Proposition 2.3. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ admit a Sasakian duo $(\varphi_r, \varphi_{r+1})$, $r \in \{0, 1, \dots, m-1\}$, with characteristic numbers (c_+, c_-) . Then*

- (1) $(\nabla_\xi \varphi_r, \varphi_r) = (\nabla_\xi \varphi_{r+1}, \varphi_{r+1}) = 0$.
- (2) Both (φ_r, φ_r) and $(\varphi_{r+1}, \varphi_{r+1})$ are non-vanishing on an open dense subset of M^{2m+1} .
- (3) If $2r \neq m-1$, then $(M^{2m+1}, \phi, \xi, \eta, g)$ must be eta-Einstein and it holds that

$$\begin{aligned} c_+c_- &= \frac{(r+1)(m-r)}{m(m+1)}(S+2m), \\ D_0\varphi_r &= (-1)^{m+r} \frac{2r+1-m}{8m(m+1)}(S+2m)\varphi_r, \\ D_0\varphi_{r+1} &= (-1)^{m+r+1} \frac{2r+1-m}{8m(m+1)}(S+2m)\varphi_{r+1}. \end{aligned}$$

- (4) If $2r = m-1$ (m must be odd) and $c_+c_- > 0$ is positive, then $(M^{2m+1}, \phi, \xi, \eta, g)$ is eta-Einstein and it holds that

$$c_+c_- = \frac{m+1}{4m}(S+2m), \quad D_0\varphi_r = D_0\varphi_{r+1} = 0.$$

Proof. Part (1) of the proposition is immediate from (2.1) and (2.4). We now use the relation

$$\begin{aligned} &\operatorname{div} \left(\sum_{u=1}^{2m} (\varphi_1, P_-(E_u) \cdot \varphi_2) E_u \right) \\ &= -(D_+\varphi_1, \varphi_2) + (\varphi_1, D_-\varphi_2), \quad \varphi_1, \varphi_2 \in \Gamma(\Sigma(M)), \end{aligned}$$

and

$$(\nabla_\xi \varphi_r, \varphi_r) = (\nabla_\xi \varphi_{r+1}, \varphi_{r+1}) = 0,$$

to check that

$$\begin{aligned} -(D_+\varphi_r, D_+\varphi_r) + (\varphi_r, D_-D_+\varphi_r) &= (r+1)\Delta(\varphi_r, \varphi_r), \\ -(D_-\varphi_{r+1}, D_-\varphi_{r+1}) + (\varphi_{r+1}, D_+D_-\varphi_{r+1}) &= (m-r)\Delta(\varphi_{r+1}, \varphi_{r+1}), \end{aligned}$$

where $\Delta := -\operatorname{div} \circ \operatorname{grad}$. It follows that the functions (φ_r, φ_r) , $(\varphi_{r+1}, \varphi_{r+1})$ satisfy

$$\begin{aligned} \Delta(\varphi_r, \varphi_r) &= \frac{c_+c_-}{r+1}(\varphi_r, \varphi_r) - \frac{c_+\bar{c}_+}{r+1}(\varphi_{r+1}, \varphi_{r+1}), \\ \Delta(\varphi_{r+1}, \varphi_{r+1}) &= -\frac{c_-\bar{c}_-}{(m-r)}(\varphi_r, \varphi_r) + \frac{c_+c_-}{m-r}(\varphi_{r+1}, \varphi_{r+1}), \end{aligned}$$

where \bar{c}_+ (resp. \bar{c}_-) denotes the complex conjugate of c_+ (resp. c_-). This completes the proof of part (2) of the proposition. To prove part (3), we note that, for any $\varphi \in \Gamma(\Sigma(M))$ and $V \in \Gamma(\xi^\perp)$, the identity

$$(2.8) \quad \begin{aligned} \frac{1}{2}\text{Ric}(V) \cdot \varphi &= -\bar{\nabla}_V(\bar{C}\varphi) + \bar{C}(\bar{\nabla}_V\varphi) - V \cdot \varphi - \phi(V) \cdot \Phi \cdot \varphi + \phi(V) \cdot \nabla_\xi \varphi \\ &\quad - \sum_{u=1}^{2m} E_u \cdot \nabla_{\nabla_{E_u} V} \varphi + \frac{1}{2} \sum_{u=1}^{2m} E_u \cdot \phi(\nabla_{E_u} V) \cdot \xi \cdot \varphi \end{aligned}$$

holds. Applying (2.7) to (2.8), we obtain

$$(2.9) \quad \begin{aligned} \frac{1}{2}\text{Ric}(V) \cdot \varphi_r &= \left[-1 + \frac{c_+c_-(m+1)}{4(r+1)(m-r)} \right] V \cdot \varphi_r \\ &\quad + \frac{c_+c_-(2r+1-m)}{4(r+1)(m-r)} \sqrt{-1} \phi(V) \cdot \varphi_r \\ &\quad - 2(-1)^{m+r} \sqrt{-1} \phi(V) \cdot D_0\varphi_r, \end{aligned}$$

and

$$(2.10) \quad \begin{aligned} \frac{1}{2}\text{Ric}(V) \cdot \varphi_{r+1} &= \left[-1 + \frac{c_+c_-(m+1)}{4(r+1)(m-r)} \right] V \cdot \varphi_{r+1} \\ &\quad + \frac{c_+c_-(2r+1-m)}{4(r+1)(m-r)} \sqrt{-1} \phi(V) \cdot \varphi_{r+1} \\ &\quad + 2(-1)^{m+r} \sqrt{-1} \phi(V) \cdot D_0\varphi_{r+1}. \end{aligned}$$

Using (2.1) and (2.4), we can now rewrite (2.9) and (2.10) as

$$(2.11) \quad \begin{aligned} &(m-2r) \left[2(r+1)(m-r)\text{Ric}(V) + [4(r+1)(m-r) - c_+c_-(m+1)] V \right] \cdot \varphi_r \\ &= [-c_+c_-m(m+1) + (r+1)(m-r)(S+2m)] \sqrt{-1} \phi(V) \cdot \varphi_r \end{aligned}$$

and

$$(2.12) \quad \begin{aligned} &(m-2-2r) \left[2(r+1)(m-r)\text{Ric}(V) + [4(r+1)(m-r) - c_+c_-(m+1)] V \right] \cdot \varphi_{r+1} \\ &= [-c_+c_-m(m+1) + (r+1)(m-r)(S+2m)] \sqrt{-1} \phi(V) \cdot \varphi_{r+1}, \end{aligned}$$

respectively. Since both (φ_r, φ_r) and $(\varphi_{r+1}, \varphi_{r+1})$ are non-vanishing on an open dense subset of M^{2m+1} , we can compare (2.11) with (2.12) to conclude that part (3) of the proposition is true. However, it should be noted here that if $2r = m-1$, then combing (2.11) with (2.12) alone does not suffice to prove part (4). In the case $2r = m-1$, we make use of the fact that one can equivalently rewrite

$$\begin{aligned} \bar{\nabla}_V \varphi_l &= -\frac{c_+}{m+1} P_-(V) \cdot \varphi_{l+1}, \\ \bar{\nabla}_V \varphi_{l+1} &= -\frac{c_-}{m+1} P_+(V) \cdot \varphi_l, \quad l := \frac{m-1}{2}, \end{aligned}$$

