

Colourings and the Alexander Polynomial

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ABSTRACT. Using a combination of calculational and theoretical approaches, we establish results that relate two knot invariants, the Alexander polynomial, and the number of quandle colourings using any finite linear Alexander quandle. Given such a quandle, specified by two coprime integers n and m , the number of colourings of a knot diagram is given by counting the solutions of a matrix equation of the form $AX = 0 \pmod n$, where A is the m -dependent colouring matrix. We devised an algorithm to reduce A to echelon form, and applied this to the colouring matrices for all prime knots with up to 10 crossings, finding just three distinct reduced types. For two of these types, both upper triangular, we found general formulae for the number of colourings. This enables us to prove that in some cases the number of such quandle colourings cannot distinguish knots with the same Alexander polynomial, whilst in other cases knots with the same Alexander polynomial can be distinguished by colourings with a specific quandle. When two knots have different Alexander polynomials, and their reduced colouring matrices are upper triangular, we find a specific quandle for which we prove that it distinguishes them by colourings.

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Received January 13, 2015, accepted June 17, 2015

2010 Mathematics Subject Classification: 57M27, 57M25.

Key words and phrases: knot invariant, quandles, Alexander polynomial.

This work was supported by Fundação para a Ciência e a Tecnologia.

1. Introduction

The theory of knots has a long and rich history, giving rise to a large number of different invariants coming from many diverse perspectives. For this reason it is very important to understand the interconnections between different knot invariants, normally not an easy task. In the present study, using a combination of calculational and theoretical approaches, we establish results that relate two knot invariants, namely the classical Alexander polynomial on the one hand, and the number of quandle colourings using any finite linear Alexander quandle, on the other.

The number of quandle colourings of a knot diagram is a well known and rich invariant of a knot, introduced independently in [12] and [18] - see also [4] for a recent survey. An interesting class of quandles are the finite linear Alexander quandles, which are given by two coprime integers n and m . Thus we can consider the information contained in the number of such quandle colourings for arbitrary choices of n and m . A separate invariant of the knot is its Alexander polynomial [1], and in this article we clarify a number of points about the precise relationship between these two invariants. Note that if the Alexander polynomial is replaced by the, much harder to compute, collection of all Alexander polynomials (which includes the Alexander polynomial of a knot as its first element), then by [11] this collection completely determines the number of quandle colourings for any Alexander quandle, linear or otherwise.

Our work started with various attempts to calculate the number of quandle colourings for specific choices of quandles and particular knots, following on from an article by one of us with P. Lopes [8]. During this process, certain patterns emerged which we wanted to understand by means of a systematic approach. We also wanted to streamline the calculations by finding more efficient algorithms, since even with clever methods the specific calculations could take a long time. This motivated us to look for more general and better methods.

Given a finite linear Alexander quandle, the number of quandle colourings of a knot diagram is given by counting the solutions of a matrix equation of the form $AX = 0$, where the entries of the colouring matrix A are Laurent polynomials in m , and the corresponding linear equations are all taken modulo n . Thus a natural strategy is to try and reduce A to echelon form. We devised an algorithm (using the Mathematica programming environment) which does this in such a way as to preserve a property of the matrix A , namely that its columns add up to zero. Applying this algorithm to the colouring matrices for all prime knots with up to 10 crossings we found that all the reduced matrices were of just three distinct and highly specific types, two of which were upper triangular with mostly 1's on the diagonal (which we call Type I and Type II), and the third of which was non-triangular, but upper block triangular, with mainly 1's on the diagonal along with a single 2 by 2 non-triangular block. There were only 12 out of 249 prime knots with up to 10 crossings that gave non-triangular reduced matrices.

Using theoretical methods, we were then able to prove general formulae for the

number of solutions of $AX = 0$ as a function of n and m , when the reduced matrix is of Type I or Type II. We note that whilst various ingenious methods may be used to find the number of quandle colourings of a knot diagram in particular cases, having a general formula giving that number is far more powerful (and faster). However we did use various direct calculations via other methods to check that the number of colourings given by our formulae were in agreement with those results.

The formulae that we obtained for the Type I or Type II reduced colouring matrices either involve the Alexander polynomial of the knot, or factors of the Alexander polynomial, both evaluated at m (which is not surprising since the Alexander polynomial is given by the determinant of a minor of A). This opens the way for proving general results which relate two different invariants: the Alexander polynomial, and the number of quandle colourings using any finite linear Alexander quandle.

In particular, we can show in many cases that the number of such quandle colourings cannot distinguish knots with the same Alexander polynomial, confirming indications coming from previous calculations, which failed to distinguish these cases, despite using exhaustive batteries of quandles. On the other hand, we find cases of pairs of knots with the same Alexander polynomial which can be distinguished by counting quandle colourings for appropriate finite linear Alexander quandles.

When two knots have different Alexander polynomials, and their colouring matrices can be triangularized (not necessarily only as Type I or Type II matrices), we prove that they can be distinguished by colourings using a suitable finite linear Alexander quandle. These include instances which in the light of the previous test calculations appeared to be indistinguishable by quandle colourings. The quandles that do distinguish them did not show up in the calculations simply because they were too large. We conjecture that, in general, when two knots have different Alexander polynomials, it is possible to distinguish them with finite linear Alexander quandle colourings, irrespective of whether their colouring matrices can be triangularized.

We also use information coming from the Alexander polynomial, namely that in some cases it is not factorizable into proper factors, to prove that in those cases it is impossible to reduce the corresponding colouring matrix to triangular form. Curiously, in some analogous cases with a non-factorizable Alexander polynomial, the same reasoning fails, since despite the fact that the colouring matrix has a non-triangular reduced form, in a large number of test calculations the number of colourings always obeys the formula for a Type I reduced matrix. These and other issues deserve further investigation by calculational and theoretical means.

The structure of this article is as follows. In section 2 we recall the necessary background for quandles [12, 18] and quandle colourings of knot diagrams [2, 4, 10, 11, 17, 19], focussing on the case of finite linear Alexander quandles. We also introduce the colouring matrix A associated to a knot diagram, and its role both in computing the number of quandle colourings and in getting the Alexander polynomial [1, 16].

In section 3 we describe our computations which reduce A to specific echelon

or upper triangular forms, for all prime knots with up to 10 crossings. We observe that precisely three patterns occur for the reduced matrices: two types of upper triangular matrices (which we call Type I and Type II) and a non-triangular form which is block upper triangular containing a single non-triangular 2×2 block. We employ an algorithm which preserves a useful property of the colouring matrix, namely that the sum of its columns is equal to zero, but we also describe the reduction using a more general algorithm and compare all results with an analogous computation in the literature [15]. We illustrate the computation of the number of colourings for a specific knot and choice of quandle.

In section 4 we define a notion of equivalence between matrices based on the operations used in the algorithm of section 3 for reducing A . When the colouring matrix of a knot diagram is equivalent to a Type I or Type II upper triangular matrix, we prove that the number of quandle colourings, using a finite linear Alexander quandle specified by two coprime integers m and n , obeys a general formula. These central results, which use the well-known linear congruence theorem, are given in Propositions 4.5 and 4.6. Both formulae involve the Alexander polynomial of the knot, or factors of the Alexander polynomial, evaluated at m .

Section 5 exploits the two formulae for the number of colourings to show that, in suitable circumstances, the number of quandle colourings using any finite linear Alexander quandle is no stronger than the Alexander polynomial in distinguishing knots. In particular we list many cases of knots with up to 10 crossings, which are indistinguishable by means of such quandle colourings. On the other hand we also find cases, amongst knots with up to 10 crossings, of pairs of knots with the same Alexander polynomial, but which can be distinguished by such a quandle colouring.

In section 6 we examine the cases where our calculations reduced the colouring matrix to a non-triangular form. For five cases we are able to prove that it is impossible to reduce the colouring matrix to triangular form, by exploiting a feature of the corresponding Alexander polynomial, namely that it cannot be factorized into proper factors. We comment on a remarkable phenomenon for a further four cases, which also have non-factorizable Alexander polynomials, where the number of colourings agrees with the Type I formula in test calculations, despite the fact that we were unable to reduce the colouring matrix to Type I form.

Finally, in section 7 we prove that, for a class of knots, the number of quandle colourings using any finite linear Alexander quandle is as strong an invariant as the Alexander polynomial when it comes to distinguishing knots. These knots are those for which the colouring matrix can be reduced to upper triangular form using the operations of our algorithm - see Proposition 7.3. We conjecture that the same holds true even when the colouring matrix cannot be reduced to triangular form. We then give some examples of quandle colourings which distinguish knots with differing Alexander polynomials.

In section 8 we present some conclusions. The two appendices give the output of our calculations for prime knots with up to 10 crossings by displaying the relevant part of the Type II or non-triangularized reduced colouring matrices (for Type I knots this output is given simply by the corresponding Alexander polynomial for

which listings already exist).

2. Background

In this section we recall the definition of a quandle and the notion of quandle coloring of a diagram. Since the knot quandle is a classifying invariant for knots (introduced independently by Joyce and Matveev - see [12] and [18]), the number of quandle colorings associated to a knot diagram, for any fixed quandle, is a knot invariant. Below we also present the notions of finite Alexander quandle, Alexander polynomial and linear finite Alexander quandle.

2.1. Quandles and colourings

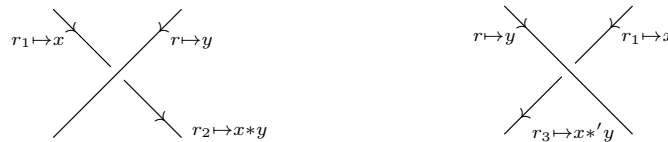
Colourings of the arcs of oriented knot diagrams with elements of a quandle generalize *mod p* labellings of the arcs, that, in turn, generalize the colorability invariant of R. Fox (with $p = 3$ colours). They are also a generalization of arc labellings of oriented knot diagrams with group elements (see, for instance [16]). At each crossing the quandle elements labelling the arcs are related by the quandle operation $*$. The number of colourings is a knot invariant since different diagrams of the same knot have the same number of colourings using a given quandle. Indeed, the definition of a quandle consists of precisely those properties of the binary operation $*$ that ensure that colourings are preserved under the Reidemeister moves.

Definition 2.1. (Quandle) A quandle is a set X endowed with a binary operation, denoted $*$, such that:

- (a) for any $a \in X, a * a = a$
- (b) for any a and $b \in X$, there is a unique $x \in X$ such that $a = x * b$. This element x is denoted by $a *' b$.
- (c) for any a, b and $c \in X, (a * b) * c = (a * c) * (b * c)$

We may use the elements of a quandle to colour the arcs of a knot diagram.

