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Evaluating and improving system reliability of bridge structure using gamma distribution

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Abstract: In this paper, we study a system of five components. One of them is a bridge network component. Each of these components is identical and has a failure rate as a function of time. The system components have non-constant failure rates. The given system is improved by using the reduction, hot duplication, and cold duplication methods. We derive the equivalence factors of the bridge structure system to be as another system improved according to these different methods. The β- fractiles are obtained to compare the original system with these improved systems. Finally, we present numerical results to show the difference between these methods.

Key Words: *Bridge structure, improving methods, reliability engineering, reliability equivalence*

1. INTRODUCTION

In our life, there are many systems of a bridge network structure and then we were in need of discussing these types of systems and improving their performance. Many papers studied reliability and equivalence factor for simple and complex systems with constant failure rate and changeable failure rate as a function in time.
Råde (1990, 1993), Sarhan and Mustafa (2006), Mustafa and El-Bassoiuny (2009) and Mustafa and El-Faheem (2014) improved various systems by applying such concept. Mustafa et al. (2009) studied series system such that each component has mixing constant failure rates. Mustafa and El-Faheem (2011) improved the performance of a system has m

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delayed lifetimes distributions with mixed constant failures by applying reliability equivalence technique.

All articles mentioned above introduced various systems that system components have constant failure rates. Xia and Zhang (2007) improved a parallel system consists of n components each component has Gamma lifetime distribution. Mustafa (2009) improved the performance for a series system with non-constant failure rates. Mustafa and El-Bassiouny (2009) studied the system with non-constant failure rates. They introduced two cases (i) two stages lifetime distributions for each component with increasing failure rates, (ii) two stages failure rates for each system components. Ezzati and Rasouli (2015) studied the Radar system with linear-exponential distribution function.

Also, Sarhan (2004) studied a system of a bridge network system of five components with constant failure rate.

In this paper, we study the bridge network system with failure rate as a function of time such that each component is distributed as Gamma distribution and improve it according to three different methods:

(1) Reduction method.
(2) Hot duplication method.
(3) Cold duplication method.

If $\alpha = 1,$ Gamma distribution reduced to exponential with $1/\lambda$ and Sarhan (2004) is a special case from our article.

The reliability function and mean time to failure for the bridge structure system are calculated in Section 2. In Section 3, the original system improved according to reduction, hot and cold methods. The reliability equivalence factor are obtained in Section 4, also β- fractiles for the original and improved systems are calculated in Section 5. Finally, numerical results are presented in Section 6.

2. THE BRIDGE SYSTEM

The original system consists of five components connected in series and parallel as in Figure 1. These components are assumed to be independent and identical, have the lifetime times gamma distribution, with parameters α, λ, that is $T_i \sim$ Gamma (α, λ).

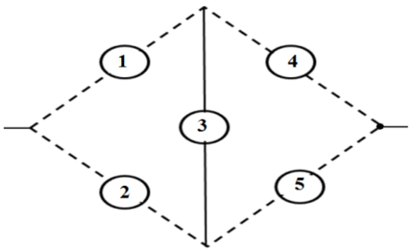

Figure 1. The bridge structure diagram

The reliability function of the component i denoted by $R_i(t)$, $i = 1, 2, \cdots, 5$, can be obtained as follows

$$R_i(t) = P[T_i > t] = \int_t^{\infty} \frac{x^{\alpha-1}}{\lambda^{\alpha}\Gamma(\alpha)} e^{-\frac{x}{\lambda}} dx = 1 - \varphi(\alpha, \lambda t), \tag{1}$$

where

$$\varphi(\alpha, \lambda t) = \int_0^{\lambda t} \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u} du,$$

and assume that N= {1, 2, 3, 4, 5} be the set of all components and N_i is the set of all components except component i, then the reliability function of the system can be obtained by using the minimal paths techniques, see Figure 2.

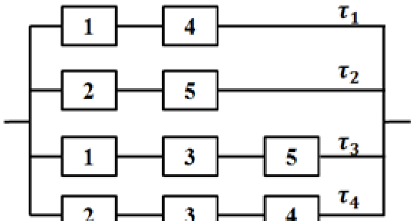

Figure 2. Minimal paths for bridge structure

Where τ_i is the minimal tie set, $i = 1, 2, 3, 4$. Let R(t) denote to the system reliability function that can be given by

$$\begin{aligned} R(t) &= \sum_{i=1}^{4} P(\tau_i) - \sum_{i=1}^{3}\sum_{j=i+1}^{4} P(\tau_i \cap \tau_j) + \sum_{i=1}^{2}\sum_{j=i+1}^{3}\sum_{k=j+1}^{4} P(\tau_i \cap \tau_j \cap \tau_k) - P\left(\bigcap_{i=1}^{4} \tau_i\right) \\ &= R_1(t)R_4(t) + R_2(t)R_5(t) + R_1(t)R_3(t)R_5(t) + R_2(t)R_3(t)R_4(t) \\ &\quad - \sum_{i=1}^{5}\prod_{j \in N_i} R_i(t) + 2\prod_{j \in N} R_j(t). \end{aligned} \tag{2}$$

Substituting from Eq. (1) into Eq. (2), we have

$$R(t) = 1 - \varphi^2(\alpha, \lambda t)[2 + 2\varphi(\alpha, \lambda t) - 5\varphi^2(\alpha, \lambda t) + 2\varphi^3(\alpha, \lambda t)]. \tag{3}$$

The mean time to failure (MTTF) can be obtained as follows.

$$MTTF = \int_0^{\infty} R(t)\, dt \tag{4}$$

The MTTF can be calculated numerically by using some Numerical Programs.

3. IMPROVED SYSTEMS

We improve the performance of the studied system by improving some of its components according to improved methods as follows.