as

$$\begin{aligned} \bar{\nabla}_V \psi_l &= -(-1)^{l+1} \frac{\sqrt{c_+c_-}}{m+1} P_-(V) \cdot \psi_{l+1}, \\ \bar{\nabla}_V \psi_{l+1} &= -(-1)^{l+1} \frac{\sqrt{c_+c_-}}{m+1} P_+(V) \cdot \psi_l, \end{aligned}$$

where $\psi_l := (-1)^{l+1} \sqrt{c_+c_-} \varphi_l$ and $\psi_{l+1} := c_+ \varphi_{l+1}$. Then, $\psi = \psi_l + \psi_{l+1}$ is a Sasakian Killing spinor with characteristic number $(-1)^{l+1} \sqrt{c_+c_-}$. A simple computation shows that

$$\begin{aligned} R(V, \xi)(\psi) &= -\frac{1}{2} V \cdot \xi \cdot \psi \\ &= \nabla_V \nabla_\xi \psi - \nabla_\xi \nabla_V \psi - \nabla_{[V, \xi]} \psi \\ &= -\frac{1}{2} V \cdot \xi \cdot \psi + \frac{1}{8} (-1)^{l+1} dS(V) \xi \cdot \psi \\ &\quad - \frac{\sqrt{c_+c_-}}{8(m+1)} \left(S + 2m - \frac{4mc_+c_-}{m+1} \right) [\phi(V) + (-1)^l V \cdot \xi] \cdot \psi, \end{aligned}$$

from which it follows that $dS(V) = 0$ and $S + 2m - \frac{4mc_+c_-}{m+1} = 0$ vanish identically. We finally conclude from (2.9)-(2.12) that part (4) of the proposition is true. \square

Recall [21] that a Sasakian manifold $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 2$, is called a *Sasakian space form* of constant ϕ -sectional curvature $\kappa \in \mathbb{R}$ if, at any point of M^{2m+1} , the sectional curvature $g(R(V, \phi(V))\phi(V), V) = \kappa$ is constant and independent of the choice of unit vector $V \in \xi^\perp$. A complete simply-connected Sasakian space form of constant ϕ -sectional curvature κ is isometric to one of the following three model spaces:

- (1) A sphere $S^{2m+1}(\kappa)$ with constant ϕ -sectional curvature $\kappa > -3$,
- (2) A real vector space $\mathbb{R}^{2m+1}(-3)$ with constant ϕ -sectional curvature $\kappa = -3$,
- (3) A principal \mathbb{R}^1 -bundle $(B^{2m} \times \mathbb{R}^1)(\kappa)$ over B^{2m} with constant ϕ -sectional curvature $\kappa < -3$, where $B^{2m} \subset \mathbb{C}^m$ denotes a simply-connected bounded complex domain with constant holomorphic curvature $\kappa + 3 < 0$.

We now prove that each subbundle $\Sigma_r \oplus \Sigma_{r+1}$ of the spinor bundle Σ over a simply-connected Sasakian space form admits a Sasakian duo.

Theorem 2.1. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a simply-connected Sasakian space form of constant ϕ -sectional curvature $\kappa \in \mathbb{R}$. Then, for any $r \in \{0, 1, \dots, m-1\}$ and for any pair (c_+, c_-) of complex numbers satisfying $c_+c_- = (r+1)(m-r)(\kappa+3)$, there exists a Sasakian duo $(\varphi_r, \varphi_{r+1}) \in \Gamma(\Sigma_r \oplus \Sigma_{r+1})$ with characteristic numbers (c_+, c_-) .*

Proof. Since the Riemann curvature tensor R of $(M^{2m+1}, \phi, \xi, \eta, g)$ is given by

$$R(X, Y)(Z) = \frac{\kappa + 3}{4} \left[g(Y, Z)X - g(X, Z)Y \right]$$

$$\begin{aligned}
 & + \frac{\kappa - 1}{4} \left[g(\phi Y, Z)(\phi X) - g(\phi X, Z)(\phi Y) - 2g(\phi X, Y)(\phi Z) \right] \\
 & + \frac{\kappa - 1}{4} \left[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi \right. \\
 & \quad \left. - \eta(X)g(Y, Z)\xi \right]
 \end{aligned}$$

and κ is related to the scalar curvature S by

$$S = m(m + 1)(\kappa + 3) - 2m,$$

we see that, for all vector fields $V, W \in \Gamma(\xi^\perp)$, the action of the Riemann curvature $R(V, W)$ on a spinor field φ is given by

$$\begin{aligned}
 & R(V, W)(\varphi) \\
 & = -\frac{S + 2m}{8m(m + 1)}V \cdot W \cdot \varphi - \left[\frac{S + 2m}{8m(m + 1)} - \frac{1}{2} \right] \phi(V) \cdot \phi(W) \cdot \varphi \\
 (2.13) \quad & - \left[\frac{S + 2m}{4m(m + 1)} - \frac{1}{2} \right] g(V, W)\varphi - \left[\frac{S + 2m}{4m(m + 1)} - 1 \right] g(V, \phi(W))\Phi \cdot \varphi.
 \end{aligned}$$

Define a connection $\tilde{\nabla}$ acting on sections $(\varphi_r, \varphi_{r+1}) \in \Gamma(\Sigma_r \oplus \Sigma_{r+1})$ of the subbundle $\Sigma_r \oplus \Sigma_{r+1}$ of $\Sigma(M)$, $r \in \{0, 1, \dots, m - 1\}$, by

$$\begin{aligned}
 \tilde{\nabla}_V \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} & = \begin{pmatrix} \bar{\nabla}_V, & \frac{c_+}{2(r+1)}P_-(V) \\ \frac{c_-}{2(m-r)}P_+(V), & \bar{\nabla}_V \end{pmatrix} \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} \\
 & = \begin{pmatrix} \bar{\nabla}_V \varphi_r + \frac{c_+}{2(r+1)}P_-(V) \cdot \varphi_{r+1} \\ \frac{c_-}{2(m-r)}P_+(V) \cdot \varphi_r + \bar{\nabla}_V \varphi_{r+1} \end{pmatrix}
 \end{aligned}$$

and

$$\begin{aligned}
 \tilde{\nabla}_\xi \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} & = \left(\nabla_\xi - \frac{1}{2}\Phi + \nu\sqrt{-1} \right) \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} \\
 & = \begin{pmatrix} \nabla_\xi \varphi_r + \frac{1}{2}(m - 2r + 2\nu)\sqrt{-1}\varphi_r \\ \nabla_\xi \varphi_{r+1} + \frac{1}{2}(m - 2 - 2r + 2\nu)\sqrt{-1}\varphi_{r+1} \end{pmatrix},
 \end{aligned}$$

where $\nu \in \mathbb{R}$ is an arbitrary real number. Then, a direct computation show that, for any pair $(\varphi_r, \varphi_{r+1}) \in \Gamma(\Sigma_r \oplus \Sigma_{r+1})$,

$$\begin{aligned}
 \tilde{R}(V, \xi) \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} & = \tilde{\nabla}_V \tilde{\nabla}_\xi \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} - \tilde{\nabla}_\xi \tilde{\nabla}_V \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} - \tilde{\nabla}_{[V, \xi]} \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} \\
 & = 0
 \end{aligned}$$