Definition 2.2. (quandle colouring of a knot diagram) Let X be a fixed finite quandle, K a knot (assumed to be oriented), \vec{D} a diagram of K and $R_{\vec{D}}$ the set of arcs of \vec{D} . A quandle colouring of a diagram \vec{D} is a map $C : R_{\vec{D}} \rightarrow X$ such that, at each crossing:



i.e. if $C(r_1) = x$ and $C(r) = y$, then $C(r_2) = x * y$ for the crossing on the left, and if $C(r_1) = x$ and $C(r) = y$, then $C(r_3) = x *' y$ for the crossing on the right.

Colourings of knot diagrams using quandles are knot invariants in the following sense.

Theorem 2.3. *Let X be a fixed finite quandle, K a knot and D and D' oriented diagrams of K . Then the number of colourings $C : R_D \rightarrow X$ is equal to the number of colourings $C' : R_{D'} \rightarrow X$.*

For a more complete discussion of the results above and related topics see [4, 11, 13, 17, 19].

2.2. Finite Alexander quandles

Finite Alexander quandles have the form $\mathbb{Z}_n[t, t^{-1}]/h(t)$ where n is an integer and $h(t)$ is a monic polynomial in t . These quandles have as elements equivalence classes of Laurent polynomials with coefficients in \mathbb{Z}_n , where two polynomials are equivalent if their difference is divisible by $h(t)$. The quandle operation is

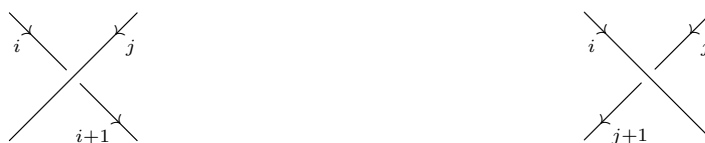
$$(2.1) \quad a * b = ta + (1 - t)b.$$

Note that this means equality of quandle elements, i.e. equivalence classes of Laurent polynomials. Recall that $c = a *' b$ is defined to mean the same as $a = c * b$. From this it follows easily that

$$(2.2) \quad a *' b = t^{-1}a + (1 - t^{-1})b.$$

For finite Alexander quandles the colouring condition at each crossing states that the label of the emerging arc is expressed as a linear combination of the labels of the other two arcs. Therefore one uses matrices to organize the colouring conditions (equations).

For that purpose we need an enumeration of the arcs and the crossings. Any enumeration will do, but to fix ideas we describe one possibility. We choose a starting arc, labelled 1, and use an enumeration that assigns $i + 1$ to the emerging arc where i is the number assigned to the incoming arc (see figure below), except for the last crossing when the emerging arc is already labelled (by 1). For crossings we use the enumeration suggested by the enumeration of arcs, i.e. the k -th crossing is the one with under arc also labelled k .



Let X_k be the label (in the quandle) of arc k . Then the colouring conditions of Definition 2.2, using (2.1) and (2.2), applied to the figure below:



are, respectively,

$$\begin{aligned} X_k &= tX_i + (1-t)X_j \\ X_k &= t^{-1}X_j + (1-t^{-1})X_i \end{aligned}$$

The second condition $X_k = X_j *' X_i$ is equivalent to $X_j = X_k * X_i$, i.e. $X_j = tX_k + (1-t)X_i$, and thus the colouring conditions for the crossings in the figure above can also be expressed as:

$$\begin{aligned} tX_i + (1-t)X_j - X_k &= 0 \\ (1-t)X_i - X_j + tX_k &= 0 \end{aligned}$$

It may happen that two of the arcs labelled i, j, k at a crossing are actually the same arc, e.g. $i = j$. In this case, the corresponding terms in the equations above are combined.

Thus, given an oriented diagram D of a knot K , we can write the colouring conditions as a matrix equation

$$AX = 0$$

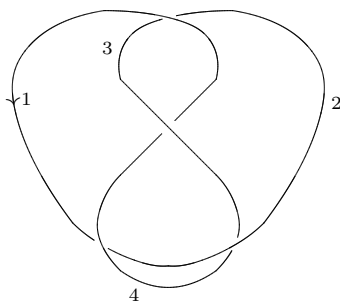
where X is the vector of colouring unknowns $(X_1, X_2, \dots, X_i, \dots)$ and each row in the matrix A represents a colouring condition for one crossing in D .

Obviously the number of colourings of a diagram in a linear Alexander quandle is the number of solutions of $AX = 0$. We will call the matrix A a *colouring matrix*.

For example,

$$\begin{bmatrix} -1 & t & 0 & 1-t \\ 1-t & t & -1 & 0 \\ 0 & 1-t & -1 & t \\ -1 & 0 & 1-t & t \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is the matrix equation corresponding to the following diagram of the knot 4_1 (the figure-8 knot):



Remark 2.4 We note for future reference that the sum of the columns of any colouring matrix is the zero column (this is obvious from the coefficients in the colouring conditions).

2.3. The Alexander polynomial

The Alexander polynomial $\text{Alex}_K(t)$ [1] of a knot K is a knot invariant that may be computed in a number of ways. We will follow the approach by Livingston [16]. First choose a diagram for K and pick an orientation for that diagram; we denote the oriented diagram by D . Enumerate the arcs of the diagram, and separately enumerate the crossings. If the number of crossings is 0, i.e. K is the unknot O , we define the Alexander polynomial to be $\text{Alex}_O(t) = 0$. Otherwise, define an $N \times N$ matrix, where N is the number of crossings (and arcs) in the diagram, according to the following procedure.

Any crossing is of one of two types:



- If the crossing numbered l is like the diagram on the left above, enter t in column i of row l , enter $1-t$ in column j of that row, and enter -1 in column k of the same row.
- If the crossing numbered l is like the diagram on the right above, enter $1-t$ in column i of row l , enter -1 in column j of that row, and enter t in column k of the same row.
- An exceptional case occurs if any of i , j or k are equal. In this case the sum of the entries described above is put in the appropriate column.
- All of the remaining entries of row l are 0.

Remark 2.5. Notice that the matrix thus obtained is exactly the same as the colouring matrix A from the previous subsection, assuming we use the same enumerations for arcs and crossings. The only difference is the orientation convention used in the two procedures. In any case, the Alexander polynomial, to be defined next, is independent of all choices, including the choice of orientation for the diagram.

As we have noted before, the sum of the columns of A is the zero column, i.e. the determinant of A is 0. The Alexander polynomial is essentially obtained by removing the final column and row of A to obtain a reduced matrix A_r , which is non-singular, and taking its determinant. However different choices for the enumerations and orientation may lead to different polynomials, which however are always related by multiplying by a sign or an integer power of t . Thus the Alexander polynomial in the definition to follow is, in fact, an equivalence class of polynomials.

Definition 2.6. (from [16]) The $(N-1) \times (N-1)$ matrix A_r obtained by removing the final row and column from the $N \times N$ matrix A described above is called the Alexander matrix of K . The determinant of the Alexander matrix is called the Alexander polynomial of K , regarded up to equivalence, where two polynomials are equivalent if they are obtained from each other by multiplying by a sign and/or by an integer power of t . It is customary to normalize the Alexander polynomial ([7]) by choosing the representative with no negative powers of t and a positive constant term.

Example 2.7. Applying the definition to the colouring matrix for the knot 4_1 from the previous subsection, the Alexander polynomial is $-1 + 3t - t^2$, or in normalized form $1 - 3t + t^2$.

3. Computations with Linear Finite Alexander Quandles

From now on we will be concentrating on quandle colourings using a special class of quandles, called linear finite Alexander quandles. These are finite Alexander quandles (see subsection 2.2), of the form $\mathbb{Z}_n[t, t^{-1}] / (t - m)$, where n and m are integers and n, m are coprime. Recall that the elements are equivalence classes of Laurent polynomials having the same remainder when divided by $t - m$. Obviously the polynomial t is in the same equivalence class as the constant polynomial m , since $t = (t - m) + m$. Similarly t^{-1} is equivalent to m^{-1} (the inverse of m in \mathbb{Z}_n), since $t^{-1} - m^{-1} = -m^{-1}t^{-1}(t - m)$. (Note that m is invertible since $\gcd(m, n) = 1$). It follows that any polynomial is equivalent to some number in \mathbb{Z}_n and that one can identify $\mathbb{Z}_n[t, t^{-1}] / (t - m)$ with \mathbb{Z}_n . The quandle operation can be written as $a * b = ma + (1 - m)b \pmod{n}$ and $a *' b = m^{-1}a + (1 - m^{-1})b \pmod{n}$.

Thus the colourings of any knot diagram with elements of a linear finite Alexander quandle are the solutions of the matrix equation

$$AX = 0,$$

where A is the colouring matrix of subsection 2.2 with t replaced by m , X is the vector of colouring unknowns $(X_1, X_2, \dots, X_i, \dots)$, belonging to \mathbb{Z}_n , and equalities hold in \mathbb{Z}_n (i.e. equality mod n).

In this section, we describe computations that we carried out in order to solve this equation, using two slightly different algorithms, and we compare the results with an analogous computation in the literature. We also give an example of calculating the number of colourings for a specific choice of m and n . The colouring matrices were obtained from Rolfsen's standard knot diagrams [21]. Thus when we refer to the colouring matrix of a knot K , we mean the colouring matrix obtained from the diagram of K in this list.

3.1. Reduction of A - first algorithm

In order to solve the equation $AX = 0$ for general m and n , we wrote algorithms in Mathematica that reduce the colouring matrix to a standard echelon form¹. The

¹The source code is available on request from the first author.

operations used in this reduction process were of four types:

- 1) multiplying rows by -1 , m and m^{-1}
- 2) adding rows
- 3) swapping rows
- 4) swapping columns

Note that these operations all preserve the property that the sum of the columns of the matrix is the zero vector, a feature that will be useful in the next section when we derive general expressions for the number of colourings for certain classes of knots. The algorithms were also designed to yield as output an echelon matrix such that, eliminating the final row and column and taking the determinant, gives the Alexander polynomial in normalized form, a feature that will be exploited later on.

Applying these algorithms to the prime knots with up to 10 crossings, we reduced their colouring matrices to three kinds of echelon form, which we call Type I, Type II and non-triangular.