3.1 Reduction method

We assume that in this method, the system can be improved by reducing the failure rates of some of its components by a factor $\rho, 0 < \rho < 1$. Let A be a set of components that are improved according to reduction method. Let $R_{A,\rho}(t)$ be the reliability function of the improved system by reducing the failure rates of the components belong to set A. $R_{A,\rho}(t)$ can obtained as follows.

1. $A \in S_1 = \{\{3\}\}$:

$$R_{A,\rho}(t) = 1 - \varphi^2(\alpha,\lambda t)\{2 + 2\varphi(\alpha,\rho\lambda t)[1 - 2\varphi(\alpha,\lambda t)] - \varphi^2(\alpha,\lambda t)[1 - 2\varphi(\alpha,\rho\lambda t)]\}, \quad (5)$$

2. $A \in S_2 = \{\{1\},\{2\},\{4\},\{5\}\}$:

$$R_{A,\rho}(t) = 1 - \varphi(\alpha,\lambda t)\{\varphi(\alpha,\lambda t)[1 + \varphi(\alpha,\lambda t) - \varphi^2(\alpha,\lambda t)] + \varphi(\alpha,\rho\lambda t)[1 + \varphi(\alpha,\lambda t) - 4\varphi^2(\alpha,\lambda t) + 2\varphi^3(\alpha,\lambda t)]\}, \quad (6)$$

3. $A \in S_3 = \{\{1,3\},\{2,3\},\{3,4\},\{3,5\}\}$:

$$R_{A,\rho}(t) = 1 - \varphi(\alpha,\lambda t)\{\varphi(\alpha,\lambda t) + \varphi(\alpha,\rho\lambda t)[1 + \varphi(\alpha,\lambda t) - 2\varphi^2(\alpha,\lambda t)] + \varphi^2(\alpha,\rho\lambda t)[1 - 3\varphi(\alpha,\lambda t) + 2\varphi^2(\alpha,\lambda t)]\}, \quad (7)$$

4. $A \in S_4 = \{\{1,5\},\{2,4\}\}$:

$$R_{A,\rho}(t) = 1 - \varphi(\alpha,\lambda t)\{\varphi^2(\alpha,\lambda t) + 2\varphi(\alpha,\rho\lambda t)[1 + \varphi^2(\alpha,\lambda t)] + \varphi^2(\alpha,\rho\lambda t)[1 - 3\varphi(\alpha,\lambda t) + 2\varphi^2(\alpha,\lambda t)]\}, \quad (8)$$

5. $A \in S_5 = \{\{1,2\},\{4,5\}\}$:

$$R_{A,\rho}(t) = 1 - \varphi^2(\alpha,\lambda t) - 2\varphi(\alpha,\rho\lambda t)\varphi^2(\alpha,\lambda t)[1 - \varphi(\alpha,\lambda t)] - \varphi^2(\alpha,\rho\lambda t)[1 - 3\varphi^2(\alpha,\lambda t) + 2\varphi^3(\alpha,\lambda t)], \quad (9)$$

6. $A \in S_6 = \{\{1,4\},\{2,5\}\}$:

$$R_{A,\rho}(t) = 1 - 2\varphi(\alpha,\rho\lambda t)\varphi(\alpha,\lambda t)[1 + \varphi(\alpha,\lambda t) - \varphi^2(\alpha,\lambda t)] + \varphi^2(\alpha,\rho\lambda t)\varphi^2(\alpha,\lambda t)[3 - 2\varphi(\alpha,\lambda t)]. \quad (10)$$

The mean time to failure can be obtained as follows.

$$MTTF_A = \int_0^\infty R_A(t)\ dt. \quad (11)$$

3.2 Hot duplication method

Let $R_i^H(t)$ denote the reliability function of component i when it is improved by hot duplication method, as shown in Figure 3.

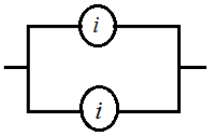

Figure 3. Hot duplication of component i

The reliability function of the duplicated component is given by

$$R_i^H(t) = 1 - \varphi(\alpha, \lambda t)^2. \quad (12)$$

Let $R_B^H(t)$ be the system reliability function when the components belong to the set B are improved by hot duplication method. $R_B^H(t)$ can be obtained as follows:

1. $B \in S_1 = \{\{3\}\}$:

$$R_B^H(t) = 1 - \varphi^2(\alpha, \lambda t)\{2 + \varphi^2(\alpha, \lambda t) - 4\varphi^3(\alpha, \lambda t) + 2\varphi^4(\alpha, \lambda t)\}, \quad (13)$$

2. $B \in S_2 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$:

$$R_B^H(t) = 1 - \varphi^2(\alpha, \lambda t)\{1 + 2\varphi(\alpha, \lambda t) - 4\varphi^3(\alpha, \lambda t) + 2\varphi^4(\alpha, \lambda t)\}, \quad (14)$$

3. $B \in S_3 = \{\{1,3\}, \{2,3\}, \{3,4\}, \{3,5\}\}$:

$$R_B^H(t) = 1 - \varphi^2(\alpha, \lambda t)\{1 + \varphi(\alpha, \lambda t) + \varphi^2(\alpha, \lambda t) - \varphi^3(\alpha, \lambda t) - 3\varphi^4(\alpha, \lambda t) + 2\varphi^5(\alpha, \lambda t)\} \quad (15)$$

4. $B \in S_4 = \{\{1,5\}, \{2,4\}\}$:

$$R_B^H(t) = 1 - \varphi^3(\alpha, \lambda t)\{3 - \varphi^2(\alpha, \lambda t) - 3\varphi^3(\alpha, \lambda t) + 2\varphi^4(\alpha, \lambda t)\}, \quad (16)$$

5. $B \in S_5 = \{\{1,2\}, \{4,5\}\}$:

$$R_B^H(t) = 1 - \varphi^2(\alpha, \lambda t)\{1 + 3\varphi^2(\alpha, \lambda t) - 2\varphi^3(\alpha, \lambda t) - 3\varphi^4(\alpha, \lambda t) + 2\varphi^5(\alpha, \lambda t)\}, \quad (17)$$

6. $B \in S_6 = \{\{1,4\}, \{2,5\}\}$:

$$R_B^H(t) = 1 - \varphi^3(\alpha, \lambda t)\{2 + 2\varphi(\alpha, \lambda t) - 2\varphi^2(\alpha, \lambda t) - 3\varphi^3(\alpha, \lambda t) + 2\varphi^4(\alpha, \lambda t)\}. \quad (18)$$

The mean time to failure can be obtained as follows.

$$MTTF_B = \int_0^\infty R_B^H(t)\ dt. \quad (19)$$

3.3 Cold duplication method

This method assumed that we improve some components according to cold duplication method. The reliability function of component i when it is improved according to cold duplication method denoted by $R_i^C(t)$, see Figure 4.