vanishes identically. Moreover,

$$\begin{aligned}
 \tilde{R}(V, W) \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} & = \tilde{\nabla}_V \tilde{\nabla}_W \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} - \tilde{\nabla}_W \tilde{\nabla}_V \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix} \\
 & \quad - \tilde{\nabla}_{[V, W]} \begin{pmatrix} \varphi_r \\ \varphi_{r+1} \end{pmatrix}
 \end{aligned}$$

$$= 0$$

vanishes identically if and only if the relations

$$\begin{aligned}
 0 &= R(V, W)(\varphi_r) - \left[\frac{1}{2}\phi(V) \cdot \phi(W) + \frac{1}{2}g(V, W) \right. \\
 &\quad \left. - (m - 2r + 2\nu)g(V, \phi(W))\sqrt{-1} \right] \cdot \varphi_r \\
 &\quad + \frac{c_+c_-}{8(r+1)(m-r)} \left[V \cdot W + \phi(V) \cdot \phi(W) + 2g(V, W) \right. \\
 &\quad \left. - 2g(V, \phi(W))\sqrt{-1} \right] \cdot \varphi_r \\
 &= R(V, W)(\varphi_{r+1}) - \left[\frac{1}{2}\phi(V) \cdot \phi(W) + \frac{1}{2}g(V, W) \right. \\
 &\quad \left. - (m - 2 - 2r + 2\nu)g(V, \phi(W))\sqrt{-1} \right] \cdot \varphi_{r+1} \\
 &\quad + \frac{c_+c_-}{8(r+1)(m-r)} \left[V \cdot W + \phi(V) \cdot \phi(W) \right. \\
 (2.14) \quad &\quad \left. + 2g(V, W) + 2g(V, \phi(W))\sqrt{-1} \right] \cdot \varphi_{r+1}
 \end{aligned}$$

hold. Let us choose the constant ν as

$$\nu = -\frac{m-1-2r}{8m(m+1)}(S+2m).$$

Then, one finally checks that (2.13) satisfies (2.14). □

Using the *contact Bochner curvature tensor* B [19] defined by

$$\begin{aligned}
 &B(X, Y)(Z) \\
 = &R(X, Y)(Z) + \frac{1}{2m+4} \left[g(X, Z)\text{Ric}(Y) - g(Y, Z)\text{Ric}(X) + \text{Ric}(X, Z)Y \right. \\
 &- \text{Ric}(Y, Z)X + g(\phi X, Z)\text{Ric}(\phi Y) - g(\phi Y, Z)\text{Ric}(\phi X) + 2g(\phi X, Y)\text{Ric}(\phi Z) \\
 &+ \text{Ric}(\phi X, Z)(\phi Y) - \text{Ric}(\phi Y, Z)(\phi X) + 2\text{Ric}(\phi X, Y)(\phi Z) \\
 &+ \eta(Y)\eta(Z)\text{Ric}(X) - \eta(X)\eta(Z)\text{Ric}(Y) + \eta(X)\text{Ric}(Y, Z)\xi \\
 &\left. - \eta(Y)\text{Ric}(X, Z)\xi \right] + \frac{S+2m-8(m+1)}{4(m+1)(m+2)} \left[g(Y, Z)X - g(X, Z)Y \right] \\
 &+ \frac{S+2m+4m(m+1)}{4(m+1)(m+2)} \left[g(\phi Y, Z)(\phi X) - g(\phi X, Z)(\phi Y) - 2g(\phi X, Y)(\phi Z) \right] \\
 &+ \frac{S+2m}{4(m+1)(m+2)} \left[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi \right]
 \end{aligned}$$

$$- \eta(X)g(Y, Z)\xi \Big],$$

one can slightly generalize Theorem 2.1 as follows. Note that, over any Sasakian space form, the contact Bochner curvature tensor $B = 0$ vanishes identically. Since the proof is straightforward, we omit it.

Theorem 2.2. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a simply-connected eta-Einstein Sasakian spin manifold of dimension $2m + 1 \geq 5$ and let $r \in \{0, 1, \dots, m - 1\}$. Suppose that*

$$B(V, W)(\varphi) := -\frac{1}{4} \sum_{u,v=1}^{2m+1} g(B(V, W)(E_v), E_u)E_u \cdot E_v \cdot \varphi = 0$$

vanishes identically for all $V, W \in \Gamma(\xi^\perp)$ and $\varphi \in \Gamma(\Sigma_r \oplus \Sigma_{r+1})$. Then, for any pair (c_+, c_-) of complex numbers satisfying $c_+c_- = \frac{(r+1)(m-r)}{m(m+1)}(S + 2m)$, there exists a Sasakian duo $(\varphi_r, \varphi_{r+1}) \in \Gamma(\Sigma_r \oplus \Sigma_{r+1})$ with characteristic numbers (c_+, c_-) .

Let $(\varphi_r, \varphi_{r+1})$ be a Sasakian duo with characteristic numbers (c_+, c_-) on a Sasakian spin manifold $(M^{2m+1}, \phi, \xi, \eta, g)$. Then the spinor field $\varphi = \varphi_r + \varphi_{r+1}$ is generally not an eigenspinor of the generalized Dirac operator D_{ab} . More precisely, one verifies using Proposition 2.3 that $\varphi = \varphi_r + \varphi_{r+1}$ is an eigenspinor of D_{ab} if and only if the relation $(-1)^{m+r}c_- - (-1)^{m+r}c_+ = \left[\frac{a(S+2m)}{4m(m+1)} + 1 - 2b \right] (m - 1 - 2r)$ holds. If this is the case and $\varphi = \varphi_r + \varphi_{r+1}$ is associated with eigenvalue λ , then one checks that

$$\begin{aligned} (-1)^{m+r}\lambda + \frac{1}{2} - b &= (-1)^{m+r}c_+ + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b \right] (m - 1 - 2r) \\ &= (-1)^{m+r}c_- - \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b \right] (m - 1 - 2r). \end{aligned}$$

Inserting this into $c_+c_- = \frac{(r+1)(m-r)}{m(m+1)}(S + 2m)$, one can now explicitly express the characteristic numbers c_+, c_- in terms of the scalar curvature. In what follows, we fix the notation $E^\lambda(D_{ab})$ to denote the space of all eigenspinors of D_{ab} with eigenvalue $\lambda \in \mathbb{C}$.

Proposition 2.4. *Let $(\varphi_r, \varphi_{r+1})$ be a Sasakian duo with characteristic numbers (c_+, c_-) and let $\varphi = \varphi_r + \varphi_{r+1} \in E^\lambda(D_{ab})$. Then all the possible values for c_+, c_- are:*

$$\begin{aligned} &(1) \\ &(-1)^{m+r}c_\pm \\ &= \mp \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b \right] (m - 1 - 2r) \end{aligned}$$

$$+ \sqrt{\frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b\right]^2} (m-1-2r)^2.$$

In this case λ is equal to

$$\begin{aligned} & (-1)^{m+r}\lambda \\ &= b - \frac{1}{2} + \sqrt{\frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b\right]^2} (m-1-2r)^2. \end{aligned} \tag{2}$$

$$\begin{aligned} & (-1)^{m+r}c_{\pm} \\ &= \mp \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b\right] (m-1-2r) \\ & \quad - \sqrt{\frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b\right]^2} (m-1-2r)^2. \end{aligned}$$

In this case λ is equal to

$$\begin{aligned} & (-1)^{m+r}\lambda \\ &= b - \frac{1}{2} - \sqrt{\frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b\right]^2} (m-1-2r)^2. \end{aligned}$$

We remark that if $S + 2m \geq 0$ in Proposition 2.4, then all of c_+, c_-, λ are real numbers. However, if $S + 2m < 0$, then it may happen that all of c_+, c_-, λ are complex numbers with a non-trivial imaginary part. The following two propositions are easy to verify and we omit the proofs.

