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A New Kind of Slant Helix in Lorentzian $(n+2)-$ Spaces

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Abstract.In this paper, we introduce a new kind of slant helix for null curves called null W_n-slant helix and we give a definition of new harmonic curvature functions of a null curve in terms of W_n in $(n+2)-$dimensional Lorentzian space M_1^{n+2} (for $n>3$). Also, we obtain a characterization such as:

"The curve α is a null W_n-slant helix $\Leftrightarrow H_n^{'}-k_1H_{n-1}-k_2H_{n-3}=0$"

where H_n, H_{n-1} and H_{n-3} are harmonic curvature functions and k_1, k_2 are the Cartan curvature functions of the null curve α.

1. Introduction

A null curve create many different difficulties because its arc length vanishes. So, it is impossible to do normalize the tangent vector field in the usual way and null curves have differences according to spacelike curves and timelike curves.

W. B. Bonnor [17], studied geometry of null curves in Minkowski spacetime and he proved the fundamental existence and congruence theorems of the null curves in

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Minkowski spacetime. Later, A. Bejancu gave a method for the general situation of the null curves in Lorentzian manifold. Ferrandez, *et. al.* [2], studied null curves in the Lorentz-Minkowski spaces by analysing the Frenet equations associated to different screen distributions. They found the general frame for the null curves with reference of Bonnor's study and called Cartan frame in the Minkowski spacetime. Moreover, they characterized the null helices in n-dimensional Lorentzian space form. Çöken and Çiftçi [1], reconstructed the Cartan frame of the null curve for an arbitrary parameter and characterized pseudo-spherical null curves in Minkowski spacetime .

In mathematical aspects of relativity theory, a null geodesic is the path that a massless particle, such as a photon, follows. Physical significance of null curves is obvious from [15]. They analyzed the motion of classical relativistic string in flat complex ten-dimensional spacetime and found general solution for the equations of motion and then they established two parametrization theorems for null curves in ten dimensions.

Izumiya and Takeuchi [16], introduced the new concept of slant helix in Euclidean 3−space. It was defined by the property that its principal normal vector field makes a constant angle with a fixed direction. Then Camcı [7], gave some characterizations of generalized helices and Turgut [6], generalized slant helices in higher dimensional Euclidean space. Gök, *et. al.* characterized V_n-slant helices in Euclidean $n-$space [12] and Minkowski $n-$space [13] using harmonic curvature functions defined by Özdamar and Hacisalihoğlu [8]. In [5], the authors gave some new characterizations for inclined curves and slant helices in $n-$dimensional Euclidean space E^n. Many papers about helices and slant helices have been published for null and non-null curves in Minkowski space ([2, 3, 4]).

In the present study, we define a new type slant helices called null W_n-slant helix and we give a characterization Theorem (3.8) for null W_n-slant helix using its harmonic curvature functions. Also, we previously gave several examples of the null $W-$slant helices in E_1^3 (*see* [9]).

2. Preliminaries

In this section, we recall the notion of the null curve in the Lorentzian manifold [14].

Let be a real $(n+2)-$dimensional Lorentzian manifold $M_1^{n+2} = (M, g)$ is called Minkowski space defined by a Minkowski metric

$$g(X,Y) = -x_0y_0 + \lim_{i=1}^{n+1} x_iy_i.$$

for each $X = (x_0, x_1, ..., x_{n+1})$, $Y = (y_0, y_1, ..., y_{n+1})$.

Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{n+2}$ be a null curve in M locally given by

$$\alpha^i = \alpha^i(t), \quad t \in I \subset \mathbb{R}, \quad i \in \{1, 2, ..., n+2\}$$

for a coordinate neighbourhood U on α. Then the tangent vector field

$$\frac{d\alpha}{dt} = (\frac{d\alpha^1}{dt}, ..., \frac{d\alpha^{n+2}}{dt})$$

on U satisfies the condition

$$g\left(\frac{d\alpha}{dt}, \frac{d\alpha}{dt}\right) = 0.$$

We denote by $T\alpha$ the tangent bundle of α and $T\alpha^{\perp}$ is defined as follows

$$T\alpha^{\perp} = \bigcup T_p\alpha^{\perp} \quad , \quad T_p\alpha^{\perp} = \{v_p \in T_pM;\ g(v_p, \xi_p) = 0\},$$

where ξ_p is a null tangent vector field at any $p \in \alpha$. Clearly, $T\alpha^{\perp}$ is a vector bundle over α of *rank* $(n+1)$. Since ξ_p is a null, it follows that the tangent bundle $T\alpha$ is a vector sub bundle of $T\alpha^{\perp}$.