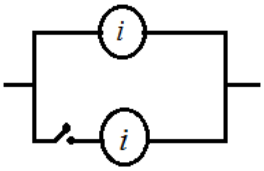

Figure 4. Cold duplication of component i

The function $R_i^C(t)$ can be given as

$$R_i^C(t) = 1 - \varphi(2\alpha, \lambda t). \tag{20}$$

Let $R_B^C(t)$ be the reliability function of the improved system. That is obtained by cold duplication of the set B of the system components. That can be obtained as follows:

1. $B \in S_1 = \{\{3\}\}$:

$$R_B^C(t) = 1 - \varphi^2(\alpha, \lambda t)\{2 + 2\varphi(2\alpha, \lambda t)[1 - \varphi(\alpha, \lambda t)]^2 - \varphi^2(\alpha, \lambda t)\}, \tag{21}$$

2. $B \in S_2 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$:

$$R_B^C(t) = 1 - \varphi(\alpha, \lambda t)\{\varphi(\alpha, \lambda t)[1 + \varphi(\alpha, \lambda t) - \varphi^2(\alpha, \lambda t)] + \varphi(2\alpha, \lambda t)[1 + \varphi(\alpha, \lambda t) - 4\varphi^2(\alpha, \lambda t) + 2\varphi^3(\alpha, \lambda t)]\}, \tag{22}$$

3. $B \in S_3 = \{\{1,3\}, \{2,3\}, \{3,4\}, \{3,5\}\}$:

$$R_B^C(t) = 1 - \varphi(\alpha, \lambda t)\{\varphi(\alpha, \lambda t) + \varphi(2\alpha, \lambda t)[1 + \varphi(\alpha, \lambda t) - 2\varphi^2(\alpha, \lambda t)] + \varphi^2(\alpha, \lambda t)[1 - 3\varphi(\alpha, \lambda t) + 2\varphi^2(\alpha, \lambda t)]\}, \tag{23}$$

4. $B \in S_4 = \{\{1,5\}, \{2,4\}\}$:

$$R_B^C(t) = 1 - \varphi(\alpha, \lambda t)\{\varphi^2(\alpha, \lambda t) + 2\varphi(2\alpha, \lambda t)[1 - \varphi(\alpha, \lambda t)^2] + +\varphi^2(2\alpha, \lambda t)[1 - 3\varphi(\alpha, \lambda t) + 2\varphi^2(\alpha, \lambda t)]\}, \tag{24}$$

5. $B \in S_5 = \{\{1,2\}, \{4,5\}\}$:

$$R_B^C(t) = 1 - \varphi^2(\alpha, \lambda t) - 2\varphi^2(\alpha, \lambda t)\varphi(2\alpha, \lambda t)[1 - \varphi(\alpha, \lambda t)] - \varphi^2(2\alpha, \lambda t)[1 - 3\varphi^2(\alpha, \lambda t) + 2\varphi^3(\alpha, \lambda t)], \tag{25}$$

6. $B \in S_6 = \{\{1,4\}, \{2,5\}\}$:

$$R_B^C(t) = 1 - 2\varphi(\alpha, \lambda t)\varphi(2\alpha, \lambda t)\{1 + \varphi(\alpha, \lambda t) - \varphi^2(\alpha, \lambda t)\} + \varphi^2(\alpha, \lambda t)\varphi^2(2\alpha, \lambda t)\{3 - 2\varphi(\alpha, \lambda t)\}. \tag{26}$$

We can obtain the mean time to failure for the improved as follows.

$$MTTF_A = \int_0^\infty R_B^C(t)\ dt. \tag{27}$$

4. RELIABILITY EQUIVALENCE FACTOR

The reliability equivalence factor can be computed by equating the equations of the system improved according to the reduction method with equations by hot and cold improving methods. So the reliability equivalence factor is defined as the factor by which the failure rates of some of the system's components should be reduced in order to reach equality of the reliability of another better system.

Since the failure rate of Gamma distribution is non-constant. The failure rate of Gamma distribution,

$$\lambda(t) = \frac{1}{\int_0^\infty \left(1+\frac{u}{t}\right)^{\alpha-1} e^{-\lambda u} du}.$$

This implies that, the reliability equivalence factors of Gamma distribution is a function about time t. For convenience of calculation, while failure rate is reduced by factor $r(t)$, we consider the scale parameter of Gamma distribution reduced from λ to $\rho\lambda$, only. From the failure rate of Gamma distribution, we know

$$r(t)\lambda(t) = \frac{1}{\int_0^\infty \left(1+\frac{u}{t}\right)^{\alpha-1} e^{-\rho\lambda u} du}. \quad (28)$$

Obviously, $r(t)$ will increase as ρ increases, and they fall in interval (0, 1) also.

In what follows, we will present how to calculate ρ only, and we obtain $r(t)$ by taking ρ in Eq. (28). Next, we present some of reliability equivalence factors of the improved bridge structure system studied here.