Proposition 2.5. *Let $(\varphi_r, \varphi_{r+1})$ be a Sasakian duo with characteristic numbers (c_+, c_-) and let $\varphi = \varphi_r + \varphi_{r+1} \in E^\lambda(D_{ab})$. Then the pair $(\varphi_r^* := c_- \varphi_r, \varphi_{r+1}^* := -c_+ \varphi_{r+1})$ is a Sasakian duo with characteristic numbers $(c_+^* := -c_-, c_-^* := -c_+)$ and $\varphi^* = \varphi_r^* + \varphi_{r+1}^*$ is an eigenspinor D_{ab} with eigenvalue $\lambda^* = -\lambda + (-1)^{m+r}(2b - 1)$.*

The above proposition means that if $\varphi = \varphi_r + \varphi_{r+1} \in E^\lambda(D_{ab})$ is the spinor field described in part (1) (resp. part (2)) of Proposition 2.4, then $\varphi^* = \varphi_r^* + \varphi_{r+1}^* \in E^{\lambda^*}(D_{ab})$ is the one described in part (2) (resp. part (1)).

Proposition 2.6. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be an eta-Einstein Sasakian spin manifold with $S + 2m > 0$ and let $(\varphi_r, \varphi_{r+1}), r \in \{0, 1, \dots, m - 1\}$, be a Sasakian duo with characteristic numbers (c_+, c_-) . If we rescale the length of φ_r and φ_{r+1} as $\psi_r := (-1)^{m+r}c_- \varphi_r$ and $\psi_{r+1} := \tau \varphi_{r+1}$,*

$$\tau := \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b\right] (m-1-2r)$$

$$\pm \sqrt{\frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b\right]^2} (m-1-2r)^2,$$

then (ψ_r, ψ_{r+1}) is a Sasakian duo with characteristic numbers $((-1)^{m+r} \frac{c_+c_-}{\tau}, (-1)^{m+r} \tau)$ and $\psi = \psi_r + \psi_{r+1}$ is an eigenspinor of D_{ab} .

As a consequence of Proposition 2.6, we may assume without loss of generality, by rescaling the length of spinor fields if necessary, that, for any Sasakian duo $(\varphi_r, \varphi_{r+1})$ on an eta-Einstein Sasakian spin manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ with $S + 2m > 0$, $\varphi = \varphi_r + \varphi_{r+1}$ is an eigenspinor of D_{ab} .

Consider the Tanno deformation [20] of Sasakian structure (ϕ, ξ, η, g)

$$\tilde{\phi} := \phi, \quad \tilde{\xi} := t^2\xi, \quad \tilde{\eta} := t^{-2}\eta, \quad \tilde{g} := t^{-2}g + (t^{-4} - t^{-2})\eta \otimes \eta,$$

where $t > 0$ is a positive real number. Then the Ricci tensor $\widetilde{\text{Ric}}$ of $(M^{2m+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is related to the Ricci tensor Ric of $(M^{2m+1}, \phi, \xi, \eta, g)$ by

$$\widetilde{\text{Ric}} = \text{Ric} + 2(1 - t^{-2})g + (2mt^{-4} + 2t^{-2} - 2m - 2)\eta \otimes \eta.$$

It follows that if $(M^{2m+1}, \phi, \xi, \eta, g)$ is eta-Einstein with $S + 2m > 0$ and the parameter t is chosen to be $t^2 = \frac{4m(m+1)}{S+2m}$, then $(M^{2m+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ has to be Einstein with positive scalar curvature $\tilde{S} = 2m(2m + 1)$. Now observe that Sasakian duos are invariant under the Tanno deformation. Namely, if $(\varphi_r, \varphi_{r+1})$ is a Sasakian duo on $(M^{2m+1}, \phi, \xi, \eta, g)$ with characteristic numbers (c_+, c_-) , then the corresponding pair $(\tilde{\varphi}_r, \tilde{\varphi}_{r+1})$ on $(M^{2m+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian duo with characteristic numbers $(\tilde{c}_+ := tc_+, \tilde{c}_- := tc_-)$. By Propositions 2.5 and 2.6, we can choose the characteristic numbers $(\tilde{c}_+, \tilde{c}_-)$ to be

$$\begin{aligned} & (-1)^{m+r} \tilde{c}_\pm \\ &= \mp \left[\frac{a(\tilde{S} + 2m)}{8m(m+1)} + \frac{1}{2} - b \right] (m-1-2r) \\ &+ \sqrt{\frac{(r+1)(m-r)}{m(m+1)}(\tilde{S} + 2m) + \left[\frac{a(\tilde{S} + 2m)}{8m(m+1)} + \frac{1}{2} - b\right]^2} (m-1-2r)^2. \end{aligned}$$

In that case, $\tilde{\varphi} = \tilde{\varphi}_r + \tilde{\varphi}_{r+1}$ is an eigenspinor of D_{ab} with eigenvalue $\tilde{\lambda}$,

$$\begin{aligned} & (-1)^{m+r} \tilde{\lambda} \\ &:= b - \frac{1}{2} + \sqrt{\frac{(r+1)(m-r)}{m(m+1)}(\tilde{S} + 2m) + \left[\frac{a(\tilde{S} + 2m)}{8m(m+1)} + \frac{1}{2} - b\right]^2} (m-1-2r)^2. \end{aligned}$$

Note that the spinor field $\tilde{\varphi} = \tilde{\varphi}_r + \tilde{\varphi}_{r+1}$ becomes a Killing spinor satisfying

$$\nabla_X \tilde{\varphi} = \frac{(-1)^{m+r+1}}{2} X \cdot \tilde{\varphi}, \quad X \in \Gamma(T(M)),$$

when $(M^{2m+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is Einstein and the parameters a, b for D_{ab} are chosen as $(a = 1, b = 0)$. Suppose that $(M^{2m+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a simply-connected closed Sasaki-Einstein spin manifold that is neither 3-Sasakian nor the round sphere. Then we know that M^{2m+1} admits exactly two Killing spinors ψ_0, ψ_m with $\psi_0 \in \Gamma(\Sigma_0)$ and $\psi_m \in \Gamma(\Sigma_m)$ [4, 8, 12]. Thus we have proved:

Theorem 2.3. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a simply-connected closed eta-Einstein Sasakian spin manifold with $S + 2m > 0$ and let M^{2m+1} admit a Sasakian duo $(\varphi_r, \varphi_{r+1})$.*

(1) *If $m \geq 3$ is odd, then $(M^{2m+1}, \phi, \xi, \eta, g)$ must be isometric to either a $(2m + 1)$ -dimensional spherical Sasakian space form or a Tanno deformation of 3-Sasakian manifold.*

(2) *If $m \geq 2$ is even, then $(M^{2m+1}, \phi, \xi, \eta, g)$ must be isometric to a $(2m+1)$ -dimensional spherical Sasakian space form.*

We close the section with a lemma which is useful in studying the limiting case of (3.5) and (3.13).