Type I matrices: These are upper triangular, with 1's on the diagonal except in the penultimate row where the entry is denoted $\alpha(m)$, and in the last row which has all entries equal to zero. The entry $\alpha(m)$ is the normalized Alexander polynomial of the knot for t equal to m .

$$\begin{matrix} & & & \text{Type I} & & \\ \left[\begin{array}{cccccc} 1 & \lambda_{12}(m) & \cdots & \cdots & \cdots & \lambda_{1N}(m) \\ 0 & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & 0 & 1 & \lambda_{N-2\ N-1}(m) & \lambda_{N-2\ N}(m) & \\ \vdots & \vdots & 0 & \alpha(m) & -\alpha(m) & \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{array} \right] \end{matrix}$$

Type II matrices: These are upper triangular, with 1's on the diagonal except in the antepenultimate and penultimate rows, where the diagonal entries are denoted $\alpha_1(m)$ and $\alpha_2(m)$, respectively, and in the last row which has all entries equal to zero. The product $\alpha_1(m)\alpha_2(m)$ is the normalized Alexander polynomial of the knot for t equal to m .

$$\begin{matrix} & & & \text{Type II} & & \\ \left[\begin{array}{cccccc} 1 & \lambda_{12}(m) & \cdots & \cdots & \cdots & \lambda_{1N}(m) \\ 0 & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & 0 & 1 & \lambda_{N-3\ N-2}(m) & \lambda_{N-3\ N-1}(m) & \lambda_{N-3\ N}(m) \\ \vdots & \vdots & 0 & \alpha_1(m) & \beta_1(m) & -(\alpha_1(m) + \beta_1(m)) \\ \vdots & \vdots & \vdots & 0 & \alpha_2(m) & -\alpha_2(m) \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{array} \right] \end{matrix}$$

Non-triangular matrices: These are just like Type II matrices, except for replacing the triangular 2×2 array

$$(2.1) \quad \begin{bmatrix} \alpha_1(m) & \beta_1(m) \\ 0 & \alpha_2(m) \end{bmatrix}$$

with a non-triangular 2×2 array, which has determinant equal to the normalized Alexander polynomial.

We now summarize the results of our computations.

Type I The vast majority of colouring matrices for knots with up to 10 crossings could be reduced to a matrix of Type I. We do not list these 216 knots, or the special entry $\alpha(m)$, since it is just the corresponding normalized Alexander polynomial with t equal to m .²

Type II There were 21 colouring matrices that could be reduced to Type II matrices, namely those coming from the following knots:

$$8_{18}; 9_{37}; 9_{40}; 9_{46}; 10_{61}; 10_{63}; 10_{65}; 10_{74}; 10_{75}; 10_{98}; 10_{99}; 10_{103};$$

$$10_{106}; 10_{122}; 10_{123}; 10_{140}; 10_{142}; 10_{144}; 10_{147}; 10_{155}; 10_{164}$$

In Appendix A we list their characteristic 2 by 2 arrays.

Non-triangular Finally there were 12 colouring matrices that could be reduced to the non-triangular echelon form described above. These came from the following knots:

$$9_{35}; 9_{38}; 9_{41}; 9_{47}; 9_{48}; 9_{49}; 10_{69}; 10_{101}; 10_{108}; 10_{115}; 10_{157}; 10_{160}$$

In Appendix B we list their characteristic 2 by 2 arrays.

Remark 3.1. Using the general results that will be developed starting in the next section, we were able to prove (in section 6), that for some of these cases (corresponding to the 5 knots $9_{35}, 9_{47}, 9_{48}, 9_{49}$ and 10_{157}) it is impossible to triangularize the colouring matrix into Type I or Type II form. This shows that, at least in these cases, the absence of a triangular form is not due to some shortcoming in the algorithm. In the same section, we also conjecture that it is impossible to triangularize the colouring matrices for a further 4 knots, $10_{69}, 10_{101}, 10_{115}$ and 10_{160} , on the basis of the fact that they have non-factorizable Alexander polynomials. As regards the distinction between Type I and Type II, in the next section we will provide general formulae for the number of colourings using any linear Alexander quandle, for knots with colouring matrices that reduce to Type I or Type II matrices. Therefore, for this purpose it is irrelevant if a Type II matrix may be further simplified to Type I, since a formula exists in either case.

²For up to 9 crossings these are listed in Livingston [16]. For up to 10 crossings they are given in shorthand form in Rolfsen [21] or Kawauchi [15], e.g. for the knot 6_3 , Rolfsen gives $[5 - 3 + 1]$, meaning that the Alexander polynomial is $m^{-2} - 3m^{-1} + 5 - 3m + m^2$ or in normalized form $1 - 3m + 5m^2 - 3m^3 + m^4$.

3.2. Reduction of A - second algorithm

As noted earlier, in reducing the colouring matrices to echelon form, we restricted ourselves to operations which preserve the sum of the columns (equal to zero). For the purpose of obtaining the number of solutions, i.e. colourings, rather than the solutions themselves, there is no need to exclude more general column operations, such as summing columns and multiplying columns by -1 , m and m^{-1} . We developed a second algorithm, allowing these more general column operations, as well as the previous operations. In this way, some of the colouring matrices could be reduced to a simpler type compared to the first algorithm, as follows:

1) The colouring matrices corresponding to knots 10_{106} and 10_{147} , which formerly could only be reduced to Type II matrices, can be reduced to Type I matrices with more general column operations.

2) The colouring matrices corresponding to knots 9_{41} and 10_{108} , which formerly could only be reduced to non-triangular matrices, can be reduced to a Type II matrix and a Type I matrix, respectively, with more general column operations.

3.3. Comparison with presentation matrices

It is worthwhile to compare our results with Kawauchi's presentation matrices [15], since these are also obtained from colouring matrices by simplification operations, which include the row operations and column swaps of our algorithm, as well as more general operations, in particular on columns, and other operations which can reduce the size of the matrix. The presentation matrices obtained by Kawauchi for the colouring matrices coming from prime knots with up to 10 crossings are either 1×1 , comparable to the special diagonal entry $\alpha(m)$ in our Type I reduced matrices, or 2×2 , comparable to the special 2×2 arrays in our Type II and non-triangular reduced matrices. The 29 latter cases are listed in Appendix F.4 of [15]. To a very large extent, we find a correspondence between our Type I, Type II, and non-triangular reduced matrices, and Kawauchi's 1×1 , upper triangular 2×2 , and non-triangular 2×2 presentation matrices, respectively. The only six cases for which a discrepancy occurs are listed below:

$10_{65}, 10_{106}, 10_{147}$	Type II	1×1
$9_{38}, 9_{41}$	nontriang.	2×2 upp. triang.
10_{108}	nontriang.	1×1

When more general column operations are allowed, as in subsection 3.2, the only discrepancies that persist correspond to two knots, 9_{38} and 10_{65} . We emphasize once more that the operations allowed in obtaining presentation matrices are more general than the operations in our algorithms, so there is no inconsistency.

3.4. The number of colourings for a specific m and n

Having reduced the colouring matrices to echelon form, obtaining the number of colourings for any specific choice of m and n becomes a simple calculational task. We give an example below that can be solved by hand, before proceeding in the next

section to develop powerful general methods for obtaining the number of colourings for arbitrary m and n .

For the knot 9_{35} our program gave as output the following non-triangular echelon matrix:

$$\begin{bmatrix} 1 & h_{1j} & \cdots & \cdots & \cdots & h_{1r} \\ 0 & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & 0 & 1 & h_{r-3 \ r-2} & h_{r-3 \ r-1} & h_{r-3 \ r} \\ \vdots & \vdots & 0 & 2 - m & -1 - m & -1 + 2m \\ \vdots & \vdots & \vdots & -3 & -2 + 7m & 5 - 7m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we choose $m = 2$ and $n = 3$, we get a matrix with the 3 final rows consisting only of zeros, after reducing modulo 3. Thus, for this case, the number of colourings is equal to $3^3 = 27$, since we can choose the final 3 unknowns, X_7, X_8 and X_9 , freely in \mathbb{Z}_3 , and the remaining unknowns are uniquely determined after this choice.

4. Formulae for the Number of Colourings

Recall from section 2 that for an $N \times N$ matrix A , we denote by A_r the reduced $(N - 1) \times (N - 1)$ matrix with the last row and column of A removed. If K is a knot and A is the colouring matrix coming from a diagram of K , then the Alexander polynomial is defined to be the determinant of A_r , i.e. $\text{Alex}_K = |A_r|$, up to equivalence (see Def. 2.6).

Definition 4.1. Let A and B be $N \times N$ matrices with entries in $\mathbb{Z}_n[m, m^{-1}]$. We say B is equivalent to A iff the following two conditions hold:

- 1) B is obtained from A by a sequence of the following operations:
 - a) multiplication of a row by m, m^{-1} or -1 ; b) replacing a row by its sum with some row; c) swapping two rows; d) swapping two columns,
- 2) $|A_r| = |B_r|$, up to equivalence.

Remark 4.2. This is an equivalence relation since, for 1), all operations a)-d) can be inverted using the same operations, and for 2), it is obvious. Note that if A is the colouring matrix of some knot, or a matrix obtained from such a colouring matrix by means of operations a)-c), then a column swap d) will automatically satisfy the condition 2): since the sum of the columns is zero, it is easy to see that $|A_r| = -|B_r|$, i.e. $|A_r| = |B_r|$ up to equivalence, for column swaps involving any of the columns including the last one.

Definition 4.3. A knot is said to be of Type I, or of Type II, if the colouring matrix of one of its diagrams is equivalent to a matrix of the corresponding type.

A result that we will use several times is the well known linear congruence theorem.

Proposition 4.4. *If a and b are integers and n is a positive integer, then the congruence $ax = b \pmod n$ has a solution for x if and only if b is divisible by the greatest common divisor of a and n , $d = \gcd(a, n)$. When this is the case, and x_0 is a solution of $ax = b \pmod n$, then the set of all solutions is given by $\{x_0 + k\frac{n}{d}, k \in \mathbb{Z}\}$. In particular, there will be exactly $d = \gcd(a, n)$ solutions in the set $\{0, \dots, n - 1\}$.*

When a knot is Type I, it is possible to find a general expression for the number of linear quandle colourings, using an arbitrary linear quandle, as the following proposition shows.