The hot (cold) reliability equivalence factor, $\rho^{D}_{A,B}(\beta)$, D = H (C), is the factor by which the failure rates of the set A components should be reduced to improve the system reliability to be as the system reliability improved by hot (cold) duplication of the components belong to the set B. Then, $\rho^{D}_{A,B}(\beta)$ is the solution of the following system of two non-linear equations:

$$R_B^D(t) = \beta, \quad R_{A,\rho}(t) = \beta, \quad D = H\ (C). \quad (29)$$

Thus, the hot (cold) reliability equivalence factor can be obtained by solving these system of equations when $A, B \in S_i$, $i = 1, \dots, 6$.

1. When $A \in S_1$: The factor $\rho^{H}_{A,B}(\beta)$ can be calculated by solve the following system of equations with respect to ρ:

$$\left.\begin{aligned} 1-\varphi^2(\alpha,\lambda t)\{2+2\varphi(\alpha,\rho\lambda t)[1-2\varphi(\alpha,\lambda t)]-\varphi^2(\alpha,\lambda t)[1-\varphi(\alpha,\rho\lambda t)]\} &= \beta, \\ R_B^D(t) &= \beta. \end{aligned}\right\} \quad (30)$$

2. When $A \in S_2$: $\rho^{H}_{A,B}(\beta)$ can be obtained by solve the following system of equations with respect to ρ:

$$\left.\begin{array}{c} 1-\varphi(\alpha,\lambda t)\{\varphi(\alpha,\lambda t)[1+\varphi(\alpha,\lambda t)-\varphi^2(\alpha,\lambda t)]+ \\ \varphi(\alpha,\rho\lambda t)[1+\varphi(\alpha,\lambda t)-4\varphi^2(\alpha,\lambda t)+2\varphi^3(\alpha,\lambda t)]\}=\beta, \\ \\ R_B^D(t)=\beta. \end{array}\right\} \quad (31)$$

3. When $A \in S_3$: $\rho_{A,B}^H(\beta)$ can be calculated as a solution of the following system of equations:

$$\left.\begin{array}{c} 1-\varphi(\alpha,\lambda t)\{\varphi(\alpha,\lambda t)+\varphi(\alpha,\rho\lambda t)[1+\varphi(\alpha,\lambda t)-2\varphi^2(\alpha,\lambda t)]+ \\ \varphi^2(\alpha,\rho\lambda t)[1-3\varphi(\alpha,\lambda t)+2\varphi^2(\alpha,\lambda t)]\}=\beta, \\ \\ R_B^D(t)=\beta. \end{array}\right\} \quad (32)$$

4. When $A \in S_4$: $\rho_{A,B}^H(\beta)$ is the solution of the system of nonlinear equations:

$$\left.\begin{array}{c} 1-\varphi(\alpha,\lambda t)\{\varphi^2(\alpha,\lambda t)+2\varphi(\alpha,\rho\lambda t)[1+\varphi^2(\alpha,\lambda t)]+ \\ \varphi^2(\alpha,\rho\lambda t)[1-3\varphi(\alpha,\lambda t)+2\varphi^2(\alpha,\lambda t)]\}=\beta, \\ \\ R_B^D(t)=\beta. \end{array}\right\} \quad (33)$$

5. When $A \in S_5$: $\rho_{A,B}^H(\beta)$ when we solve the following nonlinear equations with respect to ρ:

$$\left.\begin{array}{c} 1-\varphi^2(\alpha,\lambda t)-2\varphi(\alpha,\rho\lambda t)\varphi^2(\alpha,\lambda t)[1-\varphi(\alpha,\lambda t)]- \\ \varphi^2(\alpha,\rho\lambda t)[1-3\varphi^2(\alpha,\lambda t)+2\varphi^3(\alpha,\lambda t)]=\beta, \\ \\ R_B^D(t)=\beta. \end{array}\right\} \quad (34)$$

6. $\rho_{A,B}^H(\beta)$ can be obtained when $A \in S_6$ by solving the following system of equations:

$$\left.\begin{array}{c} 1-2\varphi(\alpha,\rho\lambda t)\varphi(\alpha,\lambda t)[1+\varphi(\alpha,\lambda t)-\varphi^2(\alpha,\lambda t)]+ \\ \varphi^2(\alpha,\rho\lambda t)\varphi^2(\alpha,\lambda t)[3-2\varphi(\alpha,\lambda t)]=\beta, \\ \\ R_B^D(t)=\beta. \end{array}\right\} \quad (35)$$

where $R_B^D(t)$ be the reliability function of the improved system by hot (cold) duplication method for different set of system components see, Eqs. (13) – (18) ((21) – (26)).
Given $\beta, A, B, \alpha, \lambda$, we can obtain the $\rho=\rho_{A,B}^D(\beta)$ by solve the non-linear systems of equation (30) – (35) using some Numerical technique. The reliability equivalence factor

$$r_{A,B}^D(\beta,t)=\frac{\int_0^\infty\left(1+\frac{u}{t}\right)^{\alpha-1}e^{-\lambda u}du}{\int_0^\infty\left(1+\frac{u}{t}\right)^{\alpha-1}e^{-\rho\lambda u}du}, \quad (36)$$

where $\rho=\rho_{A,B}^D(\beta)$.

5. β −FRACTILES

Let $L(\alpha,\beta)$ be the β-fractile of the original system in distribution of $\Gamma(\alpha,1)$, so $\int_{L(\alpha,\beta)}^{\infty}\frac{u^{\alpha-1}}{\Gamma(\alpha)}e^{-u}du=\beta$. $L_B^D(\alpha,\beta)$ denote to the β-fractile of the improved system obtained by improving the set of components B according to duplication methods.
The fractile $L(\alpha,\beta)$ can be found by solving the following equation with respect to L:

$$R\left(\frac{L}{\lambda}\right)=\beta, \tag{37}$$

substituting from Eq. (3) into Eq. (37), we have

$$\varphi^2(\alpha,L)[2+2\varphi(\alpha,L)-5\varphi^2(\alpha,L)+2\varphi^3(\alpha,L)]=1-\beta. \tag{38}$$

We can obtain the fractile $L_B^D(\alpha,\beta)$, by solve Eq. (38) with respect to L,

$$R_B^D\left(\frac{L}{\lambda}\right)=\beta, \tag{39}$$

substituting from Eqs. (13) – (18) into Eq. (39), we can obtain the hot β −fractiles for different set of components. Also substituting from Eqs. (21) – (26) into Eq. (39), we obtain the β −fractiles for different set of cold duplications.
The above equations obtained from Eq. (39) have no closed form solution in L. Thus, we have to use the numerical technique method to find out L.