Suppose $S(T\alpha^{\perp})$ is the complementary vector sub bundle to $T\alpha$ in $T\alpha^{\perp}$.

$$T\alpha^{\perp} = T\alpha \perp S(T\alpha^{\perp}) \tag{2.1}$$

where $\perp$ means the orthogonal direct sum. It follows that $S(T\alpha^{\perp})$ is a non-degenerate $n-$dimensional vector sub bundle of the null curve α which is said to be the screen vector bundle of α. We have

$$TM\,|_{\alpha} = S(T\alpha^{\perp}) \perp S(T\alpha^{\perp})^{\perp},$$

where $S(T\alpha^{\perp})^{\perp}$ is a $2-$dimensional complementary orthogonal vector sub bundle to $S(T\alpha^{\perp})$ in $TM\,|_{\alpha}$.

Theorem 2.1.([14]) *Let α be a null curve of a semi-Riemannian manifold $(M, \langle , \rangle)$ and $S(T\alpha^{\perp})$ be a screen vector bundle of α. Then there exists a unique vector bundle $ntr(\alpha)$ over α of rank* 1, *such that on each coordinate neighbourhood $U \subset \alpha$ there is a unique $N \in \Gamma(ntr(\alpha)\,|_U)$ satisfying*

$$\langle \alpha'(t), N\rangle = 1, \quad \langle N, N\rangle = 0, \quad \langle N, X\rangle = 0, \quad \forall X \in \Gamma(S(T\alpha^{\perp})\,|_U).$$

We consider the equality

$$tr(\alpha) = ntr(\alpha) \perp S(T\alpha^{\perp}). \tag{2.2}$$

from Eq.(2.1) and Eq.(2.2) then we have the following sum

$$TM\,|_{\alpha} = T\alpha \oplus tr(\alpha) = (T\alpha \oplus ntr(\alpha)) \perp S(T\alpha^{\perp}).$$

Theorem 2.2.([14]) *Let $\alpha : I \subset \mathbb{R} \to M_1^{n+2}$ be a null curve parametrized by the pseudo-arc s (that is $g(\alpha''(s), \alpha''(s)) = 1$) such that $\{\alpha', \alpha'', ..., \alpha^{(n+2)}\}$ is a basis of*

$TM\big|_{\alpha(s)}$ for all s and $\{L, N, W_1, ..., W_n\}$ be the Cartan frame along α where L, N are null vector fields, $\{W_i\}_{|i=1,..,n}$ are spacelike vector fields. Then there exists only one Cartan frame satisfying the equations

$$\left\{\begin{array}{c} L' = \alpha'' = W_1, \\ N' = k_1 W_1 + k_2 W_2, \\ W_1' = -k_1 L - N, \\ W_2' = -k_2 L + k_3 W_3, \\ \vdots \\ W_i' = -k_i W_{i-1} + k_{i+1} W_{i+1}, \quad i \in \{3, ..., n-1\} \\ W_n' = -k_n W_{n-1}, \end{array}\right. \tag{2.3}$$

where the functions $\{k_1, k_2, k_3, ..., k_n\}$ are called the Cartan curvatures of the null curve α.

3. Null W_n- Slant Helix and Its Harmonic Curvature Functions

In this section, we give some characterizations for a null W_n-slant helix in terms of the n_th Cartan vector field W_n of the null curve α in M_1^{n+2} (for $n > 3$).

Definition 3.1. Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{n+2}$ be a null curve in M_1^{n+2} and X be a fixed vector field. If the following equality is provided for all $s \in I$,

$$g\left(W_n(s), X\right) = f(\varphi)\ , \ \ f(\varphi) = \text{constant},$$

then the curve α is called a null W_n-slant helix in M_1^{n+2} where $W_n(s)$ is the last Cartan vector field of the curve α at its point $\alpha(s)$ and $f(\varphi)$ is a constant function between the vectors W_n and X.