Proposition 4.5. *Let K be a Type I knot, and Q be the linear finite Alexander quandle $Q = \mathbb{Z}_n[t, t^{-1}]/(t - m)$. Then $C_Q(K)$, the number of Q -colourings of K , is*

$$C_Q(K) = n \times \gcd(\text{Alex}_K(m), n).$$

Proof. By assumption the $N \times N$ colouring matrix A of K is equivalent to a matrix B of Type I form (see Section 3.1). This may be expressed as follows: there exist $N \times N$ matrices C and P , with C non-singular and P a permutation matrix, such that

$$CAP = B$$

The matrix C captures the row operations a) - c), and the matrix P captures the column operations d) of Definition 4.1. Thus the equation $AX = 0$ is equivalent to the equation $BX' = 0$, where $X' = P^{-1}X$, and hence there is a one-to-one correspondence between their respective sets of solutions. We solve the latter equation by solving first for X'_N , then for X'_{N-1} , and so on. X'_N satisfies: $0X'_N = 0 \pmod n$, thus there are n solutions for X'_N in \mathbb{Z}_n . For X'_{N-1} we have the equation $\alpha(m)(X'_{N-1} - X'_N) = 0 \pmod n$, and thus, by proposition 4.4, there are $d = \gcd(\alpha(m), n)$ solutions for X'_{N-1} , namely $X'_{N-1} = X'_N + k\frac{n}{d}, k = 0, \dots, d - 1$. Recall that $\alpha(m)$ is the normalized Alexander polynomial of K , $\text{Alex}_K(m)$. The remaining unknowns, X'_{N-2}, \dots, X'_1 each have a unique solution in terms of X'_N and X'_{N-1} . In conclusion, the number of solutions of $AX = 0$, i.e. $C_Q(K)$, is given by the product of n and $\gcd(\text{Alex}_K(m), n)$. \square

Remark 4.6. An immediate consequence of proposition 4.5 is that Type I knots with the same Alexander polynomial cannot be distinguished by colourings of finite linear Alexander quandles. We will make a more detailed statement in the next section.

For Type II knots the process of calculating the number of solutions is similar to that of Type I knots. In this case, apart from the equations coming from the final two rows of the Type II matrix (see Section 3.1), a third equation has to be considered.

Proposition 4.7. *Let K be a Type II knot, and Q be the linear finite Alexander quandle $Q = \mathbb{Z}_n[t, t^{-1}]/(t - m)$. Then $C_Q(K)$, the number of Q -colourings of K , is*

$$C_Q(K) = n \times \gcd(\alpha_2(m), n) \times \gcd(\beta_1(m) \frac{n}{\gcd(\alpha_2(m), n)}, \gcd(\alpha_1(m), n)).$$

Proof. As in the proof of the previous proposition, there exist matrices C and P , with C non-singular and P a permutation matrix, such that $CAP = B$, where B is of Type II form, and hence the equation $AX = 0$ is equivalent to the equation $BX' = 0$, where $X' = P^{-1}X$. Following the reasoning of the previous proposition, the number of solutions of the equations corresponding to the final two rows of B , involving only X'_N and X'_{N-1} , is $n \cdot d_2$, where $d_2 = \gcd(\alpha_2(m), n)$. These solutions are $X'_N = 0, 1, \dots, n - 1, \text{ mod } n$ and $X'_{N-1} = X'_N + k \frac{n}{d_2}, \text{ mod } n, k = 0, \dots, d_2 - 1$.

The row of B above the final two rows yields the equation $\alpha_1(m)X'_{N-2} + \beta_1(m)X'_{N-1} - (\alpha_1(m) + \beta_1(m))X'_N = 0 \text{ mod } n$, that can be rewritten as $\alpha_1(m)(X'_{N-2} - X'_N) + \beta_1(m)(X'_{N-1} - X'_N) = 0 \text{ mod } n$. Letting $Y_{N-2} = X'_{N-2} - X'_N$ and $Y_{N-1} = X'_{N-1} - X'_N$ we obtain

$$\alpha_1(m)Y_{N-2} + \beta_1(m)Y_{N-1} = 0 \text{ mod } n$$

From before we have $Y_{N-1} = k \frac{n}{d_2}$, for $k = 0, \dots, d_2 - 1$. Substituting above one obtains $\alpha_1(m)Y_{N-2} + \beta_1(m)k \frac{n}{d_2} = 0 \text{ mod } n$. This equation only has solutions for those values of k such that $\beta_1(m)k \frac{n}{d_2}$ is divisible by $d_1 = \gcd(\alpha_1(m), n)$. And, if there is one solution then there will be d_1 solutions.

We check how many of the values $\beta_1(m)k \frac{n}{d_2}$ for $k = 0, \dots, d_2 - 1$ are multiples of d_1 . This is equivalent to $\beta_1(m) \frac{n}{d_2} k = 0 \text{ mod } d_1$ that has $d_3 = \gcd(\beta_1(m) \frac{n}{d_2}, d_1)$ solutions and these are $k = t \times \frac{d_1}{d_3}$ for $t = 0, \dots, d_3 - 1$. Now we have to check which of these k 's are in $0, \dots, d_2 - 1$, i.e. such that $t \times \frac{d_1}{d_3} < d_2$. It is easy to check that there are $c_3 = d_2 \times \frac{d_3}{d_1}$ possible values for t namely $0, \dots, c_3 - 1$. Therefore there are c_3 values of k such that $\beta_1(m)k \frac{n}{d_2}$ is a multiple of d_1 .

Summing up we have that for each value of X'_N in $\{0, \dots, n - 1\}$ there will be $d_2 = \gcd(\alpha_2(m), n)$ values for X'_{N-1} that are solutions of the final equations but only $c_3 = d_2 \times \frac{d_3}{d_1}$ of them lead to solutions for X'_{N-2} . Each of these, however, gives d_1 solutions for X'_{N-2} . Therefore there are $n \times c_3 \times d_1$ solutions since the other X'_j s with $j < N - 2$ are uniquely determined by the values of these three. Substituting one gets $C_Q(K) = n \times d_2 \times \frac{d_3}{d_1} \times d_1 = n \times d_2 \times d_3 = n \times \gcd(\alpha_2(m), n) \times \gcd(\beta_1(m) \frac{n}{d_2}, d_1) = n \times \gcd(\alpha_2(m), n) \times \gcd(\beta_1(m) \frac{n}{\gcd(\alpha_2(m), n)}, \gcd(\alpha_1(m), n))$. \square

Example 4.8. For the knot 8_{18} and choosing the quandle $\mathbb{Z}_{15}[t, t^{-1}]/(t - 8)$ (i.e. $m = 8$ and $n = 15$), the significant 2×2 block of the Type II matrix B is: .

$$\begin{bmatrix} 3 & 6 \\ 0 & 12 \end{bmatrix}$$

The number of colourings is given by

$$n \times \gcd(\alpha_2(m), n) \times \gcd(\beta_1(m) \frac{n}{\gcd(\alpha_2(m), n)}, \gcd(\alpha_1(m), n)).$$

Substituting one obtains $15 \times \gcd(12, 15) \times \gcd(6 \frac{15}{\gcd(12, 15)}, \gcd(3, 15)) = 15 \times 3 \times \gcd(6 \times 5, 3) = 15 \times 3 \times 3$. The number of colourings is 135.

Remark 4.9. Note that we could extend these results to more general triangular matrices. In that case, however, it is difficult to find an expression for the solutions, and, therefore for the number of solutions. A general algorithm could work by finding the solutions for X_i using previous solutions of X_j , with $j > i$. For each equation it is easy to check what values of the previous X_j 's give solutions (the independent term must be a multiple of the gcd of the entry on the diagonal and n). The solutions themselves, that are needed for rows above, can be computed with the extended Euclidean algorithm (available in some computer systems, such as Mathematica). We have written such a program and it agrees with the values for Type I and II matrices. However, for prime knots up to 10 crossings this is not relevant because such more general triangular matrices do not occur.

5. Comparing Knots with the Same Alexander Polynomial

The results of the previous section open the way for making comparisons between two separate knot invariants: the Alexander polynomial, on the one hand, and the number of quandle colourings for any linear Alexander quandle, on the other. In this section we will focus on the case where two knots have the same Alexander polynomial, and investigate whether or not they can be distinguished by linear Alexander quandle colourings (see Section 7 for the case where the Alexander polynomials are different). Our first statement is a direct corollary of propositions 4.5 and 4.6.

Corollary 5.1. *A pair of Type I knots with the same Alexander polynomial cannot be distinguished by counting linear Alexander quandle colourings. Likewise, a pair of Type II knots with the same characteristic triangular 2×2 array (2.1), hence the same Alexander polynomial, cannot be distinguished by counting linear Alexander quandle colourings.*

Thus, from our calculations for knots with up to 10 crossings, we can list pairs of knots, and even one triple of knots, which are indistinguishable by counting linear Alexander quandle colourings. These cases all involve Type I knots, since our calculations produced no examples of Type II pairs with the same characteristic triangular array.

$$\begin{aligned} &5_1, 10_{132}; 7_4, 9_2; 7_5, 10_{130}; 7_6, 10_{133}; 8_3, 10_1; 8_5, 10_{141}; 8_8, 10_{129}; 8_{10}, 10_{143}; \\ &8_{16}, 10_{156}; 8_{21}, 10_{136}; 9_{15}, 10_{166}; 9_{20}, 10_{149}; 9_{28}, 9_{29}; 10_{10}, 10_{165}; 10_{12}, 10_{54}; \\ &10_{18}, 10_{24}; 10_{20}, 10_{163}; 10_{23}, 10_{52}; 10_{25}, 10_{56}; 10_{28}, 10_{37}; 10_{31}, 10_{68}; \\ &10_{34}, 10_{135}; 10_{127}, 10_{150}; 8_{14}, 9_8, 10_{131}. \end{aligned}$$

It is also interesting to understand if knots with the same Alexander polynomial, but different types of reduced colouring matrix can be distinguished by linear quandle colourings. Our calculations give a number of examples, mainly Type I - Type II pairs, but also one case of a non-triangular - Type II pair (9_{38} and 10_{63}). We start by considering pairs of knots where one of the colouring matrices reduces to Type I and the other to Type II. First there are a number of cases of such pairs that can be distinguished by counting linear quandle colourings, simply by finding values for n and m that give a different number of colourings in the two formulae of Propositions 4.5 and 4.6. The following proposition gives the result of this search.

Proposition 5.2. *The following pairs of knots, where the first knot is Type I and the second is Type II, have the same Alexander polynomial but can be distinguished by counting linear Alexander quandle colourings, using a quandle with n and m as specified below:*

$$\begin{aligned} &6_1, 9_{46} (n = 3, m = 2); & 8_9, 10_{155} (n = 5, m = 4); \\ &9_{24}, 8_{18} (n = 6, m = 5); & 10_{40}, 10_{103} (n = 5, m = 4); \\ &10_{42}, 10_{75} (n = 3, m = 2); & 10_{59}, 9_{40} (n = 5, m = 4); \\ &10_{67}, 10_{74} (n = 3, m = 2); & 10_{87}, 10_{98} (n = 3, m = 2). \end{aligned}$$

However, it is not always possible to distinguish by linear quandle colourings knots that have the same Alexander polynomial and are of different types, as the following proposition shows.