6. NUMERICAL RESULTS

In this section, we introduce a numerical example to explain the previous theoretical results. In such example, we calculate the reliability equivalence factors (REF) of a bridge structure system under the following assumptions:

(1) The scale parameter $\lambda=0.5$.
(2) The set B of system components are improved according to one of the previous duplication methods, where $B\in S_i, i=1,2,\cdots,6$ to improve the system reliability.
(3) In the reduction method, we reduce the failure rates of the set $A\in S_i,\ i=1,2,\cdots,6$, by the same factor ρ.

Table 1. The MTTF, $MTTF_B^D, D=H,C,B\in S_i$

α	MTTF	Hot						Cold					
		S_1	S_2	S_3	S_4	S_5	S_6	S_1	S_2	S_3	S_4	S_5	S_6
2	3.613	3.717	3.975	4.084	4.342	4.230	4.488	3.812	4.346	4.611	5.145	4.787	6.099
3	5.607	5.739	6.063	6.199	6.525	6.379	6.704	5.887	6.646	7.045	7.804	7.233	9.542
4	7.604	7.759	8.139	8.298	8.678	8.506	8.886	7.954	8.914	9.442	10.402	9.625	13.061
5	9.603	9.777	10.205	10.384	10.813	10.616	11.045	10.013	11.154	11.804	12.945	11.975	16.638

For this example, Table 1 contains the MTTF and $MTTF_B^D$ for the original and improved systems.

The β-fractiles $L(\alpha, \beta)$, $L_B^D(\alpha, \beta)$ and $\rho_{A,B}^D(\beta)$, $D = H, C$ are calculated using some numerical techniques according to the previous theoretical formulae. In such calculations the level β is chosen to be 0.1, 0.2, …, 0.9. Table 2 represents the β-fractiles, of the original and improved systems, $L(\alpha, \beta)$ and $L_B^D(\alpha, \beta)$, when $\alpha = 2$.

Table 2. The $L(\alpha, \beta)$ and $L_B^D(\alpha, \beta), D = H, C, B \in S_i$

β	L	Hot						Cold					
		S_1	S_2	S_3	S_4	S_5	S_6	S_1	S_2	S_3	S_4	S_5	S_6
0.1	2.914	2.973	3.133	3.187	3.314	3.295	3.409	3.069	3.488	3.663	3.937	3.834	4.694
0.2	2.433	2.498	2.643	2.705	2.826	2.799	2.912	2.579	2.920	3.100	3.361	3.233	3.992
0.3	2.125	2.191	2.327	2.393	2.511	2.478	2.591	2.258	2.556	2.730	2.986	2.839	3.535
0.4	1.885	1.949	2.079	2.146	2.264	2.224	2.339	2.005	2.271	2.436	2.690	2.527	3.178
0.5	1.678	1.740	1.865	1.930	2.050	2.003	2.121	1.785	2.026	2.177	2.433	2.254	2.870
0.6	1.488	1.545	1.666	1.728	1.851	1.795	1.919	1.579	1.798	1.934	2.194	1.997	2.586
0.7	1.300	1.351	1.468	1.526	1.653	1.586	1.717	1.376	1.574	1.691	1.955	1.741	2.306
0.8	1.101	1.143	1.256	1.306	1.441	1.359	1.501	1.159	1.335	1.428	1.699	1.464	2.008
0.9	0.858	0.888	0.993	1.031	1.177	1.071	1.232	0.895	1.042	1.103	1.382	1.123	1.642

Figures 5–7 show the comparing the reliability for the original and improved systems, when $B \in S_i, i = 1, 2, \cdots, 6$ and $\alpha = 2$.

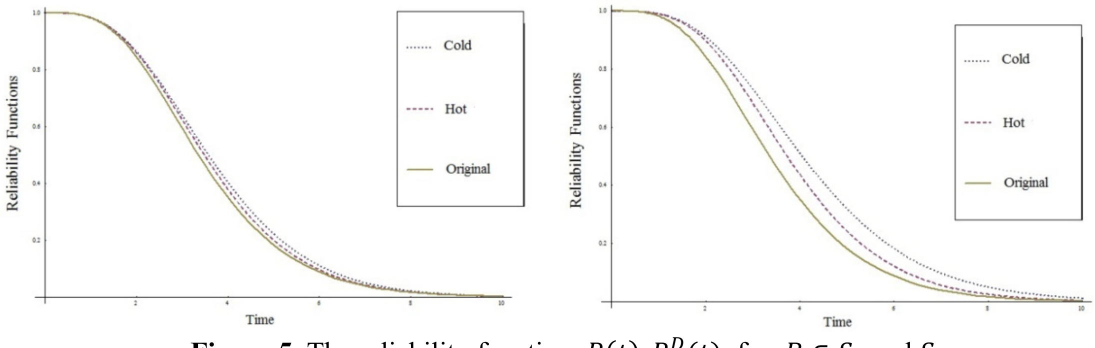

Figure 5. The reliability function, $R(t), R_B^D(t)$, for $B \in S_1$ and S_2.