Definition 3.2. Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{n+2}$ be a null curve parametrized by the pseudo arc-lengthed and $\{L, N, W_1, ..., W_n\}$, $\{k_1(s), k_2(s), ..., k_n(s)\}$ be the Cartan frame and Cartan curvature functions of the null curve α, respectively. Harmonic curvature functions of the null curve α are defined by

$$H_i : I \subset \mathbb{R} \longrightarrow \mathbb{R}$$

$$H_i = \left\{\begin{array}{ccc} 0 & , & i = 0 \\ \frac{k_n}{k_{n-1}} & , & i = 1 \\ -\frac{1}{k_{n-i}}\left\{H_{i-1}^{'} - k_{n-i+1} H_{i-2}\right\} & , & i = 2, 3, ..., n-2 \\ H_{n-2}' & , & i = n-1 \\ -\left\{H_{n-1}^{'} + k_1 H_{n-2}\right\} & , & i = n. \end{array}\right. \tag{3.1}$$

Theorem 3.3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{n+2}$ be a null W_n-slant helix in M_1^{n+2}. There is the following relationship between the functions H_i and k_i*

$$k_1 = -\frac{\{H'_{n-1} + H_n\}}{H_{n-2}},$$

$$k_{n-r} = -\frac{\left\{\sum_{i=1}^{r-1} H_i^2\right\}'}{2H_{r-1}H_r}, \quad r = 2, 3, ..., n-2.$$

where H_i and k_i denote the Harmonic curvature functions and the Cartan curvature functions of the curve α, respectively.

Proof. We will prove the relation theorem by induction method.

For $r = 2$, from Definition (3.2), we have

$$k_{n-2} = -\frac{H'_1}{H_2}.$$

Using the last equation, we obtain

$$k_{n-3} = -\frac{\{H_1H'_1 + H_2H'_2\}}{H_2H_3}.$$

Now, assume that the theorem is true for $r = p$ and then let us prove that the theorem is also true for $r = p + 1$.

For $r = p$,

$$k_{n-p} = -\frac{\left\{\sum_{i=1}^{p-1} H_i^2\right\}'}{2H_{p-1}H_p}.$$

From eq. (3.1), we give

$$k_{n-(p+1)} = -\left\{\frac{H'_p - k_{n-p}H_{p-1}}{H_{p+1}}\right\}.$$

The curvature function k_{n-p} is placed on the above equation, we obtain

$$k_{n-(p+1)} = -\frac{\left\{\sum_{i=1}^{p} H_i^2\right\}'}{2H_pH_{p+1}}$$

which completes the proof. □

Theorem 3.4. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{n+2}$ be a null curve with the harmonic curvature functions $\{H_1, H_2, ..., H_n\}$ and $\{H'_1, H'_2, ..., H'_n\}$ be the differentiation with*

respect to s of them. There is the following relationship between the harmonic curvature functions and their derivatives

$$\begin{bmatrix} H_1' \\ H_2' \\ H_3' \\ H_4' \\ \vdots \\ H_{n-4}' \\ H_{n-3}' \\ H_{n-2}' \\ H_{n-1}' \\ H_n' \end{bmatrix} = \begin{bmatrix} 0 & -k_{n-2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ k_{n-2} & 0 & -k_{n-3} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{n-3} & 0 & -k_{n-4} & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{n-4} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & k_3 & 0 & -k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -k_1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & k_2 & 0 & k_1 & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ H_{n-4} \\ H_{n-3} \\ H_{n-2} \\ H_{n-1} \\ H_n \end{bmatrix}.$$

Proof. It is obvious from the eq. (3.1). □

Theorem 3.5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{n+2}$ be a null curve given by the Cartan frame $\{L, N, W_1, ..., W_n\}$ and harmonic curvature functions $\{H_1, H_2, ..., H_n\}$.*
If the curve α is a null W_n-slant helix with its axis X, then we have

$$\begin{cases} g\left(W_{n-(i+1)}, X\right) = H_i g\left(W_n, X\right) \quad , \quad 0 \leq i \leq n-3 \\ g\left(W_1, X\right) = H_{n-1} g\left(W_n, X\right), \\ g\left(L, X\right) = H_{n-2} g\left(W_n, X\right), \\ g\left(N, X\right) = H_n g\left(W_n, X\right). \end{cases} \tag{3.2}$$

Proof. First, we prove the first equality in the eq. (3.2). Thus, we apply the induction method for the proof.

