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Mathematical Structures of Polynomials in Jeong Yag-yong's Gugo Wonlyu

丁若鏞의 算書 勾股源流의 多項式의 數學的 構造

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Dedicated to the memory of our teacher, Bumwoo(凡愚)

Kim Chi Young(金致榮) on the centennial of his birth

This paper is a sequel to our paper [3]. Although polynomials in the tianyuanshu induce perfectly the algebraic structure of polynomials, the tianyuan(天元) is always chosen by a specific unknown in a given problem, it can't carry out the role of the indeterminate in ordinary polynomials. Further, taking the indeterminate as a variable, one can study mathematical structures of polynomials via those of polynomial functions. Thus the theory of polynomials in East Asian mathematics could not be completely materialized. In the previous paper [3], we show that Jeong Yag-yong disclosed in his Gugo Wonlyu(勾股源流) the mathematical structures of Pythagorean polynomials, namely polynomials $p(a, b, c)$ where a, b, c are the three sides gou(勾), gu(股), xian(弦) of a right triangle, respectively. In this paper, we show that Jeong obtained his results through his recognizing Pythagorean polynomials as polynomial functions of three variables a, b, c.

Keywords: Jeong Yag-yong, Gugo Wonlyu, Pythagorean polynomials, polynomial functions; 丁若鏞(1762–1836), 勾股源流, 피타고라스 多項式, 多項函數.

MSC: 01A13, 01A55, 06-03, 08-03, 12-03, 26-03, 26C05

1 Introduction

Since this paper is a sequel to our previous paper [3], we refer to it for the personal history of Jeong Yag-yong (丁若鏞, 1762–1836) and the bibliographical setting and contents of his book, Gugo Wonlyu (勾股源流) [4, 5], written during his exile of

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18 years (1801–1818). We also omit a short history of the theory of right triangles (勾股術) and its contributions to the development of the theory of equations [2].

Throughout this paper, a, b, c *always* denote a shorter base, gou (勾), longer height, gu (股) and hypotenuse, xian (弦) of a right triangle, respectively. Since Jiuzhang Suanshu (九章筭術), $a < b$ has been assumed. As is well known, the case of $a = b$, i.e., the right isosceles triangles, implies $c = \sqrt{2}a$ and hence one of their three sides must be irrational which does not belong to the basic field $\mathbb{Q}$ of rational numbers in East Asian mathematics. We don't know whether the commentator of Jiuzhang Suanshu, Liu Hui (劉徽, fl. the 3rd C.) did notice that $\sqrt{2}$ is irrational but he included the condition $a < b$. Along this tradition, our set of Pythagorean triples is precisely $\{(a, b, c) \in \mathbb{Q}^3 | a^2 + b^2 = c^2, 0 < a < b < c\}$, which will be denoted by $\mathbb{P}$.

In Xiangjie Jiuzhang Suanfa (詳解九章算法, 1261), Yang Hui (楊輝) introduced *ten terms* for $(a, b, c) \in \mathbb{P}$ as follows:

gougujiao (勾股較, $b - a$), gouxianjiao (勾弦較, $c - a$), guxianjiao (股弦較, $c - b$),
gouguhe (勾股和, $b + a$), gouxianhe (勾弦和, $c + a$), guxianhe (股弦和, $c + b$),
xianjiaohe (弦較和, $c + (b - a)$), xianhehe (弦和和 $c + (b + a)$),
xianhejiao (弦和較, $(a + b) - c$), xianjiaojiao (弦較較, $c - (b - a)$).

The sum of five terms ending with he (和) (jiao (較), resp.) is called wuhe (五和) (wujiao (五較), resp.).