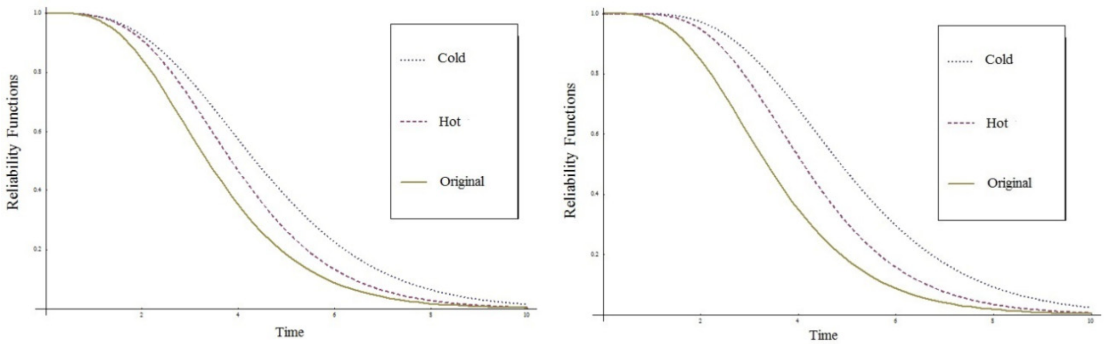

Figure 6. The reliability function, $R(t), R_B^D(t)$, for $B \in S_3$ and S_4

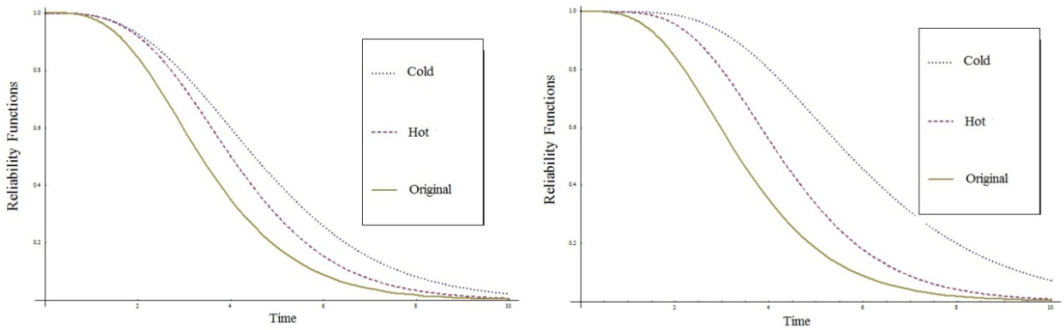

Figure 7. The reliability function, $R(t), R_B^D(t)$, for $B \in S_5$ and S_6.

Figure 8 shows comparing the reliability for different sets of system components for the improved methods $\mathrm{D} = \mathrm{H}, \mathrm{C}$ with the original system.

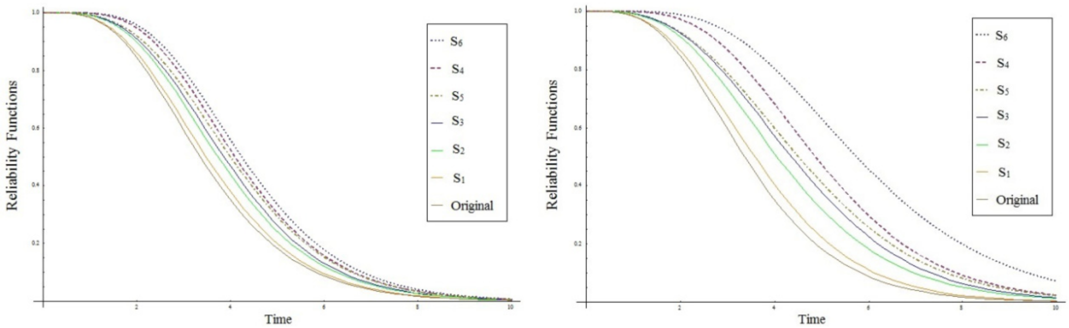

Figure 8. The reliability function, $R(t), R_B^D(t)$ for $B \in S_i$ and $D = H$ and C.

It seems from the results shown in Tables 1, 2 and Figures 5 to 8 that:
$R(t) < R_B^H(t) < R_B^C(t)$ for all $B \in S_i$.
$R(t) < R_{S_1}^D(t) < R_{S_2}^D(t) < R_{S_3}^D(t) < R_{S_5}^D(t) < R_{S_4}^D(t) < R_{S_6}^D(t)$, for $D = H, C$.
$MTTF < MTTF_B^H < MTTF_B^C$, in all studied cases.

$$MTTF < MTTF_{S_1}^D < MTTF_{S_2}^D < MTTF_{S_3}^D < MTTF_{S_5}^D < MTTF_{S_4}^D < MTTF_{S_6}^D.$$

$L(\alpha, \beta) < L_B^H(\alpha, \beta) < L_B^C(\alpha, \beta)$ in all studied cases.
As α increases, the mean time to failure increases too.
The cold duplication method improves system reliability much better than hot duplication method.
Table 3 shows the reliability equivalence factors of the improved systems using each duplication method and A, B.
According to the results presented in Tables 2 and 3, at $\alpha = 2$, it may be observed that:
Hot duplication of the set B of system's components belong to S_1, will increase $L(2, 0.1)$ from $\frac{2.9143}{\lambda}$ to $\frac{2.9732}{\lambda}$, see Table 2. The same effect can occur by reducing the failure rates of set A of system's components belong to (i) S_1 by the factor $\rho^H = 0.7256$, (ii) S_2 by the factor $\rho^H = 0.9191$, (iii) S_3 by the factor $\rho^H = 0.9347$, (iv) S_4 by the factor $\rho^H = 0.9582$, (v) S_5 by the factor $\rho^H = 0.9578$, (vi) S_6 by the factor $\rho^H = 0.9595$, see Table 3.
Cold duplication of the set of system's components, B belong to S_1, will increase $L(2, 0.1)$ from $\frac{2.9143}{\lambda}$ to $\frac{3.06851}{\lambda}$, see Table 2 .The same effect on $L(2,0.1)$ can occur by reducing the failure rates of the set A of components belong to (i) S_1 by $\rho^C = 0.4190$, (ii) S_2 by $\rho^C = 0.8056$, (iii) S_3 by $\rho^C = 0.8417$, (iv) S_4 by $\rho^C = 0.8948$, (v) S_5 by $\rho^C = 0.8925$, (vi) S_6 by $\rho^C = 0.9029$, see Table 3.
In the same manner, we can read the rest of results presented in Tables 2 and 3.
The notation NA means that there is no equivalence between the two improved systems: one obtained by reducing the failure rates of the set A of system components and the other obtained by improving the set B of the system components according to the duplication methods.