Lemma 2.1. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed Sasakian spin manifold with constant scalar curvature $S + 2m > 0$ and let $\psi := \psi_r + \psi_{r+1} \in E^\lambda(D_{ab})$, $r \in \{0, 1, \dots, m - 1\}$. Assume that ψ_r, ψ_{r+1} satisfy $D_0\psi_r = (-1)^{m+r}\nu\psi_r$ and $D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}$ with $\nu = -\frac{m-1-2r}{8m(m+1)}(S + 2m)$.*

(1) *If ψ_r is a left Sasakian twistor spinor, then ψ_{r+1} is a right Sasakian twistor spinor and hence (ψ_r, ψ_{r+1}) is a Sasakian duo with characteristic numbers (c_+, c_-) described in Proposition 2.4*

(2) *If ψ_{r+1} is a right Sasakian twistor spinor, then ψ_r is a left Sasakian twistor spinor and hence (ψ_r, ψ_{r+1}) is a Sasakian duo with characteristic numbers (c_+, c_-) described in Proposition 2.4.*

Proof. We prove part (1). The proof for part (2) is similar. Let $\psi = \psi_r + \psi_{r+1} \in E^\lambda(D_{ab})$ and let ψ_r be a left Sasakian twistor spinor. Denote

$$\theta^2 := \left[(-1)^{m+r}\lambda + \frac{1}{2} - b \right]^2 - \left[a\nu - \left(\frac{1}{2} - b \right) (m - 1 - 2r) \right]^2.$$

Then, by (2.1), we have

$$(2.15) \quad \theta^2 = \frac{r + 1}{2(2r + 1)}(S + 2m) + \frac{4(r + 1)(m - 2r)\nu}{2r + 1}.$$

Combining (2.15) with $\nu = -\frac{m-1-2r}{8m(m+1)}(S + 2m)$ leads immediately to

$$(2.16) \quad \theta^2 - \frac{m - r}{2(2m - 2r - 1)}(S + 2m) - \frac{4(m - r)(m - 2 - 2r)\nu}{2m - 2r - 1} = 0.$$

From (2.16) we now conclude that

$$\sum_{u=1}^{2m} \int_{M^{2m+1}} \left(\nabla_{E_u} \psi_{r+1} + \frac{1}{2(m - r)} P_+(E_u) \cdot D_- \psi_{r+1}, \right.$$

$$\bar{\nabla}_{E_u}\psi_{r+1} + \frac{1}{2(m-r)}P_+(E_u) \cdot D_-\psi_{r+1} \Big) \mu = 0,$$

where μ denotes the volume form, and so ψ_{r+1} is a right Sasakian twistor spinor. □

3. Eigenvalue estimates for the generalized Dirac operator D_{ab}

In this section we prove the main results (Theorems 3.1 and 3.2) of the paper. All the Sasakian spin manifolds $(M^{2m+1}, \phi, \xi, \eta, g)$ considered in the section are closed manifolds. Let $\text{Spec}(D_{ab})$, $a \neq 0$, denote the set of all eigenvalues of the generalized Dirac operator D_{ab} . Then, for each $\lambda \in \text{Spec}(D_{ab})$, at least one of the following two statements is true.

(1) There exists a non-trivial eigenspinor $\psi \in E^\lambda(D_{ab})$ such that $\psi = \psi_p \in \Gamma(\Sigma_p)$ is a section in Σ_p for some $p \in \{0, 1, \dots, m\}$.

(2) There exists an eigenspinor $\psi \in E^\lambda(D_{ab})$ such that $\psi = \psi_r + \psi_{r+1} \in \Gamma(\Sigma_r \oplus \Sigma_{r+1})$ is a section in $\Sigma_r \oplus \Sigma_{r+1}$ for some $r \in \{0, 1, \dots, m-1\}$ and all of $\psi_r, \psi_{r+1}, D_+\psi_r, D_-\psi_{r+1}$ are non-trivial.

An eigenspinor $\psi \in E^\lambda(D_{ab})$ described in part (1) (resp. part (2)) is called a *one-component* (resp. *two-component*) *eigenspinor* of D_{ab} . Let us fix the notations

$$\text{Spec}(D_{ab}, \text{I}) := \{ \lambda \in \text{Spec}(D_{ab}) \mid \text{There exists a one-component eigenspinor } \psi = \psi_p \in E^\lambda(D_{ab}) \},$$

$$\text{Spec}(D_{ab}, \text{II}) := \{ \lambda \in \text{Spec}(D_{ab}) \mid \text{There exists a two-component eigenspinor } \psi = \psi_r + \psi_{r+1} \in E^\lambda(D_{ab}) \}.$$

Then it holds that $\text{Spec}(D_{ab}) = \text{Spec}(D_{ab}, \text{I}) \cup \text{Spec}(D_{ab}, \text{II})$.

Recall that, for any $\lambda \in \text{Spec}(D_{ab}, \text{I})$, $a \neq 0$, there exists some $r \in \{0, 1, \dots, m\}$ such that the inequality

$$(3.1) \quad a(-1)^{m+r+1}(m-2r)\lambda \geq \frac{a^2}{8}(S_{\min} + 2m) + a\left(\frac{1}{2} - b\right)(m-2r)^2$$

holds. Equality in (3.1) occurs if and only if there exists a one-component eigenspinor $\psi = \psi_r \in E^\lambda(D_{ab})$ such that ψ is a ϕ -Killing spinor, i.e., ψ satisfies

$$\bar{\nabla}_V\psi = \nabla_V\psi - \frac{1}{2}\phi(V) \cdot \xi \cdot \psi = 0, \quad V \in \Gamma(\xi^\perp).$$

Denote K the operator

$$K := \sum_{u=1}^{2m} (\bar{\nabla}_{E_u} + tE_u \circ \bar{C} + (-1)^{m+r}t\phi(E_u) \circ \xi \circ \bar{C})^* \circ \sum_{u=1}^{2m} (\bar{\nabla}_{E_u} + tE_u \circ \bar{C} + (-1)^{m+r}t\phi(E_u) \circ \xi \circ \bar{C}),$$

where $t \in \mathbb{R}$ is a real number and $(\cdot)^*$ denotes the adjoint operator of (\cdot) with respect to the L^2 -Hermitian product (see (3.1)-(3.3) in [13]). Then, for any two-component eigenspinor $\psi = \psi_r + \psi_{r+1} \in E^\lambda(D_{ab})$, we have

$$\begin{aligned} & \frac{2r+2}{2r+1} K(\psi_r) \\ &= \overline{C}^2(\psi_r) - \frac{r+1}{2(2r+1)} (S+2m)\psi_r - (-1)^{m+r} \frac{4(r+1)(m-2r)}{2r+1} D_0\psi_r \end{aligned}$$

and

$$\begin{aligned} \frac{2m-2r}{2m-2r-1} K(\psi_{r+1}) &= \overline{C}^2(\psi_{r+1}) - \frac{m-r}{2(2m-2r-1)} (S+2m)\psi_r \\ &\quad - (-1)^{m+r+1} \frac{4(m-r)(m-2-2r)}{2m-2r-1} D_0\psi_{r+1}, \end{aligned}$$

which respectively yield (3.2) and (3.3) in the following proposition.