Most of problems in the theory of right triangles in East Asia are related to the above ten terms with their sums, differences, and products. Using tianyuanshu (天元術) and up to siyuanshu (四元術), Song–Yuan mathematicians have easily solved problems of right triangles without referring to identities involving the ten terms. Since tianyuanshu was completely forgotten in the Ming dynasty (1368–1643) and the Western mathematics was introduced in the last decade of the 16th century, the theory of right triangles was completely occupied by the geometrical proofs of identities. These results were eventually collected in the sections gouguxianhejiao xiangqiufa (勾股弦和較相求法) and gougujiyugouguxianhejiao xiangqiufa (勾股積與勾股弦和較相求法) of Chapter 12 and 13 of Shuli Jingyun (數理精蘊, 1723).

Jeong Yag-yong was leaning to the Western science and technology in the Qing dynasty (1644–1910). Jeong was indeed exiled because of his connection with the Catholic belief. We don't have any information on his contact with the traditional mathematics in the East Asia. He studied Shuli Jingyun which might be his unique reference for mathematics. He noticed that the cumbersome geometrical proofs for identities in Shuli Jingyun could be neatly replaced by algebraic structures of Pythagorean polynomials $p(a, b, c), (a, b, c) \in \mathbb{P}$. Thus he wrote Gugo Wonlyu, literally, "The origin or source of the theory of right triangles".

The purpose of this paper is to reveal that Jeong Yag-yong obtained the mathematical structures, namely algebraic and order structures of Pythagorean polynomials by his perceiving them as polynomial functions.

In the first section, we discuss the reason why polynomials and polynomial functions were not related each other in East Asian mathematics and then investigate Jeong's perception of polynomial functions. In the subsequent two sections, we discuss how Jeong Yag-yong applied his perception to obtain mathematical structures of Pythagorean polynomials. Indeed, Jeong could have the algebraic and order structures of Pythagorean polynomials, in particular the latter by the pointwise order of polynomial functions. Finally, we deal with Jeong's qiajin (恰盡), which indicates precisely $p(a,b,c)=0$ for a Pythagorean polynomial and leads to the cases of infinite solution sets of equations.

The reader may find all the Chinese sources of this paper in [1] and [6], and hence they will not be numbered as an individual reference.

2 Polynomial functions

In this section, we are concerned with the history of polynomial and polynomial functions in East Asian mathematics.

We first recall that the tianyuanshu (天元術) exhibits perfectly the algebraic structure of polynomials but in practice, the tianyuan (天元) represents always a specific unknown and hence it can't take the role of the indeterminate, let alone a variable of the function defined by a polynomial. In Shuli Jingyun, jiegenfang (借根方) is introduced instead of tianyuanshu, where the signs $\perp, -, =$ together with algebraic operations are introduced for the plus $+$, minus and equality, respectively. As is well known, $\perp$ was taken for the plus sign in order to evade the confusion between the Chinese ten (十) and the usual plus sign. Algebraic operations for polynomials in jiegenfang are exactly the same with those used in the present day. The authors of Shuli Jingyun explained the validity of operations by taking the values of polynomials at a specific number. We quote the following example where gen (根) is not a variable for $(4x^3 - x^2) + (3x^3 - 2x^2) = 7x^3 - 3x^2$ in Problem 2 in Chapter 31.

> 如以數明之 以平方爲九 卽一立方爲二十七
> 上數四立方得一百零八 少一平方得少九 是一百零八少九爲九十九
> 下數三立方得八十一 少二平方得少十八 是八十八少十八爲六十三
> 上一百零八下八十一相加得一百八十九 卽七立方之數
> 上少九與下少十八相加得二十七 卽少三平方之數
> 蓋上數九十九下數六十三兩數相加得一百六十二 卽一百八十九少二十七也

It says that in order to verify the sum, x^2 is chosen by 9 and *hence* x^3 is 27. It is shown that the sum of the *values* of two polynomials *at* $x^2 = 9$ is equal to the *value* of the sum of two polynomials *at* $x^2 = 9$.