Since α is the null W_n-slant helix then we can write

$$g\left(W_n(s), X\right) = f(\varphi) \text{ , for } \forall s \in I \text{ .}$$

Differentiating the last equation and using the Cartan formulas, we get

$$-k_n g\left(W_{n-1}, X\right) = 0$$

where $k_n \neq 0$, then

$$g\left(W_{n-1}, X\right) = 0 \text{ .} \tag{3.3}$$

For the case $i = 1$, differentiating eq.(3.3), we have

$$-k_{n-1} g\left(W_{n-2}, X\right) + k_n g\left(W_n, X\right) = 0$$

and so eq.(3.1) gives us

$$g\left(W_{n-2},X\right)=H_1g\left(W_n,X\right).$$

Let us assume that Theorem (3.5) is truth for the case $i-1$. So,

$$g\left(W_{n-i},X\right)=H_{i-1}g\left(W_n,X\right). \tag{3.4}$$

Differentiating eq.(3.4), we have

$$g\left(W_{n-i}^{'},X\right)=H_{i-1}'g\left(W_n,X\right)$$

and using the Cartan equations

$$-k_{n-i}g\left(W_{n-i-1},X\right)+k_{n-i+1}g\left(W_{n-i+1},X\right)=H_{i-1}'g\left(W_n,X\right). \tag{3.5}$$

Consequently, Theorem (3.5) is truth for the case $i-2$, this means that

$$g\left(W_{n-i+1},X\right)=H_{i-2}g\left(W_n,X\right). \tag{3.6}$$

From equations (3.5) and (3.6), we obtain that

$$g\left(W_{n-(i+1)},X\right)=-\frac{1}{k_{n-i}}\left\{H_{i-1}'-k_{n-i+1}H_{i-2}\right\}g\left(W_n,X\right)$$

and then eq.(3.1) gives us

$$g\left(W_{n-(i+1)},X\right)=H_ig\left(W_n,X\right)\ \text{ for } i=0,1,\ldots,n-3.$$

Now, we show that the equations $g\left(W_1,X\right)=H_{n-1}g\left(W_n,X\right)$, $g\left(L,X\right)=H_{n-2}g\left(W_n,X\right)$ and $g\left(N,X\right)=H_ng\left(W_n,X\right)$.

It is obvius that $g\left(W_2,X\right)=H_{n-3}g\left(W_n,X\right)$. Differentiating this equation,we obtain

$$g\left(L,X\right)=-\frac{1}{k_2}\left\{H_{n-3}'-k_3H_{n-4}\right\}g\left(W_n,X\right)$$

and from Definition (3.2), we get

$$g\left(L,X\right)=H_{n-2}g\left(W_n,X\right),$$

it is obvious that $g\left(W_1,X\right)=H_{n-1}g\left(W_n,X\right)$. Again differentiating the last equation, we get

$$\begin{aligned}g\left(N,X\right)&=(-k_1H_{n-2}-H_{n-1}')g\left(W_n,X\right)\\&=H_ng\left(W_n,X\right).\end{aligned}$$

□

Corollary 3.6. *Let α be a null W_n-slant helix with Cartan frame $\{L, N, W_1, ..., W_n\}$ and harmonic curvature functions $\{H_1, H_2, ..., H_n\}$ in M_1^{n+2}. The axis of the null W_n-slant helix is given as follows*

$$X = \left\{ \begin{array}{c} H_nL + H_{n-2}N + H_{n-1}W_1 + H_{n-3}W_2 + \\ \cdots + H_2W_{n-3} + H_1W_{n-2} + W_n \end{array} \right\} f(\varphi). \tag{3.7}$$