Proposition 3.1. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed Sasakian spin manifold. For any $\lambda \in \text{Spec}(D_{ab}, \text{II}) \setminus \text{Spec}(D_{ab}, \text{I})$, $a \neq 0$, there exists some real number $\nu \in \mathbb{R}$ and $r \in \{0, 1, \dots, m-1\}$ such that both inequalities*

$$\begin{aligned} & \left[(-1)^{m+r} \lambda + \frac{1}{2} - b \right]^2 - \left[a\nu - \left(\frac{1}{2} - b \right) (m-1-2r) + \frac{2(r+1)(m-2r)}{a(2r+1)} \right]^2 \\ (3.2) \quad & \geq \frac{r+1}{2(2r+1)} (S_{\min} + 2m) - \frac{4(r+1)^2(m-2r)^2}{a^2(2r+1)^2} \\ & \quad + \frac{2(1-2b)(r+1)}{a(2r+1)} (m-2r)(m-1-2r) \end{aligned}$$

and

$$\begin{aligned} & \left[(-1)^{m+r} \lambda + \frac{1}{2} - b \right]^2 - \left[a\nu - \left(\frac{1}{2} - b \right) (m-1-2r) + \frac{2(m-r)(m-2-2r)}{a(2m-2r-1)} \right]^2 \\ (3.3) \quad & \geq \frac{m-r}{2(2m-2r-1)} (S_{\min} + 2m) - \frac{4(m-r)^2(m-2-2r)^2}{a^2(2m-2r-1)^2} \\ & \quad + \frac{2(1-2b)(m-r)}{a(2m-2r-1)} (m-1-2r)(m-2-2r) \end{aligned}$$

hold.

We can now prove one of the principal results of the paper.

Theorem 3.1. *Let $(M^5, \phi, \xi, \eta, g)$ be a 5-dimensional closed Sasakian spin manifold. Assume that the parameters a, b of D_{ab} and the minimum S_{\min} of the scalar curvature satisfy*

$$0 < a \leq 1, \quad 0 \leq b \leq \frac{1}{2}$$

and

$$(3.4) \quad a^2(S_{\min} + 4) > \frac{1}{2} [8 - a(1 - 2b)]^2,$$

respectively. Then, for any eigenvalue $\lambda \in \text{Spec}(D_{ab})$, the inequality

$$(3.5) \quad |\lambda| \geq \min \left\{ \omega, \frac{a(S_{\min} + 4)}{16} + 1 - 2b \right\}$$

holds, where ω is equal to

$$\omega = \begin{cases} b - \frac{1}{2} + \sqrt{\frac{S_{\min} + 4}{3} + \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b \right]^2} \\ \text{for } a^2(S_{\min} + 4) \leq 24[8 - a(1 - 2b)], \\ b - \frac{1}{2} + \sqrt{\frac{S_{\min} + 4}{2} - \frac{16}{a^2} + \frac{4(1 - 2b)}{a}} \\ \text{for } a^2(S_{\min} + 4) \geq 24[8 - a(1 - 2b)]. \end{cases}$$

Equality $|\lambda| = b - \frac{1}{2} + \sqrt{\frac{S_{\min} + 4}{3} + \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b \right]^2}$ in (3.5) occurs if and only if there exists a Sasakian duo (ψ_0, ψ_1) with characteristic numbers

$$c_{\pm} = \mp \left[\frac{a(S + 4)}{48} + \frac{1}{2} - b \right] + \sqrt{\frac{S + 4}{3} + \left[\frac{a(S + 4)}{48} + \frac{1}{2} - b \right]^2}.$$

If this is the case and M^5 is simply-connected, then $(M^5, \phi, \xi, \eta, g)$ must be isometric to a 5-dimensional spherical Sasakian space form.

Proof. As pointed out in [13] (see Remark 5.1), there exists no one-component eigenspinor ψ of D_{ab} with $\psi = \psi_1 \in \Gamma(\Sigma_1)$. Let $\lambda \in \text{Spec}(D_{ab}, \text{I})$. Then, inserting $m = 2, r = 0$ (resp. $m = 2, r = 2$) into (3.1) yields

$$(3.6) \quad |\lambda| \geq \frac{a(S_{\min} + 4)}{16} + 1 - 2b.$$

On the other hand, for any $\lambda \in \text{Spec}(D_{ab}, \text{II})$, there exists a two-component eigenspinor $\psi = \psi_0 + \psi_1 \in E^\lambda(D_{ab})$ such that $D_0\psi_0 = \nu\psi_0$ and $D_0\psi_1 = -\nu\psi_1$ for some real number $\nu \in \mathbb{R}$. Let us first consider the case that $\nu + \frac{S_{\min} + 4}{48} \leq 0$. Inserting $m = 2, r = 0$ into (3.3) yields

$$\begin{aligned} \left(\lambda + \frac{1}{2} - b \right)^2 &\geq \frac{S_{\min} + 4}{3} + \left(a\nu - \frac{1}{2} + b \right)^2 \\ &\geq \frac{S_{\min} + 4}{3} + \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b \right]^2. \end{aligned}$$

Since the scalar curvature satisfies (3.4), it follows that

$$(3.7) \quad |\lambda| \geq b - \frac{1}{2} + \sqrt{\frac{S_{\min} + 4}{3} + \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b \right]^2}.$$

Consider the next case $\nu + \frac{S_{\min}+4}{48} \geq 0$. Inserting $m = 2, r = 0$ into (3.2) and noting that $a^2(S_{\min} + 4) \leq 24[8 - a(1 - 2b)]$ implies $\frac{a(S_{\min}+4)}{48} + \frac{1}{2} - b - \frac{4}{a} \leq 0$ and

$$\begin{aligned} & \left(a\nu - \frac{1}{2} + b + \frac{4}{a} \right)^2 \\ &= \left[a\nu + \frac{a(S_{\min} + 4)}{48} - \frac{a(S_{\min} + 4)}{48} - \frac{1}{2} + b + \frac{4}{a} \right]^2 \\ &= a^2 \left(\nu + \frac{S_{\min} + 4}{48} \right)^2 - 2a \left(\nu + \frac{S_{\min} + 4}{48} \right) \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b - \frac{4}{a} \right] \\ & \quad + \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b - \frac{4}{a} \right]^2, \end{aligned}$$

we obtain

$$\begin{aligned} \left(\lambda + \frac{1}{2} - b \right)^2 &\geq \frac{S_{\min} + 4}{2} - \frac{16}{a^2} + \frac{4(1 - 2b)}{a} + \left(a\nu - \frac{1}{2} + b + \frac{4}{a} \right)^2 \\ &\geq \frac{S_{\min} + 4}{2} - \frac{16}{a^2} + \frac{4(1 - 2b)}{a} + \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b - \frac{4}{a} \right]^2 \\ &\geq \frac{S_{\min} + 4}{3} + \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b \right]^2, \end{aligned}$$

which again yields (3.7). Since

$$\frac{S_{\min} + 4}{3} + \left[\frac{a(S_{\min} + 4)}{48} + \frac{1}{2} - b \right]^2 \geq \frac{S_{\min} + 4}{2} - \frac{16}{a^2} + \frac{4(1 - 2b)}{a}$$

holds under condition (3.4), inequality (3.5) is valid indeed. The statement for the limiting case is immediate from Lemma 2.1 and Theorem 2.3. \square

In the case where $(a = 1, b = 0)$ or $(a = 1, b = \frac{1}{2})$, we can explicitly describe (3.5) as follows.