We must point out that the above verifications (以數明之) by the values of given polynomials at some chosen numbers were used at each problem of the four operations, sums, subtractions, multiplications and divisions. As the above quote indicates, they chose some number of x^2 for the cases of sums and subtractions when the lowest non zero degree of polynomials is 2. Every case of multiplications and divisions was verified by some number at x even for the lowest degree 2 of involved polynomial. Further the chosen numbers are all natural numbers ranging from 2 to 7 and for the case of x^2, only square numbers were chosen.

In all, it is too much to presume that authors of Shuli Jingyun considered values of polynomials at an *arbitrary* number but the above processes could eventually germinate the concept of polynomial functions. In the late 18th century, Chinese mathematicians revived tianyuanshu which replaced jiegenfang once again. Thus the connection between polynomials and polynomial functions has been lost until the middle of the 19th century when the modern mathematics was brought into China through western missionaries [7].

Jeong Yag-yong was one of the most favorite officials of the King Jeong Jo (正祖, r. 1776–1800) who was also very much enthusiastic to the practical utilities and theory of the western science and technology as Jeong and included Shuli Jingyun and Lixiang Kaocheng (曆象考成, 1723) as subjects of the national examination, eumyanggwa (陰陽科) of officials in the National Observatory (觀象監) [8]. One can easily assume that Jeong Yag-yong studied Shuli Jingyun. Further, he took the titles of sections gouguxianhejiao xiangqiufa (勾股弦和較相求法) and gougujiyugouguxianhejiao xiangqiufa (勾股積與勾股弦和較相求法) in chpeters 12, 13 of Shuli Jingyun as subtitles in his book Gugo Wonlyu and used duo (多) and shao (少) for the addition and subtraction of polynomials, respectively as in Shuli Jingyun.

Except the above titles and notions, Gugo Wonlyu is entirely independent of Shuli Jingyun. Gugo Wonlyu deals with the algebraic structures of Pythagorean polynomials $p(a, b, c)$ which are obtained by products, sums and subtractions of the *ten terms* for $(a, b, c) \in \mathbb{P}$ and a, b, c but it does not deal with the applications of those polynomials for solving right triangles.

Since Jeong Yag-yong is concerned with three sides of a right triangle, he wants the final results to be nonnegative. Since the *ten terms* and a, b, c for $(a, b, c) \in \mathbb{P}$ are all positive, their products and sums are also positive but it is not the case for their differences (相減). He noticed that $2a$ (二勾) and c (一弦) are not comparable as at $(3, 4, 5)$, $2a > c$ and at $(11, 60, 61) \in \mathbb{P}$, $2a < c$. We note that $2a - c = (a+b) - (c+b-a)$ as the following quote indicates [3].

勾股和與弦較和相減 卽一弦二勾之較

勾股和多 卽爲二勾內少一弦 弦較和多 卽爲一弦內少二勾

Jeong's above observation gives rise to surprising but really important two features of Pythagorean polynomials:

The first one is that the basic field $\mathbb{Q}$ of rational numbers is totally ordered but there are *non comparable* Pythagorean polynomials as $a+b$ and $c+(b-a)$ indicate.

The second one is that the value of a Pythagorean polynomial at one single choice of Pythagorean triples is *not* enough for comparisons of Pythagorean polynomials.

The second feature enforces that for a full information of a Pythagorean polynomial $p(a,b,c)$, one must have its value $p(a_0,b_0,c_0)$ at every Pythagorean triple (a_0,b_0,c_0). These assignments of values $p(a_0,b_0,c_0)$ of $p(a,b,c)$ at each triple define naturally the function p on the set $\mathbb{P}$. Thus Jeong Yag-yong could have implicitly defined Pythagorean polynomial *functions* of *three* variables.