Proposition 5.3. *The knots 8_{20} (Type I) and 10_{140} (Type II) have the same Alexander polynomial and cannot be distinguished by counting linear Alexander quandle colourings.*

Proof. Both knots have the same Alexander polynomial $\text{Alex}(m) = (1 - m + m^2)^2 = 1 - 2m + 3m^2 - 2m^3 + m^4$. Since 8_{20} is Type I the number of colourings in any linear Alexander quandle is given by $C_1(m, n) = n \times \text{gcd}(\text{Alex}(m), n)$. The relevant entries (2.1) for the Type II matrix of 10_{140} are

$$\begin{bmatrix} 1 - m + m^2 & -2m^2 \\ 0 & 1 - m + m^2 \end{bmatrix}$$

Therefore the number of colourings of 10_{140} using any linear Alexander quandle is given by $C_2(m, n) = n \times \text{gcd}(\alpha(m), n) \times \text{gcd}(\beta(m) \frac{n}{\text{gcd}(\alpha(m), n)}, \text{gcd}(\alpha(m), n))$, where $\alpha(m) = 1 - m + m^2$ and $\beta(m) = -2m^2$.

We now show that $C_2(m, n) = C_1(m, n)$. Since $\alpha(m) = 1 - m + m^2 = m(m - 1) + 1$ is odd and m and n are coprime then $\beta(m) = -2m^2$ is also coprime with $\text{gcd}(\alpha(m), n)$. Therefore $\text{gcd}(\beta(m) \frac{n}{\text{gcd}(\alpha(m), n)}, \text{gcd}(\alpha(m), n)) = \text{gcd}(\frac{n}{\text{gcd}(\alpha(m), n)}, \text{gcd}(\alpha(m), n))$. Recall that $a \times \text{gcd}(b, c) = \text{gcd}(a \times b, a \times c)$ with $a \geq 1$ and note that $\text{gcd}(\text{gcd}(a, n)^2, n) = \text{gcd}(a^2, n)$. Indeed, $\text{gcd}(a^2, n) =$

$\gcd(\frac{a^2}{\gcd(a,n)^2} \gcd(a,n)^2, n) = \gcd((\frac{a}{\gcd(a,n)})^2 \gcd(a,n)^2, n) = \gcd(\gcd(a,n)^2, n)$
 since $\frac{a}{\gcd(a,n)}$ and n are coprime. Therefore

$$\begin{aligned} C_2(m, n) &= n \times \gcd(\alpha(m), n) \times \gcd(\frac{n}{\gcd(\alpha(m), n)}, \gcd(\alpha(m), n)) \\ &= n \times \gcd(n, \gcd(\alpha(m), n)^2) \\ &= n \times \gcd(\alpha(m)^2, n) \\ &= n \times \gcd(A(m), n) = C_1(m, n), \end{aligned}$$

since $\alpha(m)^2$ is the Alexander polynomial $Alex(m)$. \square

Finally we mention the short remaining list of pairs of knots with the same Alexander polynomial and different types of reduced colouring matrices, which we could not distinguish by counting linear Alexander quandle colourings, despite using a battery of thousands of linear quandles.

$$\begin{aligned} \text{Type I - Type II: } &10_{77}, 10_{65}; 9_{28}, 10_{164}; 9_{29}, 10_{164}; 8_{11}, 10_{147} \\ \text{Non-triang. - Type II: } &9_{38}, 10_{63}. \end{aligned}$$

Thus we conjecture that they cannot be distinguished by linear quandles, although we have been unable to prove this using arguments like in Proposition 5.3.

6. Non-triangularizability Results

Our results so far clearly depend on knowing the type of matrix to which the colouring matrix can be reduced, using the operations of Definition 4.1. To show that a knot is, say, Type I, is a priori a calculational property, but to show that it is not of Type I can sometimes be proved, as we will see in this section. The reason we can do this is by noticing that some knots have Alexander polynomials that cannot be factorized in any non-trivial way. This means that, if the colouring matrix of such a knot is equivalent to a triangular matrix, the knot has to be of Type I, since there cannot be more than one non-trivial diagonal entry in the reduced triangular matrix. If, however, a linear quandle can be exhibited for which the number of colourings disagrees with the Type I formula of Proposition 4.5, then the knot cannot be of Type I, hence its colouring matrix is necessarily non-triangularizable.

We use this approach to prove that the colouring matrices of five of the twelve knots for which our calculations gave a non-triangular reduced matrix, namely 9_{35} , 9_{47} , 9_{48} , 9_{49} and 10_{157} , are indeed non-triangularizable. Curiously, a similar line of thinking should apply to four more knots, 10_{69} , 10_{101} , 10_{115} and 10_{160} , since they also have Alexander polynomials that do not factorize. However, we were unable to find linear quandles that could serve as counter examples. We comment on this at the end of the section.

We start by looking for Alexander polynomials that are not factorizable, i.e. cannot be written as products of other polynomials in a non-trivial way.

Definition 6.1. An Alexander polynomial $Alex(m)$ with integer coefficients is said to be properly factorizable if the corresponding normalized form is properly

factorizable. A polynomial (in non-negative powers of m) is properly factorizable if it can be written as the product of integer polynomials not equal to ± 1 or \pm itself.

Note, for example, that $4 + 2m^2 = 2(2 + m^2)$ is properly factorizable. The following example illustrates the fact that polynomials with non-integer roots may be properly factorizable.

Example 6.2. The polynomial $8 - 2m^2 - m^4$ has roots $m = \pm 2i, \pm\sqrt{2}$, none of which are integer. However, it can be written as the product of integer polynomials since $8 - 2m^2 - m^4 = -(m - 2i)(m + 2i)(m - \sqrt{2})(m + \sqrt{2}) = (4 + m^2)(2 - m^2)$. Therefore it is properly factorizable.

We now show that there are Alexander polynomials that are not properly factorizable.

Proposition 6.3. *The Alexander polynomials for the knots $9_{35}, 9_{47}, 9_{48}, 9_{49}, 10_{69}, 10_{101}, 10_{115}, 10_{157}$ and 10_{160} are not properly factorizable.*

Proof. The normalized Alexander polynomial of the knot 9_{35} is $\text{Alex}_{9_{35}}(m) = 7 - 13m + 7m^2$. Its roots are $r_1 = \frac{1}{14}(13 - 3i\sqrt{3})$ and $r_2 = \frac{1}{14}(13 + 3i\sqrt{3})$. The original polynomial is $7 \times (m - r_1) \times (m - r_2)$ and it is not possible to multiply these factors to yield integer polynomials other than $\text{Alex}_{9_{35}}(m)$. Therefore $\text{Alex}_{9_{35}}(m) = 7 - 13m + 7m^2$ is not properly factorizable.

The proof for the other knots is similar. We found their roots using Mathematica and tested the finite products of $(m - r_i)$ multiplied by the coefficient of the highest degree monomial of m . Only the products including all the $(m - r_i)$ factors yielded an integer polynomial (the original polynomial) and all other products did not yield integer polynomials.³ Here we list only the Alexander polynomials and omit the calculations.

- 9_{47} $(1 - 4m + 6m^2 - 5m^3 + 6m^4 - 4m^5 + m^6)$
- 9_{48} $(1 - 7m + 11m^2 - 7m^3 + m^4)$
- 9_{49} $(3 - 6m + 7m^2 - 6m^3 + 3m^4)$
- 10_{69} $(1 - 7m + 21m^2 - 29m^3 + 21m^4 - 7m^5 + m^6)$
- 10_{101} $(7 - 21m + 29m^2 - 21m^3 + 7m^4)$
- 10_{115} $(1 - 9m + 26m^2 - 37m^3 + 26m^4 - 9m^5 + m^6)$
- 10_{157} $(1 - 6m + 11m^2 - 13m^3 + 11m^4 - 6m^5 + m^6)$
- 10_{160} $(1 - 4m + 4m^2 - 3m^3 + 4m^4 - 4m^5 + m^6)$

□

³Substituting m with values 0 and 1 non-integer values were obtained.

We now show that in the case of knots with Alexander polynomials that are not properly factorizable, if their colouring matrices are triangularizable then they are equivalent to a Type I matrix.

Proposition 6.4. *Let K be a knot and let $\text{Alex}_K(m)$ be its Alexander polynomial. Assume that $\text{Alex}_K(m)$ is not properly factorizable and that the colouring matrix for K can be reduced to an equivalent triangular matrix. Then K is Type I.*

Proof. Let A denote the triangular matrix which is equivalent to the colouring matrix of K . We show first that it is possible to transform A in such a way that all entries on the diagonal (except the final one that is 0) are either 1 or $\text{Alex}_K(m)$. Recall that the entries in each row of the colouring matrix add up to zero. Since the operations that transformed the original colouring matrix into the triangular matrix A preserve this property, we know that the entries in each row of A add up to zero. Since A is assumed to be triangular its final row must consist only of zeros. This row will be left unchanged. Let N be the number of rows (and columns) of A . Each remaining diagonal entry is a polynomial in m and m^{-1} and the product of the entries on the diagonal except for the final row is $\prod_{i=1}^{N-1} a_{ii} = \pm m^k \text{Alex}_K(m)$, the Alexander polynomial possibly multiplied by $\pm m^k$. Since $\text{Alex}_K(m)$ is not properly factorizable, none of the diagonal entries can be a proper factor and they are either $a_{ii} = \pm m^{k_i}$ or $a_{ii} = \pm m^{k_i} \text{Alex}_K(m)$ (this case can occur only once). Now multiply each row by $\pm m^{-k_i}$.

Next swap the column where $\text{Alex}_K(m)$ occurs with the penultimate column and then the row where $\text{Alex}_K(m)$ occurs with the penultimate row. The non-zero entries below the diagonal can be changed to zero by repeatedly adding to the corresponding row suitable multiples of the rows above where 1 occurs in the diagonal. Therefore the final matrix is Type I. \square

Now we show that some colouring matrices cannot be triangularized.

Proposition 6.5. *The colouring matrices of the knots 9_{35} , 9_{47} , 9_{48} , 9_{49} and 10_{157} cannot be triangularized.*

Proof. Since these knots have non-properly factorizable Alexander polynomials we know that if they have a triangular colouring matrix then the number of colourings using any linear Alexander quandle $\mathbb{Z}_n[t, t^{-1}] / (t - m)$ must be given by the Type I formula $n \times \gcd(\text{Alex}_K(m), n)$ of Proposition 4.5. The proof consists now in exhibiting for each such knot K a quandle for which the true number of colourings is not equal to $n \times \gcd(\text{Alex}_K(m), n)$.