7. CONCLUSION

Reliability equality function of a bridge structure system is studied. The bridge structure system contains five independent and identical gamma lifetime components is improved. Three improvement methods, including reduction method, hot and cold duplication method are also applied to improve the reliability of mentioned system. The reliability function and mean time to failure for the original and improved systems are obtained. The β −fractiles is derived to compare the obtained and original systems.

Table 3. The $\rho^{D}_{A,B}$, when $A, B \in S_i, i = 1, 2, \cdots, 6$

β	$A \in Si$	Hot, $B \in S_i$						Cold $B \in S_i$					
		S1	S2	S3	S4	S5	S6	S1	S2	S3	S4	S5	S6
0.1	S1	0.7256	0.2383	NA	NA	NA	NA	0.4190	NA	NA	NA	NA	NA
	S2	0.9191	0.7383	0.6858	0.5763	0.5917	0.5028	0.8056	0.4461	0.3294	0.1512	0.2204	NA
	S3	0.9347	0.7859	0.7424	0.6520	0.6647	0.5920	0.8417	0.5466	0.4565	0.3377	0.3797	NA
	S4	0.9582	0.8547	0.8223	0.7516	0.7618	0.7022	0.8948	0.6635	0.5834	0.4718	0.5119	0.2148
	S5	0.9578	0.8503	0.8157	0.7391	0.7502	0.6845	0.8925	0.6411	0.5497	0.4185	0.4662	0.0399
	S6	0.9595	0.8697	0.8441	0.7919	0.7992	0.7581	0.9029	0.7330	0.6844	0.6236	0.6446	0.5092
0.2	S1	0.6814	0.1326	NA	NA	NA	NA	0.3827	NA	NA	NA	NA	NA
	S2	0.8928	0.6960	0.6235	0.4976	0.5241	0.4161	0.7777	0.4086	0.2475	0.0000	0.1139	NA
	S3	0.9164	0.7598	0.7019	0.6027	0.6234	0.5403	0.8251	0.5347	0.4212	0.2829	0.3477	NA
	S4	0.9446	0.8338	0.7905	0.7130	0.7295	0.6622	0.8812	0.6575	0.5609	0.4382	0.4961	0.1844
	S5	0.9438	0.8267	0.7791	0.6918	0.7106	0.6328	0.8775	0.6273	0.5111	0.3547	0.4301	NA
	S6	0.9464	0.8491	0.8142	0.7560	0.7680	0.7205	0.8892	0.7173	0.6554	0.5862	0.6176	0.4748
0.3	S1	0.6464	0.0000	NA	NA	NA	NA	0.3555	NA	NA	NA	NA	NA
	S2	0.8749	0.6626	0.5750	0.4313	0.4704	0.3410	0.7640	0.3808	0.1790	0.0000	0.0000	NA
	S3	0.9040	0.7382	0.6700	0.5608	0.5899	0.4958	0.8176	0.5240	0.3945	0.2327	0.3227	NA
	S4	0.9355	0.8178	0.7670	0.6827	0.7056	0.6308	0.8752	0.6534	0.5476	0.4133	0.4877	0.1601
	S5	0.9343	0.8079	0.7507	0.6520	0.6792	0.5887	0.8706	0.6165	0.4831	0.2959	0.4027	NA
	S6	0.9375	0.8332	0.7918	0.7276	0.7445	0.6909	0.8825	0.7066	0.6361	0.5583	0.5998	0.4490
0.4	S1	0.6145	NA	NA	NA	NA	NA	0.3318	NA	NA	NA	NA	NA
	S2	0.8615	0.6322	0.5316	0.3658	0.4209	0.2627	0.7576	0.3567	0.1045	NA	NA	NA
	S3	0.8947	0.7176	0.6407	0.5194	0.5586	0.4512	0.8145	0.5130	0.3709	0.1765	0.2998	NA
	S4	0.9288	0.8037	0.7472	0.6553	0.6853	0.6023	0.8731	0.6503	0.5391	0.3914	0.4832	0.1362
	S5	0.9272	0.7907	0.7252	0.6130	0.6504	0.5447	0.8677	0.6068	0.4591	0.2283	0.3785	NA
	S6	0.9309	0.8191	0.7725	0.7018	0.7242	0.6640	0.8796	0.6982	0.6216	0.5341	0.5865	0.4264
0.5	S1	0.5829	NA	NA	NA	NA	NA	0.3091	NA	NA	NA	NA	NA
	S2	0.8514	0.6022	0.4898	0.2931	0.3713	0.1643	0.7566	0.3338	NA	NA	NA	NA
	S3	0.8874	0.6962	0.6117	0.4747	0.5267	0.4021	0.8146	0.5011	0.3481	0.0962	0.2771	NA
	S4	0.9239	0.7903	0.7293	0.6283	0.6669	0.5742	0.8738	0.6480	0.5340	0.3707	0.4822	0.1096
	S5	0.9218	0.7735	0.7005	0.5708	0.6218	0.4958	NA	0.5971	0.4369	0.1270	0.3551	NA
	S6	0.9260	0.8056	0.7548	0.6765	0.7055	0.6376	0.8793	0.6912	0.6101	0.5114	0.5765	0.4049
0.6	S1	0.5496	NA	NA	NA	NA	NA	0.2861	NA	NA	NA	NA	NA
	S2	0.8444	0.5707	0.4468	0.1992	0.3179	0.0000	0.7608	0.3106	NA	NA	NA	NA
	S3	0.8819	0.6726	0.5810	0.4220	0.4921	0.3428	0.8178	0.4874	0.3247	0.0000	0.2531	NA
	S4	0.9207	0.7768	0.7124	0.6001	0.6496	0.5447	NA	0.6463	0.5323	0.3496	NA	0.0751
	S5	0.9182	0.7553	0.6751	0.5207	0.5915	0.4353	NA	0.5868	0.4149	NA	0.3310	NA
	S6	0.9226	0.7919	0.7378	0.6499	0.6876	0.6100	0.4475	0.6850	0.6013	0.4885	0.5693	0.3832
0.7	S1	0.5122	NA	NA	NA	NA	NA	0.2613	NA	NA	NA	NA	NA
	S2	0.8412	0.5354	0.3999	NA	0.2550	NA	0.7712	0.2855	NA	NA	NA	NA
	S3	0.8786	0.6446	0.5461	0.3530	0.4520	0.2608	0.8247	0.4704	0.2989	NA	0.2261	NA
	S4	0.9196	0.7625	0.6962	0.5685	0.6332	0.5117	NA	0.6454	NA	0.3268	NA	NA
	S5	0.9167	0.7345	0.6474	0.4529	0.5569	0.3468	NA	0.5752	0.3918	NA	0.3043	NA
	S6	0.9212	0.7774	0.7209	0.6202	0.6698	0.5791	NA	0.6795	0.5954	0.4640	0.5654	0.3597
0.8	S1	0.4661	NA	NA	NA	NA	NA	0.2321	NA	NA	NA	NA	NA
	S2	0.8441	0.4919	0.3444	NA	0.1460	NA	0.7910	NA	NA	NA	NA	NA
	S3	0.8790	0.6076	0.5028	0.1963	0.4009	0.0868	0.8376	0.4470	0.2679	NA	0.1931	NA
	S4	0.9216	0.7466	0.6805	0.5300	0.6182	0.4714	NA	NA	NA	0.2998	NA	NA
	S5	0.9185	0.7084	0.6143	0.3343	0.5137	0.1383	NA	0.5605	0.3655	NA	0.2723	NA
	S6	0.9228	0.7608	0.7036	0.5837	0.6522	0.5414	NA	0.6749	0.5936	0.4350	NA	0.3320
0.9	S1	0.3991	NA	NA	NA	NA	NA	0.1939	NA	NA	NA	NA	NA
	S2	0.8609	0.4283	0.2678	NA	0.0000	NA	0.8295	NA	NA	NA	NA	NA
	S3	0.8884	0.5480	0.4383	NA	0.3232	0.0000	0.8631	0.4077	0.2241	NA	0.1466	NA
	S4	0.9306	0.7269	0.6662	0.4740	0.6079	0.4133	NA	NA	NA	0.2621	NA	NA
	S5	0.9278	0.6686	0.5677	0.0000	0.4484	0.0011	NA	0.5375	0.3311	NA	0.2271	NA
	S6	0.9313	0.7396	0.6859	0.5301	0.6362	0.4865	NA	0.6719	0.6008	0.3942	NA	0.2934