Recognizing Pythagorean polynomials as polynomial functions of three variables, Jeong Yag-yong could have the *equal* sign $=$ between Pythagorean polynomials, or their *identities* via the values of polynomial functions. Thus, he perceived implicitly $(p*q)(a,b,c)=p(a,b,c)*q(a,b,c)$ for Pythagorean polynomials p,q where $*$ denotes the sum, subtraction and multiplication, respectively so that the algebraic operations for Pythagorean polynomials are well defined by the algebraic operations over the field $\mathbb{Q}$. Based on this, Jeong revealed the algebraic structure of Pythagorean polynomials (see [3]).

Moreover, by the definite *identity*, Jeong had the exact concept and practice of the *substitutions* as the following quotation from Book IV of Gugo Wonlyu indicates.

勾冪 其加減諸數與股弦和乘股弦較之加減諸數幷同
股冪 其加減諸數與勾弦和乘勾弦較之加減諸數幷同

It says that since $a^2=(c+b)(c-b)$ and $b^2=(c+a)(c-a)$, for *any* Pythagorean polynomial p, $a^2 \pm p=(c+b)(c-b)\pm p$ and $b^2 \pm p=(c+a)(c-a)\pm p$. We note that the right hand sides of the identities were already discussed before their left hand sides in Gugo Wonlyu.

3 Inequalities of Pythagorean polynomials

As we discussed in the previous section, Jeong Yag-yong obtained the concept of polynomial functions when he tried to understand inequalities of linear Pythagorean polynomials. He then extended the order structure to quadratic Pythagorean polynomials by the pointwise order of the polynomial functions, namely $p \le q$ for two Pythagorean polynomials p,q if the inequality $p(a,b,c) \le q(a,b,c)$ holds for *all* Pythagorean triples (a,b,c). Further, Jeong used the *partially* ordered ring structure of

Pythagorean polynomials. For the detail of the order structure, we refer to [3].

We point out that it is very difficult to solve inequalities of polynomial functions of a single variable in modern mathematics with the curve sketching and solving polynomial equations. We can't find any evidence in Gugo Wonlyu that Jeong studied the theory of equations. Thus it is really hard for him to solve inequalities of Pythagorean polynomials.

Investigating all inequalities in Gugo Wonlyu, we find some characteristics which Jeong might use to claim the inequalities. We recall once again that he needed to calculate the difference of p, q, i.e., $|p - q|$ for Pythagorean polynomials p, q so that he had to solve inequality $p > q$. First he divided those inequalities into $p > q$ ($p < q$), $p \geq q$ ($q \geq p$) and the case of p and q being not comparable. The second case is indicated by qiajin (恰盡) and $p-q$ ($q-p$, resp.) and the last case by $p-q$ and fanjian(反減) zeduoshaoxiangfan(則多少相反), which says that for those $(a, b, c) \in \mathbb{P}$ with $q(a, b, c) < p(a, b, c)$ ($p(a, b, c) < q(a, b, c)$, resp.), $|p - q| = p - q(q - p$, resp.). He explained that for a polynomial p, $-p$ can be obtained by the interchange of the signs $+$ and $-$ of p. The qiajin will be discussed in the next section.

Using these terminologies, we now study Jeong's characteristics for inequalities. Basically Jeong did know the fundamental relations between the three sides of a right triangle like $c < a + b < 2b$ and that $c - 2a, b - 2a$ are non comparable with 0. Further, we gather that he chose $(3, 4, 5), (11, 60, 61)$ as representatives of Pythagorean triples. In the following, we will give a few examples in which he fumbled, where X.n will indicate Problem n in the Book X.

Example II.115, II.117, II.166

弦和和乘弦較和積與弦和和乘勾股和積相減卽股乘弦較和二倍少弦乘弦和和
反減卽弦乘弦和和少股乘弦較和二倍;
弦和和乘弦較和積與弦和和乘勾弦和積相減卽股乘弦較和少勾乘勾弦和二倍
反減勾乘勾弦和二倍少股乘弦較和;
弦和和乘弦較較積與弦和和乘勾股和積相減卽弦乘弦較和多勾乘勾股較二倍