9_{35} . We have seen in section 3.4 that for $m = 2$ and $n = 3$ the number of colourings of the knot 9_{35} is 27. Its Alexander polynomial is $7 - 13m + 7m^2$ and is not properly factorizable. The number of colourings computed using the Type I formula is, however, $3 \times \gcd(7 - 26 + 28, 3) = 3 \times \gcd(9, 3) = 3 \times 3 = 9$.

9_{47} . We look for colourings using the quandle given by $m = 2, n = 3$. For $m = 2$ the Alexander polynomial $1 - 4m + 6m^2 - 5m^3 + 6m^4 - 4m^5 + m^6$ is 9. Thus the number of colourings computed from the Type I formula for 9_{47} would be

$3 \times \gcd(9, 3) = 3 \times 3 = 9$. However there are $3 \times 3 \times 3$ colourings as we can see from the relevant entries of the colouring matrix for this knot (after simplification). The second matrix below is obtained from the first by putting $m = 2$ and the third by substituting the mod 3 values of the entries.

$$\begin{bmatrix} -1 + 4m - m^2 & -2 - m - m^2 + m^3 \\ 2 - 7m & 3 + 4m + 2m^2 - m^4 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ -12 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

9₄₈. We look for colourings using the quandle given by $m = 2, n = 3$. For $m = 2$ the Alexander polynomial $1 - 7m + 11m^2 - 7m^3 + m^4$ is -9 . Thus the number of colourings computed from the Type I formula for 9₄₈ would be $3 \times \gcd(-9, 3) = 3 \times 3 = 9$. However there are $3 \times 3 \times 3$ colourings, proceeding as in the previous case:

$$\begin{bmatrix} 2 - m & 2 - 8m + 7m^2 - m^3 \\ 3 & 3 - 10m + 2m^2 + 5m^3 - m^4 \end{bmatrix} \quad \begin{bmatrix} 0 & 6 \\ 3 & 15 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

9₄₉. We look for colourings using the quandle given by $m = 4, n = 5$ (since for $m = 2, n = 3$ this case is inconclusive). The Alexander polynomial $3 - 6m + 7m^2 - 6m^3 + 3m^4$ for $m = 4$ is 475. Thus the number of colourings computed from the Type I formula for 9₄₉ would be $5 \times \gcd(475, 5) = 5 \times 5 = 25$. However there are $5 \times 5 \times 5$ colourings:

$$\begin{bmatrix} -2 - m + m^2 & 3 - m - m^2 - 2m^3 \\ 3 - 2m & -3 + 3m + m^2 \end{bmatrix} \quad \begin{bmatrix} 10 & -145 \\ -5 & 25 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

10₁₅₇. For $m = 2, n = 3$ this case is also inconclusive. But for $m = 6, n = 7$ the Alexander polynomial $1 - 6m + 11m^2 - 13m^3 + 11m^4 - 6m^5 + m^6$ is 11809. Then the number of colourings computed from the Type I formula for 10₁₅₇ would be $7 \times \gcd(11809, 7) = 7 \times 7 = 49$. However there are $7 \times 7 \times 7$ colourings:

$$\begin{bmatrix} 4 - 3m & -7 + 12m - 9m^2 + 6m^3 - m^4 \\ -1 + m^2 & 2 - 3m + m^2 - m^3 \end{bmatrix} \quad \begin{bmatrix} -14 & -259 \\ 35 & -196 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \square$$

We have already shown that the knots 10₆₉, 10₁₀₁, 10₁₁₅ and 10₁₆₀ also have Alexander polynomials that do not factorize. It is remarkable that they do not have Type I colouring matrices according to our calculations (and according to [15]) but somehow behave like Type I. Indeed the number of colourings calculated using a battery of thousands of linear quandles coincides in each case with the number obtained using the expression for Type I matrices. Presumably this means that a general formula for the number of colourings reduces to the Type I formula in these cases because of some specific feature.

7. Quandle Colourings when the Alexander Polynomials Differ

In this section we show that, given two knots with triangularizable colouring matrices but different Alexander polynomials, a linear finite Alexander quandle can be exhibited that distinguishes the two knots by the number of colourings. Note that this holds for any two knots with colouring matrices that are equivalent to a matrix of triangular form, which may be of a more general type than Type I or Type II.

First we note that there is an upper bound for the number of colourings of a triangular colouring matrix using any quandle.

Proposition 7.1. *Let K be a knot and A an $N \times N$ triangular matrix, equivalent to the colouring matrix of K , with diagonal entries $a_{ii}(m)$, $i = 1, \dots, N$, where $a_{NN} = 0$. Then the number of colourings of K using any linear finite Alexander quandle $Q = \mathbb{Z}_n[t, t^{-1}]/(t - m)$ satisfies*

$$C_Q(K) \leq n \times \prod_{i=1}^{N-1} \gcd(a_{ii}(m), n).$$

Proof. We look for solutions in the order X_N, X_{N-1}, \dots . There are n solutions in \mathbb{Z}_n of the equation $0X_N = 0$, namely $X_N = 0, \dots, n-1$. Fix one such value $X_N = v$. We now investigate for $X_N = v$ how many solutions there are for the other variables. The penultimate equation is $a_{N-1, N-1}X_{N-1} - a_{N-1, N-1}v = 0 \pmod n$, since $a_{N-1, N} = -a_{N-1, N-1}$. There are $\gcd(a_{N-1, N-1}, n)$ solutions of this equation in \mathbb{Z}_n . Therefore we have $n \times \gcd(a_{N-1, N-1}, n)$ solutions for X_N and X_{N-1} (as in the proof of Proposition 4.5). Now for each pair of values for $X_N = v_1$ and $X_{N-1} = v_2$ the equation in row $N-2$ becomes $a_{N-2, N-2}X_{N-2} + a_{N-2, N-1}v_2 + a_{N-2, N}v_1 = 0 \pmod n$. If any of these equations admits a solution then the number of solutions in \mathbb{Z}_n will be $\gcd(a_{N-2, N-2}, n)$. Therefore there are at most $n \times \gcd(a_{N-1, N-1}, n) \times \gcd(a_{N-2, N-2}, n)$ solutions for X_N, X_{N-1} and X_{N-2} . Proceeding in an analogous fashion for the remaining rows, the result follows. \square

We now show that when n is a multiple of all the $a_{ii}(m)$, $i = 1, \dots, N-1$, the inequality becomes an equality.

Proposition 7.2. *Let K be a knot and A an $N \times N$ triangular matrix, equivalent to the colouring matrix of K , with diagonal entries $a_{ii}(m)$, $i = 1, \dots, N$, where $a_{NN} = 0$. Given coprime $1 < m < n$ such that n is a multiple of $\prod_{i=1}^{N-1} |a_{ii}(m)|$ (hence $a_{ii}(m) \neq 0$ for $i = 1, \dots, N-1$), the number of colourings of K using the linear finite Alexander quandle $Q = \mathbb{Z}_n[t, t^{-1}]/(t - m)$ is $C_Q(K) = n \times \prod_{i=1}^{N-1} \gcd(a_{ii}(m), n) = n \times |\text{Alex}_K(m)|$.*

Proof. First it is convenient to rewrite the equations $AX = 0 \pmod n$ in terms of the variables $Y_i = X_i - X_N$, for $i = 1, \dots, N-1$. The final equation remains unchanged but every other equation $a_{ii}(m)X_i + \dots + a_{i, N-1}(m)X_{N-1} + a_{iN}(m)X_N = 0$ can be rewritten as $a_{ii}(m)Y_i + \dots + a_{i, N-1}(m)Y_{N-1}(m) = 0$ since the entries in each row of A add up to zero and therefore $a_{iN}(m) = -(a_{ii}(m) + \dots + a_{i, N-1}(m))$.

In the proof we use the fact that $\gcd(a_{ii}(m), n) = |a_{ii}(m)|$ since $n = c \times \prod_{i=1}^{N-1} |a_{ii}(m)|$. We now show by induction that, given solutions coming from the rows below the i^{th} row, there are $\gcd(a_{ii}(m), n) = |a_{ii}(m)|$ solutions coming from row i . It is convenient to write $i = N - j$ and use j for induction. We show that, for $j = 1, \dots, N-1$, there are $\gcd(a_{N-j, N-j}(m), n)$ solutions for Y_{N-j} and also that each solution is a multiple of $\prod_{k=1}^{N-j-1} |a_{kk}(m)|$, the product of the moduli of the diagonal entries in the rows above the $i^{\text{th}} = (N-j)^{\text{th}}$ row.

For $j = 1$ we have $\gcd(a_{N-1,N-1}(m), n)$ solutions of Y_{N-1} (recall the proof of Proposition 4.5). The solutions are $k \times \frac{n}{\gcd(a_{N-1,N-1}(m), n)} = k \times \frac{c \times \prod_{i=1}^{N-1} |a_{ii}(m)|}{|a_{N-1,N-1}(m)|} = k \times c \times \prod_{i=1}^{N-2} |a_{ii}(m)| = k_1 \times \prod_{i=1}^{N-2} |a_{ii}(m)|$.

Now for $j > 1$ the equation is $a_{N-j,N-j}(m)Y_{N-j} + a_{N-j,N-j+1}(m)Y_{N-j+1} + \dots + a_{N-j,N-1}(m)Y_{N-1} = 0 \pmod n$. Choosing a solution for each $Y_{N-j+1}, \dots, Y_{N-1}$, by the induction hypothesis we obtain $a_{N-j,N-j}(m)Y_{N-j} + a_{N-j,N-j+1}(m) \times k_{j-1} \prod_{i=1}^{N-j} |a_{ii}(m)| + \dots + a_{N-j,N-1}(m) \times k_1 \prod_{i=1}^{N-2} |a_{ii}(m)| = 0 \pmod n$ where $k_i, i = 1, \dots, j-1$ are integers. Since all terms except the first include the factor $\prod_{i=1}^{N-j} |a_{ii}(m)|$ we can rearrange this equation as $a_{N-j,N-j}(m)Y_{N-j} + \beta(m) \times \prod_{i=1}^{N-j} |a_{ii}(m)| = 0 \pmod n$. Now we have that $\gcd(a_{N-j,N-j}(m), n) = |a_{N-j,N-j}(m)|$ and the independent term $\beta(m) \times \prod_{i=1}^{N-j} |a_{ii}(m)|$ is divisible by $a_{N-j,N-j}(m)$ so there are $\gcd(a_{N-j,N-j}(m), n) = |a_{N-j,N-j}(m)|$ solutions, as follows from the linear congruence theorem. Moreover, $a_{N-j,N-j}(m)Y_{N-j} + \beta(m) \times \prod_{i=1}^{N-j} |a_{ii}(m)| = 0 \pmod n$ can be equivalently rewritten as

$$a_{N-j,N-j}(m) \left(Y_{N-j} + \beta(m) \times \prod_{i=1}^{N-j-1} |a_{ii}(m)| \right) = 0 \pmod n$$

One solution of this equation is $Y_{N-j} = w_j = -\beta(m) \times \prod_{i=1}^{N-j-1} |a_{ii}(m)|$. Any other solution is of the form $w_j + k' \frac{n}{\gcd(a_{N-j,N-j}(m), n)}$. Now it is easy to check that both w_j and $k' \frac{c \times \prod_{i=1}^{N-1} |a_{ii}(m)|}{|a_{N-j,N-j}(m)|}$ are multiples of $\prod_{i=1}^{N-j-1} |a_{ii}(m)|$ and therefore so is each solution of the equation. This ends the proof by induction.