Corollary 3.1. *Let $(M^5, \phi, \xi, \eta, g)$ be a 5-dimensional closed Sasakian spin manifold.*

(1) *If $a = 1$ and $b = 0$, then any eigenvalue λ of D_{ab} satisfies*

$$(3.8) \quad |\lambda| \geq \begin{cases} -\frac{1}{2} + \sqrt{\frac{S_{\min}+4}{3} + \frac{(S_{\min}+28)^2}{2304}} & \text{for } -4 < S_{\min} \leq 164, \\ -\frac{1}{2} + \sqrt{\frac{S_{\min}-20}{2}} & \text{for } S_{\min} \geq 164. \end{cases}$$

(2) *If $a = 1$ and $b = \frac{1}{2}$, then any eigenvalue λ of D_{ab} satisfies*

$$(3.9) \quad |\lambda| \geq \begin{cases} \frac{S_{\min}+4}{16} & \text{for } -4 < S_{\min} \leq 92, \\ \sqrt{\frac{S_{\min}+4}{3} + \frac{(S_{\min}+4)^2}{2304}} & \text{for } 92 \leq S_{\min} \leq 188, \\ \sqrt{\frac{S_{\min}-28}{2}} & \text{for } S_{\min} \geq 188. \end{cases}$$

Remark 3.1. Estimate (3.8) (resp. (3.9)) improves (5.19) (resp. (5.20)) in [13]. Moreover, (3.8) is better than the estimate (1.1). It is still an open question whether equality $|\lambda| = -\frac{1}{2} + \sqrt{\frac{S_{\min}-20}{2}}$ in (3.8) (resp. $|\lambda| = \sqrt{\frac{S_{\min}-20}{2}}$ in (3.9)) may hold.

Let $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 2$, be a closed eta-Einstein Sasakian spin manifold. Let $\psi = \psi_r + \psi_{r+1} \in E^\lambda(D_{ab})$, $r \in \{0, 1, \dots, m-1\}$, be a two-component eigenspinor such that

$$D_0\psi_r = (-1)^{m+r}\nu\psi_r, \quad D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}, \quad \nu \in \mathbb{R},$$

and

$$(3.10) \quad [4r+4+a(1-2b)(m-1-2r)][4m-4r-a(1-2b)(m-1-2r)] > 0$$

hold. Then we have [13]

$$(3.11) \quad \begin{aligned} & \left[(-1)^{m+r}\lambda + \frac{1}{2} - b \right]^2 - a^2\nu^2 \\ & \geq \frac{8(r+1)(m-r) + a(1-2b)(m-1-2r)^2}{8m(m+1)} \cdot (S+2m) \\ & \quad + \left(b - \frac{1}{2} \right)^2 (m-1-2r)^2. \end{aligned}$$

Using (3.11) we can now improve the estimate

$$|\lambda| \geq b - \frac{1}{2} + \sqrt{\frac{2m(m+2) + a(1-2b)}{8m(m+1)}(S+2m) + \left(b - \frac{1}{2} \right)^2}$$

for even $m \geq 4$ in (1.5) to (3.13) in the following theorem.

Theorem 3.2. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed eta-Einstein Sasakian spin manifold of dimension $2m+1 \geq 9$, $m \equiv 0 \pmod{2}$. Assume that the parameters a, b of D_{ab} and the scalar curvature S satisfy*

$$0 < a \leq 1, \quad 0 \leq b < \frac{1}{2}$$

and

$$(3.12) \quad 0 < S+2m \leq \frac{(1-2b)^2}{2-a(1-2b)} \cdot 2m(m+1),$$

respectively. Then, for any eigenvalue λ of D_{ab} , the inequality

$$(3.13) \quad |\lambda| \geq b - \frac{1}{2} + \sqrt{\frac{(m+2)(S+2m)}{4(m+1)} + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b \right]^2}$$

holds. The equality case in (3.13) occurs if and only if there exists a Sasakian duo $(\psi_{\frac{m}{2}-1}, \psi_{\frac{m}{2}})$ with characteristic numbers

$$(-1)^{\frac{m}{2}-1}c_{\pm}$$

$$= \mp \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b \right] + \sqrt{\frac{(m+2)(S+2m)}{4(m+1)} + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b \right]^2}.$$

If this is the case and M^{2m+1} is simply-connected, then $(M^{2m+1}, \phi, \xi, \eta, g)$ must be isometric to a $(2m+1)$ -dimensional spherical Sasakian space form.

Proof. Let $\psi = \psi_r + \psi_{r+1} \in E^\lambda(D_{ab})$, $r \in \{0, 1, \dots, \frac{m-2}{2}\}$, be a two-component eigenspinor satisfying $D_0\psi_r = (-1)^{m+r}\nu\psi_r$ and $D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}$ for some real number $\nu \in \mathbb{R}$. Then, inserting

$$\begin{aligned} & \frac{2r+2}{2r+1} K(\psi_r) \\ = & \bar{C}^2(\psi_r) - \frac{r+1}{2(2r+1)} (S+2m)\psi_r - (-1)^{m+r} \frac{4(r+1)(m-2r)}{2r+1} D_0\psi_r \end{aligned}$$

into

$$\frac{2r+2}{2r+1} \int_{M^{2m+1}} (K(\psi_r), \psi_r) \mu \geq 0$$

yields

$$\begin{aligned} & \left[(-1)^{m+r}\lambda + \frac{1}{2} - b \right]^2 \\ \geq & \frac{(r+1)(S+2m)}{2(2r+1)} - \frac{(r+1)(m-2r)(S+2m)}{2(2r+1)m(m+1)} \\ & + \left[\frac{a(S+2m)}{8m(m+1)} + \left(\frac{1}{2} - b \right) (m-1-2r) \right]^2 \\ & + \left[\nu + \frac{S+2m}{8m(m+1)} \right] \left[a^2\nu + \frac{a^2(S+2m)}{8m(m+1)} \right. \\ (3.14) \quad & \left. - \frac{a^2(S+2m)}{4m(m+1)} - a(1-2b)(m-1-2r) + \frac{4(r+1)(m-2r)}{2r+1} \right]. \end{aligned}$$