The first one is that for $|(a+b+c)(c+b-a)-(a+b+c)(a+b)| = (a+b+c)|c-2a|$, the difference is fanjian; for the second one, $|(a + b + c)(c + b - a) - (a + b + c)(a + c)| = (a + b + c)|b - 2a|$ implies the difference is fanjian as well. Since $|(a + b + c)(c - b + a) - (a + b + c)(a + b)| = (a + b + c)|c - 2b| = -(a + b + c)(2b - c)$, $(a+b+c)(a+b)-(a+b+c)(c-b+a) = (a+b+c)(2b-c)$ would be more familiar to the present readers but Jeong represented it by $c(c + b - a) + 2a(b - a)$.

Example I.7

勾弦較減弦和較卽二弦少二勾一股 若相減恰盡則二弦與二勾一股等,

which says that $(c-a)-(a+b-c)=2c-(2a+b)$ and that if their difference is 0, then $2c=2a+b$.

Here Jeong assumed that $(a+b-c)\geq(c-a)$. He might have the inequality by the value 0 of $(c-a)-(a+b-c)$ at $(3,4,5)$ and its positive value at $(11,60,61)$ but missed its negative value at $(20,21,29)$ Thus $c-a$ and $a+b-c$ are non comparable and hence the xiangjian (相減) should be indicated by fanjian.

Example I.11

弦和較減股弦較卽二股一勾少二弦

That is, $(a+b-c)-(c-b)=2b+a-2c$. Unlike the above example, $(c-b)<(a+b-c)$ holds for all $(a,b,c)\in\mathbb{P}$ because $(a+b-c)^2-(c-b)^2=(c-b)(b+c-2a)>0$.

Example II.155

弦和和乘弦較和積與勾弦和冪相減卽股冪三倍少弦乘弦較較二倍

It says that $(a+b+c)(c+b-a)-(a+c)^2 = 3b^2-2c(c-b+a)$. Although the values of the polynomial at $(3,4,5)$ and $(11,60,61)$ are both positive but its value at $(20,21,29)$ is negative and hence it is non comparable with 0.

There are numerous examples where the values of a polynomial at $(3,4,5)$ and $(11,60,61)$ are different but Jeong did not notice them as the following example shows.

Example II.98

弦和和乘弦和較積與勾股較冪相減卽勾乘股四倍少弦冪

It says that $(a+b+c)(a+b-c)-(b-a)^2=4ab-c^2$. The values of the polynomial at $(3,4,5)$ and $(11,60,61)$ are different so that xiangjian should be fanjian.

Example III.123

弦和較乘弦較較積與勾弦和乘股弦較積相減

卽股乘股弦較三倍少勾乘弦較較 恰盡卽多少相等

That is, $(a+b-c)(c-b+a)-(c+a)(c-b)=3b(c-b)-a(c-b+a)$ and if it is qiajin, then $3b(c-b)=a(c-b+a)$. Indeed, the value of the polynomial at $(3,4,5)$ is 0. Further, its values at $(11,60,61)$ and $(20,21,29)$ are both positive, but the value at $(5,12,13)$ is negative. Thus the xiangjian should be also fanjian.

We note that Jeong's choices of Pythagorean triples are completely independent of those in Shuli Jingyun except $(3,4,5)$.

Presumably Jeong might recognize that $na-b, na-c(2\leq n)$ are non comparable with 0 in the following example.

Example III.75

弦和較乘弦較和積與勾股較乘勾弦較相減卽勾乘勾弦較三倍少股乘勾弦較

It says that $(a+b-c)(c+b-a)-(b-a)(c-a) = 3a(c-a)-b(c-a)$, but the values of the polynomial at $(3,4,5),(11,60,61)$ are positive and negative, respectively. Thus it should be fanjian. Incidentally the end result is $(c-a)(3a-b)$ and hence one can also easily have the result.