The final result follows easily: there are n solutions for X_N and each solution for $Y_i = X_i - X_N$ yields a solution for X_i . Therefore there are $n \times \prod_{i=1}^{N-1} |a_{ii}(m)| = n \times |\text{Alex}_K(m)|$ solutions. \square

The main result of this section follows.

Proposition 7.3. *Let K_1 and K_2 be knots with different Alexander polynomials $\text{Alex}_{K_1}(m) \neq \text{Alex}_{K_2}(m)$. Assume furthermore that their colouring matrices are both equivalent to a triangular matrix with only zeros in the final row. Then there is a linear finite Alexander quandle that distinguishes them by counting colourings.*

Proof. Since the Alexander polynomials are different there will be an infinite number of values of m such that $|\text{Alex}_{K_1}(m)| \neq |\text{Alex}_{K_2}(m)|$. Let A denote the equivalent $N_1 \times N_1$ triangular matrix for K_1 and B denote the equivalent $N_2 \times N_2$ triangular matrix for K_2 . There will also be an infinite number of values of m such that additionally $a_{ii}(m) \neq 0$, where $a_{ii}(m), i = 1, \dots, N_1 - 1$ are all but the final diagonal entries of A and also such that $b_{ii}(m) \neq 0$, where $b_{ii}(m), i = 1, \dots, N_2 - 1$ are all but the final diagonal entries of B . Such values of m are those that are not solutions of any of the equations $a_{ii}(m) = 0, i = 1, \dots, N_1 - 1, b_{ii}(m) = 0, i = 1, \dots, N_2 - 1$, or $\text{Alex}_{K_1}(m) = \text{Alex}_{K_2}(m)$ or $\text{Alex}_{K_1}(m) = -\text{Alex}_{K_2}(m)$. We need a final condition on m , namely that m is coprime with all $a_{ii}(m), i = 1, \dots, N_1 - 1$ and all $b_{ii}(m), i =$

$1, \dots, N_2 - 1$. To find such an m multiply (if needed) the diagonal entries by a power of m so that they become polynomials without negative powers of m and a non-zero constant term. An m that is coprime with the constant term of such a normalized $p(m)$ is coprime with $p(m)$. There will also be an infinite number of such values of m , for example the prime numbers that are coprime with the constant terms of $a_{ii}(m), i = 1, \dots, N_1 - 1$ and $b_{ii}(m), i = 1, \dots, N_2 - 1$. This m is coprime with M , given by $M = \prod_{i=1}^{N_1-1} |a_{ii}(m)| \times \prod_{i=1}^{N_2-1} |b_{ii}(m)| = |\text{Alex}_{K_1}(m)| \times |\text{Alex}_{K_2}(m)|$.

Choose n to be a multiple of M bigger than m and coprime with m . If $M > m$ then choose $n = M$. Otherwise multiply M by the first prime bigger than m . We now have coprime m and n , with $1 < m < n$, satisfying the conditions of proposition 7.2 for both knots K_1 and K_2 . Therefore the number of colourings of K_1 and K_2 using the linear finite Alexander quandle $Q = \mathbb{Z}_n[t, t^{-1}]/(t - m)$ satisfies

$$C_Q(K_1) = n \times |\text{Alex}_{K_1}(m)| \neq n \times |\text{Alex}_{K_2}(m)| = C_Q(K_2). \quad \square$$

We now illustrate the previous result with some examples.

Example 7.4. First we consider distinguishing two Type I knots K_1 and K_2 with different Alexander polynomials. Their triangularized matrices have one final row of zeros, the Alexander polynomial in the penultimate diagonal entry and 1's in all other diagonal entries. We have to find an m such that a) $|\text{Alex}_{K_1}(m)| \neq |\text{Alex}_{K_2}(m)|$; b) all diagonal entries except the last are non-zero and c) m is coprime with all diagonal entries.

In this case this simplifies to a) $|\text{Alex}_{K_1}(m)| \neq |\text{Alex}_{K_2}(m)|$; b) $\text{Alex}_{K_1}(m) \neq 0$ and $\text{Alex}_{K_2}(m) \neq 0$ and c) m is coprime with $\text{Alex}_{K_1}(m)$ and $\text{Alex}_{K_2}(m)$.

Take for example knots $K_1 = 3_1$ with Alexander polynomial $1 - m + m^2$ and $K_2 = 4_1$ with Alexander polynomial $1 - 3m + m^2$. Solving $1 - m + m^2 = 1 - 3m + m^2$ and $1 - m + m^2 = -(1 - 3m + m^2)$ we obtain $m = 0$ and $m = 1$. The equations $1 - m + m^2 = 0$ and $1 - 3m + m^2 = 0$ have no integer solutions. So any value of $m > 1$ satisfies a) and b). Condition c) is fulfilled for m coprime with the constant term of each Alexander polynomials that happens to be 1 in both cases. Therefore any $m > 1$ will do. Choose $m = 2$. Now $M = |\text{Alex}_{K_1}(2)| \times |\text{Alex}_{K_2}(2)| = 3 \times 1 = 3$. Since $3 > 2$ we can choose $n = 3$ and by Proposition 7.3, the quandle with $m = 2, n = 3$ should distinguish the two knots. Indeed, the number of colourings of 3_1 is $3 \times \text{gcd}(3, 3) = 3 \times 3 = 9$ and the number of colourings of 4_1 is $3 \times \text{gcd}(-1, 3) = 3 \times 1 = 3$.

Example 7.5. Consider now knots $K_1 = 10_{137}$ (Type I) with Alexander polynomial $1 - 6m + 11m^2 - 6m^3 + m^4$ and $K_2 = 10_{155}$ (Type II) with Alexander polynomial $1 - 3m + 5m^2 - 7m^3 + 5m^4 - 3m^5 + m^6$. The relevant entries of the triangularized colouring matrix (2.1) for 10_{155} are the following:

$$\begin{pmatrix} -1 + 2m - m^2 + m^3 & 0 \\ 0 & -1 + m - 2m^2 + m^3 \end{pmatrix}$$

The equation $1 - 6m + 11m^2 - 6m^3 + m^4 = 1 - 3m + 5m^2 - 7m^3 + 5m^4 - 3m^5 + m^6$ has two integer solutions, $m = -1$ and $m = 0$, the equation $1 - 6m + 11m^2 - 6m^3 +$

$m^4 = -(1 - 3m + 5m^2 - 7m^3 + 5m^4 - 3m^5 + m^6)$ has one integer solution $m = 1$, and the equations $1 - 6m + 11m^2 - 6m^3 + m^4 = 0$, $\alpha_1(m) = -1 + 2m - m^2 + m^3 = 0$ and $\alpha_2(m) = -1 + m - 2m^2 + m^3 = 0$ have no integer solutions. We can choose $m = 2$ because it is coprime with the constant terms of $1 - 6m + 11m^2 - 6m^3 + m^4$, $-1 + 2m - m^2 + m^3$ and $-1 + m - 2m^2 + m^3$. Thus, by the proof of Proposition 7.3, we can choose the linear Alexander quandle with $m = 2$ and $n = 7$, since $M = |\text{Alex}_{K_1}(2)| \times |\text{Alex}_{K_2}(2)| = 1 \times 7 = 7 > 2$.

We now confirm that this quandle distinguishes the knots. The number of colourings of knot 10_{137} for this quandle is $C_Q(K_1) = 7 \times \gcd(1, 7) = 7$. For $m = 2$, $\alpha_1(2) = 7$ and $\alpha_2(2) = 1$. Therefore $C_Q(K_2) = 7 \times \gcd(1, 7) \times \gcd(0, \gcd(7, 7)) = 7 \times 1 \times \gcd(7, 7) = 7 \times 7 = 49$.

Example 7.6. Finally, consider knots $K_1 = 8_{18}$ (Type II) with Alexander polynomial $1 - 5m + 10m^2 - 13m^3 + 10m^4 - 5m^5 + m^6$ and $K_2 = 9_{37}$ (Type II) with Alexander polynomial $2 - 11m + 19m^2 - 11m^3 + 2m^4$. The significant parts (2.1) of their triangularized colouring matrices are the following:

$$8_{18} : \begin{pmatrix} -1 + m - m^2 & m - m^2 + m^3 \\ 0 & 1 - 4m + 5m^2 - 4m^3 + m^4 \end{pmatrix}$$

$$9_{37} : \begin{pmatrix} 1 - 2m & m + m^2 \\ 0 & -2 + 7m - 5m^2 + m^3 \end{pmatrix}$$

The equation $|\text{Alex}_{K_1}(m)| = |\text{Alex}_{K_2}(m)|$ has $m = -1$ and $m = 1$ as its only integer roots. The equations $-1 + m - m^2 = 0$, $1 - 4m + 5m^2 - 4m^3 + m^4 = 0$ and $1 - 2m = 0$ have no integer roots. The equation $-2 + 7m - 5m^2 + m^3 = 0$ has $m = 2$ as its only integer root. Therefore we may choose $m = 3$ which moreover is coprime with both constant terms in the diagonal entries of both matrices. The product $M = |\text{Alex}_{K_1}(m)| \times |\text{Alex}_{K_2}(m)|$ yields $49 \times 5 = 245 > 3$ so, according to Proposition 7.3 we can choose $m = 3$ and $n = 245$.