First consider the case that $\nu + \frac{S+2m}{8m(m+1)} \geq 0$. Note that the function

$$h_1(r) := -\frac{a^2(S+2m)}{4m(m+1)} - a(1-2b)(m-1-2r) + \frac{4(r+1)(m-2r)}{2r+1}$$

at the last line of (3.14) is decreasing on the interval $[0, \frac{m-2}{2}]$. Because of (3.12), we have $h_1(\frac{m-2}{2}) \geq 0$ and therefore $h_1(r) \geq 0$ for all $r \in \{0, 1, \dots, \frac{m-2}{2}\}$. Consequently, (3.14) becomes

$$\begin{aligned} & \left[(-1)^{m+r}\lambda + \frac{1}{2} - b \right]^2 \\ \geq & \frac{(r+1)(m^2+2r)}{2(2r+1)} \cdot \frac{S+2m}{m(m+1)} \\ (3.15) \quad & + \left[\frac{a(S+2m)}{8m(m+1)} + \left(\frac{1}{2} - b \right) (m-1-2r) \right]^2. \end{aligned}$$

Since the function

$$h_2(r) := \frac{(r+1)(m^2+2r)}{2(2r+1)} \cdot \frac{S+2m}{m(m+1)}$$

at the right side of (3.15) is decreasing on the interval $[0, \frac{m-2}{2}]$, inserting $r = \frac{m-2}{2}$ into (3.15) yields (3.13). Consider the next case that $\nu + \frac{S+2m}{8m(m+1)} \leq 0$. Observe that, if the scalar curvature satisfies (3.12), then the function

$$\begin{aligned} \hat{\chi}(r) := & \frac{8(r+1)(m-r) + a(1-2b)(m-1-2r)^2}{8m(m+1)} \cdot (S+2m) \\ & + \left(b - \frac{1}{2}\right)^2 (m-1-2r)^2 \end{aligned}$$

at the right side of (3.11) is decreasing on the interval $[0, \frac{m-2}{2}]$. Since

$$\nu^2 \geq \left[\frac{S+2m}{8m(m+1)} \right]^2,$$

(3.11) now becomes

$$(3.16) \quad \left[(-1)^{m+r}\lambda + \frac{1}{2} - b \right]^2 \geq \hat{\chi} \left(\frac{m-2}{2} \right) + \left[\frac{a(S+2m)}{8m(m+1)} \right]^2,$$

from which (3.13) is immediate. On the other hand, use Proposition 6.1 in [13] together with (3.1) to find that any $\lambda \in \text{Spec}(D_{ab}, I)$ satisfies

$$|\lambda| \geq \frac{a(S+2m)}{8m} + \frac{m(1-2b)}{2}.$$

A simple computation however shows that, if the scalar curvature satisfies (3.12), then

$$\begin{aligned} & \frac{a(S+2m)}{8m} + \frac{m(1-2b)}{2} \\ \geq & b - \frac{1}{2} + \sqrt{\frac{(m+2)(S+2m)}{4(m+1)} + \left[\frac{a(S+2m)}{8m(m+1)} + \frac{1}{2} - b \right]^2}, \end{aligned}$$

which establishes (3.13). The statement for the limiting case is immediate from Lemma 2.1 and Theorem 2.3. □

Inserting $(a = 1, b = 0)$ into (3.12)-(3.13) we obtain (3.17) in the following corollary. (3.17) improves estimate (6.15) for even $m \geq 4$ in [13].

Corollary 3.2. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed eta-Einstein Sasakian spin manifold of dimension $2m+1 \geq 9$, $m \equiv 0 \pmod{2}$. Suppose that the parameters a, b of D_{ab} are chosen to be $(a = 1, b = 0)$ and the scalar curvature S satisfies*

$$0 < S + 2m \leq 2m(m + 1).$$

Then, for any eigenvalue λ of D_{ab} , the inequality

$$(3.17) \quad |\lambda| \geq -\frac{1}{2} + \sqrt{\frac{(m+2)(S+2m)}{4(m+1)} + \left[\frac{S+2m}{8m(m+1)} + \frac{1}{2} \right]^2}$$

holds.

Acknowledgements. This work was supported by a grant from 2014 Research Fund of Andong National University.

References

- [1] I. Agricola, *The Srni lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) **42** (2006), 5–84.
- [2] I. Agricola, J. Becker-Bender, and H. Kim, *Twistorial eigenvalue estimates for generalized Dirac operators with torsion*, Adv. Math. **243** (2013), 296–329.
- [3] I. Agricola, Th. Friedrich, and M. Kassuba, *Eigenvalue estimates for Dirac operators with parallel characteristic torsion*, Differential Geom. Appl. **26** (2008), no. 6, 613–624.
- [4] C. Bär, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154** (1993), no. 3, 509–521.
- [5] H. Baum, Th. Friedrich, R. Grunewald, and I. Kath, *Twistors and Killing Spinors on Riemannian Manifolds*, Teubner, Leipzig/Stuttgart, 1991.
- [6] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nachr. **97** (1980), 117–146.
- [7] ———, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics, Vol. 25, AMS, 2000.
- [8] Th. Friedrich and E. C. Kim, *The Einstein-Dirac equation on Riemannian spin manifolds*, J. Geom. Phys. **33** (2000), no. 1-2, 128–172.
- [9] ———, *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, J. Geom. Phys. **37** (2001), no. 1-2, 114.
- [10] ———, *Eigenvalues estimates for the Dirac operator in terms of Codazzi tensors*, Bull. Korean Math. Soc. **45** (2008), no. 2, 365–373.
- [11] N. Ginoux, *The Dirac spectrum*, Lecture Notes in Mathematics, 1976, Springer-Verlag, Berlin/Heidelberg, 2009.
- [12] E. C. Kim, *Dirac eigenvalues estimates in terms of divergencefree symmetric tensors*, Bull. Korean Math. Soc. **46** (2009), no. 5, 949–966.
- [13] ———, *Eigenvalue estimates for generalized Dirac operators on Sasakian manifolds*, Ann. Global Anal. Geom. **45** (2014), no. 1, 67–93.
- [14] K.-D. Kirchberg, *The first eigenvalue of the Dirac operator on Kähler manifolds*, J. Geom. Phys. **7** (1990), no. 4, 449–468.
- [15] K.-D. Kirchberg and U. Semmelmann, *Complex contact structures and the first eigenvalue of the Dirac operator on Kähler manifolds*, Geom. Funct. Anal. **5** (1995), no. 3, 604–618.
- [16] W. Kramer, U. Semmelmann, and G. Weingart, *Quaternionic Killing spinors*, Ann. Global Anal. Geom. **16** (1998), no. 1, 63–87.
- [17] ———, *Eigenvalue estimates for the Dirac operator on quaternionic Kähler manifolds*, Math. Z. **230** (1999), no. 4, 727–751.
- [18] A. Moroianu, *Kähler manifolds with small eigenvalues of the Dirac operator and a conjecture of Lichnerowicz*, Ann. Inst. Fourier **49** (1999), no. 5, 1637–1659.
- [19] Y. Shibuya, *The spectrum of Sasakian manifolds*, Kodai Math. J. **3** (1980), no. 2, 197–211.
- [20] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. **12** (1968), 700–717.

- [21] K. Yano and M. Kon, *Structure on Manifolds*, World Scientific, Singapore, 1984.

EUI CHUL KIM
DEPARTMENT OF MATHEMATICS
COLLEGE OF EDUCATION
ANDONG NATIONAL UNIVERSITY
ANDONG 36729, KOREA
E-mail address: `eckim@andong.ac.kr`