Example III.72

弦和較乘弦較和積與勾股和乘股弦較積相減卽勾乘股及弦乘弦較較少勾冪三倍
亦卽勾乘弦及股乘弦和較少勾冪二倍

It indicates that $-(a+b-c)(c+b-a)+(a+b)(c-b) = ab+c(c-b+a)-3a^2 = ac+b(a+b-c)-2a^2$. For this result, one has to prove the final polynomial is greater than 0. Indeed, $ac+b(a+b-c)-2a^2 = ab-a^2-bc+ac+b^2-a^2 = (b-a)(2a+b-c)$ which is clearly positive. We don't know whether Jeong used the above process.

Example IV.426

弦冪與弦和較乘股弦和積相減卽弦乘勾弦較少勾乘勾股較

That is, $c^2-(a+b-c)(b+c) = c(c-a)-a(b-a)$. The expansion of $c^2-(a+b-c)(b+c) = c^2-a(b+c)-(b^2-c^2) = c^2-ac-ab+a^2 = c(c-a)-a(b-a)$. Since $c > a$ and $c-a > b-a$, $c(c-a) > a(b-a)$ so that Jeong has xiangjian as above.

We recall that there was no one dealing with inequalities of polynomials even for a single variable in East Asian mathematics. We point out that Jeong Yag-yong dealt with more than 1,500 inequalities of *three* variables. Although we just quote a few cases in the above where he made wrong results, most of Jeong's inequalities are correct and Jeong achieved really important contributions to the theory of inequalities based on the very sound understanding about them.

4 Qiajin (恰盡)

The terminology chushi qiajin (除實恰盡) was first introduced in Suanxue Qimeng (算學啓蒙, 1299) of Zhu Shijie (朱世傑) which was evolved from chushi shijin (除實適盡) in Shushu Jiuzhang (數書九章, 1247) of Qin Jiushao (秦九韶, 1202–1261). As the terminology indicates, it is related to the processes of solving polynomial equation $p(x) = 0$, namely the equation $q(y) = 0$ for the last digit α of the solution of $p(x) = 0$ has the solution at the digit, i.e., $q(\alpha) = 0$. The qiajin was also adopted in Shuli Jingyun as in Suanxue Qimeng. We will discuss the history of qiajin in another paper.

Since Jeong Yag-yong took Pythagorean polynomials p as polynomial functions, he could have the solution set $\{(a,b,c)|p(a,b,c)=0\}$ of such a p. Thus he took the terminology qiajin to indicate the solution set of $p=0$ is non empty.

The Pythagorean triple $(3,4,5)$ is the most well-known and well used one in the history of right triangles. Thus Jeong tried to find the value of $p(3,4,5)$ for every Pythagorean polynomial p in Gugo Wonlyu and hence he indicated the case of the solution set containing $(3,4,5)$. As we observed in the previous section, polynomials given by sums (相加) in Gugo Wonlyu are always positive so that their solution sets are empty. Thus the case of qiajin always appears in their differences. Further, his equations are all *homogeneous* polynomials and therefore once $(3,4,5)$ is a solution for Pythagorean polynomial equation $p=0$ in Gugo Wonlyu, then the Pythagorean triples $(3k,4k,5k)$ for any positive k are also solutions of $p=0$. Thus his qiajin at $(3,4,5)$ implies that its solution set is *infinite*.

Collecting Jeong's results, we found that the linear cases of qiajin at $(3,4,5)$ occur when one of the following four relations hold:

$$2b=a+c; 2c=2a+b; 3a=b+c; 3c=a+3b.$$

For Pythagorean triple (a,b,c), each one of the above relations also implies $(a,b,c)=(3k,4k,5k), 0<k$ so that they are indeed equivalent each other.

The following example indicates that the solution set of an equation $p-q=0$ is the whole set $\mathbb{P}$ iff $p=q$ and it was first appeared in Jiuzhang Suanshu.