We now confirm that this quandle distinguishes the knots. For the knot 8_{18} , $\alpha_1(3) = -7$, $\beta_1(3) = 21$ and $\alpha_2(3) = 7$, so $C_Q(K_1) = 245 \times \gcd(7, 245) \times \gcd(21 \frac{245}{\gcd(7, 245)}, \gcd(7, 245)) = 245 \times 7 \times \gcd(21 \times 35, 7) = 245 \times 7 \times 7 = 12005$.

On the other hand, for the knot 9_{37} , $\alpha_1(3) = -5$, $\beta_1(3) = 12$ and $\alpha_2(3) = 1$, so $C_Q(K_2) = 245 \times \gcd(1, 245) \times \gcd(12 \frac{245}{\gcd(1, 245)}, \gcd(-5, 245)) = 245 \times 1 \times \gcd(12 \frac{245}{1}, 5) = 245 \times 1 \times 5 = 1225$.

8. Conclusions and Further Work

We have presented general expressions for the number of colourings of prime knots using finite linear Alexander quandles when the colouring matrices can be triangularized into one of two forms. We have obtained such a triangular form for all but 12 knots with up to ten crossings. In 5 exceptional cases we prove that no triangular form exists. We were also able to make statements, for knots with the

same Alexander polynomial, about when the number of colourings distinguishes or does not distinguish the knots.

For knots with different Alexander polynomials and colouring matrices that are triangularizable, we show that they are distinguishable by colourings. We conjecture that the condition on the triangularizability of the colouring matrices can be dropped and that knots with different Alexander polynomials can always be distinguished by colourings. This will be investigated in future work.

Some of our results apply in complete generality, and it is clear that similar methods to those we have used in concrete examples could be applied to knots having more than ten crossings. A natural direction for future work is to try and find general expressions for the number of solutions when the reduced colouring matrix is non-triangular, or of a more general triangular type. This may also help to elucidate why we were unable to prove that four knots with non-properly factorizable Alexander polynomials have non-triangularizable colouring matrices.

Acknowledgements. We thank Pedro Lopes from the Department of Mathematics of IST for ideas that led to this research; Eduardo Marques de Sá from the Department of Mathematics of the University of Coimbra for help with results on the diagonalization of matrices and J. Scott Carter from the Department of Mathematics and Statistics of the University of South Alabama for useful discussions in Lisbon.

F. M. Dionísio was supported by Fundação para a Ciência e a Tecnologia, Portugal, through programs PEst-OE/EEI/LA0008/2013 and UID/EEA/50008/2013.

R. Picken was supported in part by Fundação para a Ciência e a Tecnologia, Portugal, through projects UID/MAT/04459/2013 and EXCL/MAT-GEO/0222/2012.

A Non-triangularized matrices

We were unable to triangularize the colouring matrices for the following 12 knots where we display the relevant entries in the penultimate two rows and columns as in section 3. Note that the colouring matrices of 9_{41} and 10_{108} can be triangularized if more general column operations are allowed yielding Type II and Type I matrices respectively.

$$\begin{aligned}
 9_{35} &: \begin{pmatrix} 2 - m & -1 - m \\ -3 & -2 + 7m \end{pmatrix} \\
 9_{38} &: \begin{pmatrix} -1 + m + m^2 & 4 - 4m \\ -5 + 7m & 15 - 19m + 5m^2 \end{pmatrix} \\
 9_{41} &: \begin{pmatrix} -1 + m^2 & 4m - 3m^2 \\ -4 + 3m & 13m - 12m^2 + 3m^3 \end{pmatrix} \\
 9_{47} &: \begin{pmatrix} -1 + 4m - m^2 & -2 - m - m^2 + m^3 \\ 2 - 7m & 3 + 4m + 2m^2 - m^4 \end{pmatrix} \\
 9_{48} &: \begin{pmatrix} 2 - m & 2 - 8m + 7m^2 - m^3 \\ 3 & 3 - 10m + 2m^2 + 5m^3 - m^4 \end{pmatrix} \\
 9_{49} &: \begin{pmatrix} -2 - m + m^2 & 3 - m - m^2 - 2m^3 \\ 3 - 2m & -3 + 3m + m^2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 10_{69} &: \begin{pmatrix} m - m^2 - m^3 & 1 - 6m + 10m^2 - 2m^3 \\ 1 - 2m + 2m^2 & -1 + 2m - 4m^2 + m^3 \end{pmatrix} \\
 10_{101} &: \begin{pmatrix} 3 - 5m + 3m^2 & -3 + 11m - 15m^2 + 7m^3 \\ 1 - m + m^3 & -1 + 3m - 5m^2 + 2m^3 \end{pmatrix} \\
 10_{108} &: \begin{pmatrix} -3m - m^2 & -3 + 8m - 10m^2 + 12m^3 - 10m^4 + 6m^5 - 2m^6 \\ -11m & -11 + 33m - 47m^2 + 57m^3 - 51m^4 + 34m^5 - 14m^6 + 2m^7 \end{pmatrix} \\
 10_{115} &: \begin{pmatrix} 1 - m + m^2 & -3 + 3m - m^2 \\ 2m & 1 - 14m + 17m^2 - 8m^3 + m^4 \end{pmatrix} \\
 10_{157} &: \begin{pmatrix} 4 - 3m & -7 + 12m - 9m^2 + 6m^3 - m^4 \\ -1 + m^2 & 2 - 3m + m^2 - m^3 \end{pmatrix} \\
 10_{160} &: \begin{pmatrix} -3m & 1 + m + 3m^2 - 3m^3 - 2m^4 + m^5 \\ -2m + m^2 & 1 - m + 3m^2 - 4m^3 + m^4 \end{pmatrix}
 \end{aligned}$$

B Type II colouring matrices

In this section we list the relevant entries of the 21 Type II matrices obtained from colouring matrices using row operations and swapping of columns. Note that the colouring matrices of knots 10₁₀₆ and 10₁₄₇ can be simplified (become Type I) if more general column operations are allowed. The relevant entries of Type II matrices are :

$$\begin{bmatrix} \alpha_1(m) & \beta_1(m) \\ 0 & \alpha_2(m) \end{bmatrix}$$

Given a linear Alexander quandle $Q = \mathbb{Z}_n[t, t^{-1}] / (t - m)$ the number of colourings is

$$C_Q(K) = n \times \gcd(\alpha_2(m), n) \times \gcd(\beta_1(m), \frac{n}{\gcd(\alpha_2(m), n)}), \gcd(\alpha_1(m), n).$$

$$\begin{aligned}
 8_{18} &: \begin{pmatrix} -1 + m - m^2 & m - m^2 + m^3 \\ 0 & 1 - 4m + 5m^2 - 4m^3 + m^4 \end{pmatrix} \\
 9_{37} &: \begin{pmatrix} 1 - 2m & m + m^2 \\ 0 & -2 + 7m - 5m^2 + m^3 \end{pmatrix} \\
 9_{40} &: \begin{pmatrix} 1 - 4m + 5m^2 - 4m^3 + m^4 & 0 \\ 0 & -1 + 3m - m^2 \end{pmatrix} \\
 9_{46} &: \begin{pmatrix} 2 - m & -3 \\ 0 & 1 - 2m \end{pmatrix} \\
 10_{61} &: \begin{pmatrix} -1 + m - m^2 & 1 + m - m^2 \\ 0 & 2 - 3m + m^2 - 3m^3 + 2m^4 \end{pmatrix} \\
 10_{63} &: \begin{pmatrix} -1 + m - m^2 & 2m^2 \\ 0 & -5 + 9m - 5m^2 \end{pmatrix} \\
 10_{65} &: \begin{pmatrix} -1 + m - m^2 & -1 + m + m^2 \\ 0 & 2 - 5m + 7m^2 - 5m^3 + 2m^4 \end{pmatrix} \\
 10_{74} &: \begin{pmatrix} -1 + 2m & 0 \\ 0 & -4 + 8m - 7m^2 + 2m^3 \end{pmatrix} \\
 10_{75} &: \begin{pmatrix} 1 - 4m + 3m^2 - m^3 & -1 + 2m \\ 0 & 1 - 3m + 4m^2 - m^3 \end{pmatrix} \\
 10_{98} &: \begin{pmatrix} -2 + 3m - 3m^2 + m^3 & 1 - m + m^2 \\ 0 & -1 + 3m - 3m^2 + 2m^3 \end{pmatrix} \\
 10_{99} &: \begin{pmatrix} 1 - 2m + 3m^2 - 2m^3 + m^4 & 0 \\ 0 & 1 - 2m + 3m^2 - 2m^3 + m^4 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
10_{103} &: \begin{pmatrix} -1 + 2m - 2m^2 & -1 + 2m - m^2 + m^3 \\ 0 & 2 - 4m + 5m^2 - 3m^3 + m^4 \end{pmatrix} \\
10_{106} &: \begin{pmatrix} 1 - m + 2m^2 - m^3 & -m + 2m^2 - 2m^3 + m^4 \\ 0 & -1 + 3m - 4m^2 + 4m^3 - 2m^4 + m^5 \end{pmatrix} \\
10_{122} &: \begin{pmatrix} 1 - 4m + 5m^2 - 4m^3 + m^4 & -1 + 3m - m^2 \\ 0 & -2 + 3m - 2m^2 \end{pmatrix} \\
10_{123} &: \begin{pmatrix} 1 - 3m + 3m^2 - 3m^3 + m^4 & 0 \\ 0 & 1 - 3m + 3m^2 - 3m^3 + m^4 \end{pmatrix} \\
10_{140} &: \begin{pmatrix} 1 - m + m^2 & -2m^2 \\ 0 & 1 - m + m^2 \end{pmatrix} \\
10_{142} &: \begin{pmatrix} -1 + m - m^2 & 1 + m - m^2 \\ 0 & -2 + m + m^2 + m^3 - 2m^4 \end{pmatrix} \\
10_{144} &: \begin{pmatrix} -1 + m - m^2 & 2m \\ 0 & -3 + 7m - 3m^2 \end{pmatrix} \\
10_{147} &: \begin{pmatrix} 1 - 2m & -1 + 4m - 3m^2 \\ 0 & 2 - 3m + 3m^2 - m^3 \end{pmatrix} \\
10_{155} &: \begin{pmatrix} -1 + 2m - m^2 + m^3 & 0 \\ 0 & -1 + m - 2m^2 + m^3 \end{pmatrix} \\
10_{164} &: \begin{pmatrix} -1 + m - m^2 & 3 - 6m + 4m^2 - m^3 \\ 0 & 1 - 4m + 7m^2 - 4m^3 + m^4 \end{pmatrix}
\end{aligned}$$

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