It is shown that the reliability and the mean time to failure of an improved systems are higher than the original system for all different cases. Furthermore, it's shown in this paper that the cold duplication method improves system reliability much better than the hot duplication method, but it's not possible to derive a general statement for a comparison between the reduction method and duplication (hot and cold) methods. If we put $\alpha = 1$, gives the results which was obtained by Sarhan (2004) as a special case of our article.

REFERENCES

Ezzati, G. and Rasouli, A. (2015). Evaluating system reliability using linear-exponential distribution function, *International Journal of Advanced Statistics and Probability*, **3**, 15-24.

Mustafa, A. (2009). Reliability equivalence factor of n- components series system with non-constant failure rates, *International Journal of Reliability and Applications*, **10**, 43-58.

Mustafa, A. and El-Bassoiuny, A. H. (2009). Reliability equivalence of some systems with mixture linear increasing failure rates, *Pakistan Journal of Statistics*, **25**, 149-163.

Mustafa, A. and El-Faheem, A. A. (2011). Reliability euivalence fctors of a sstem with m non-identical mixed of lifetimes, *American Journal of Applied Sciences*, **8**, 297-302.

Mustafa, A. and El-Faheem, A. A. (2014). Reliability equivalence factors of a system with mixture of n independent and non-identical lifetimes with delayed time, *Journal of the Egyptian Mathematical Society*, **22**, 96–101.

Mustafa, A., El-Desouky, B. S. and El-Dawoody, M. (2009). Reliability equivalence factors of Non-identical components series system with mixture failure rates, *International Journal of Reliability and Applications*, **10**, 17-32.

Råde, L. (1990). Reliability Systems of 3-state components, studies in statistical quality control and reliability, Mathematical Statistics, Chalmers University of Technology, S41296, Gothenburg, Sweden.

Råde, L. (1993a). Reliability equivalence, Microelectronics Reliability, **33**, 323-325.

Råde, L. (1993b). Reliability survival equivalence, *Microelectronics Reliability*, **33**, 881-894.

Sarhan, A. (2004). Reliability equivalence factor of a bridge network system. *International Journal of Reliability and Applications*, **5**, 81-103.

Sarhan A. M. and Mustafa, A. (2006). Reliability equivalence of a series system consists of n independent and non-identical components, *International Journal of Reliability and Applications*, **7**, 111-125.

Xia, Y. and Zhang, G. (2007). Reliability equivalence factors in gamma distribution, *Applied Mathematics and Computation*, **187**, 567-573.