Example II.52

弦和和冪與勾弦和乘股弦和積相減卽勾弦和乘股弦和 減倍積恰盡

It says that $(a+b+c)^2-(a+c)(b+c)=(a+c)(b+c)$ and hence $(a+b+c)^2-2(a+c)(b+c)=0$. This is a neat algebraic proof of $(a+b+c)^2=2(a+c)(b+c)$.

Example III.23

弦和較冪與弦較較乘股弦較積相減卽勾冪少勾乘股弦較三倍 恰盡則多少相等

That is, $(a+b-c)^2-(c-b+a)(c-b)=a(a-3c+3b)$ which implies that it is qiajin at $(3,4,5)$ and not xiangjian but fanjian. The qiajin in this example is different from that in the above example. Thus the sentence "恰盡則多少相等" is true only at $(3k,4k,5k)$.

We add another example among others where the difference is a quadratic polynomials but qiajin at $(3,4,5)$. There are a few cases like the above which are also obtained by one of the above four linear relations but in the following example, we couldn't verify that the end result can be derived by one of the four linear relations.

Example IV.169

弦較較乘股弦較積與勾弦較冪相減卽弦乘勾股較二倍及勾乘弦和較少股冪

恰盡則多少相等

It says that $-(c-b+a)(c-b)+(c-a)^2 = 2c(b-a)+a(a+b-c)-b^2$ which is qiajin at $(3,4,5)$.

5 Conclusions

In Gugo Wonlyu, Jeong Yag-yong found that mathematical structures of Pythagorean polynomials could be realized by polynomial functions and their mathematical structures. His perceptions led to the perfect understanding of the algebraic structure of Pythagorean polynomials and the definition of their order structure together with inequalities of polynomials. Finally, he figured out the solutions of an equation $p = 0$ for a Pythagorean polynomial p are precisely those Pythagorean triples (a,b,c) with $p(a,b,c) = 0$. By this observation, he could have equations with infinitely many solutions and the identities $p = q$ for Pythagorean polynomials are exactly those with the the whole set $\mathbb{P}$ as the solution set of $p - q = 0$.

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References

1. Guo Shuchun ed. *Zhongguo Kexue Jishu Dianji Tonghui* Shuxuejuan, Henan Jiaoyu Pub. Co., 1993. 郭書春 主編, 《中國科學技術典籍通彙》 數學卷, 河南教育出版社, 1993.
2. Hong Sung Sa, Hong Young Hee, Kim Chang Il, Gougushu in the 18th century Chosun, *The Korean Journal for History of Mathematics* 20(4) (2007), 1–22. 홍성사, 홍영희, 김창일, 18 世紀 朝鮮의 句股術, 한국수학사학회지 20(4) (2007), 1–22.
3. Hong Sung Sa, Hong Young Hee, Lee Seung On, Mathematical Structures of Jeong Yag-yong's Gugo Wonlyu, *Journal for History of Mathematics*, 28(6) (2015), 301–310.
4. Jeong Yag-yong, *Gugo Wonlyu*, Yeoyudang Jeonseo Boyu Four, Gyeongin Munhwasa, 1975. 丁若鏞, 《勾股源流》, 與猶堂全書補遺 四, 景仁文化社, 1975.
5. Jeong Yag-yong, *Gugo Wonlyu*, Book I-III, National Library of Korea. 丁若鏞, 《勾股源流》 I-III卷, 國立中央圖書館, http://www.nl.go.kr.
6. Jing Yushu ed. *Zhongguo Lidai Suanxue Jicheng*, Shandong Renmin Pub. Co., 1994. 靖玉樹 編勘, 《中國歷代算學集成》, 山東人民出版社, 1994.
7. Wu Wenjun ed. *Zhongguo Shuxueshi Daxi* Vol. 8, Beijing Normal Univ. Pub. Co., 2000. 吳文俊 主編, 《中國數學史大系》 第八卷, 北京師範大學出版社, 2000.
8. *The Annals of the Joseon Dynasty*. 朝鮮王朝實錄. http://sillok.history.go.kr