Proof. If the null W_n-slant helix α has the axis X in M_1^{n+2}, then we can write

$$X = \delta L + \eta N + \sum_{i=1}^{n} \lambda_i W_i.$$

By using the definition of harmonic curvature functions, we get

$$\begin{gathered} \delta = g(N, X) = H_n g(W_n, X), \\ \eta = g(L, X) = H_{n-2} g(W_n, X), \\ \lambda_1 = g(W_1, X) = H_{n-1} g(W_n, X), \\ \lambda_2 = g(W_2, X) = H_{n-3} g(W_n, X), \\ \vdots \\ \lambda_{n-2} = g(W_{n-2}, X) = H_1 g(W_n, X), \\ \lambda_{n-1} = 0, \\ \lambda_n = g(W_n, X). \end{gathered}$$

Hence, the eq. (3.7) can be easily seen. □

Definition 3.7. Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{n+2}$ be a null curve. The sets $\{L, N, W_1, W_2, ..., W_n\}$ and $\{H_1, H_2, ..., H_n\}$ denote the Cartan frame, the harmonic curvature functions of the curve α, respectively. The vector

$$D = H_nL + H_{n-2}N + H_{n-1}W_1 + H_{n-3}W_2 + ... + H_2W_{n-3} + H_1W_{n-2} + W_n$$

is called the Darboux vector of the null W_n-slant helix α.

Theorem 3.8. (Main Theorem) *Let α be a null curve in M_1^{n+2} and $\{H_1, H_2, ..., H_n\}$ be harmonic curvature functions of the curve α. Then the following conditions are equivalent*

(1) *The curve α is a null W_n-slant helix.*

(2) *The Darboux vector D is a constant vector field.*

(3) $H_n^{'} - k_1 H_{n-1} - k_2 H_{n-3} = 0.$

(4) $2H_{n-2}H_n + H_{n-1}^2 + \sum_{i=1}^{n-3} H_i^2$ *is a nonzero constant and* $H_{n-2} \neq 0.$

Proof. (1) $\Rightarrow$ (2) : We assume that α is the null W_n-slant helix. From Corollary (3.6), we have

$$\begin{aligned} X &= \left\{ \begin{array}{c} H_nL + H_{n-2}N + H_{n-1}W_1 + H_{n-3}W_2 + ... \\ +H_2W_{n-3} + H_1W_{n-2} + W_n \end{array} \right\} f(\varphi) \\ &= Df(\varphi) \end{aligned}$$

where $f(\varphi)$ is a constant function and X is a fixed direction. Hence, D is a constant vector field.

(2) $\Rightarrow$ (3) : Since D is a constant vector field then, $D' = 0$. From Definition (3.7), we get derivative of the Darboux vector D with respect to s

$$\begin{aligned} D' &= H_n'L + H_nL' + H_{n-2}'N + H_{n-2}N' + H_{n-1}'W_1 + H_{n-1}W_1' + ... \\ &\quad +H_1'W_{n-2} + H_1W_{n-2}' + W_n'. \end{aligned}$$

By using the Cartan formulas eq. (2.3) and Definition (3.2), we have

$$H_n^{'} - k_1H_{n-1} - k_2H_{n-3} = 0.$$

(3) $\Rightarrow$ (4) : We assume that $H_n^{'} - k_1H_{n-1} - k_2H_{n-3} = 0$.
From eq. (3.1),

$$H_i = -\frac{1}{k_{n-i}} \left\{ H_{i-1}^{'} - k_{n-i+1}H_{i-2} \right\} \quad , \quad i \in \{2, 3, ..., n-2\}.$$

From the last equality, we can write

$$-k_{n-i}H_i = H_{i-1}^{'} - k_{n-i+1}H_{i-2} \quad , \quad i \in \{2, 3, ..., n-2\}$$

where if we take $i+1$ instead of i, we obtain

$$H_i^{'} = k_{n-i}H_{i-1} - k_{n-i-1}H_{i+1} \quad , \quad i \in \{1, 2, ..., n-3\}. \tag{3.8}$$

In eq. (3.8), we product with H_i

$$H_iH_i^{'} = k_{n-i}H_{i-1}H_i - k_{n-i-1}H_iH_{i+1} \quad , \quad i \in \{1, 2, ..., n-3\}. \tag{3.9}$$

So, we can write

$$\begin{aligned} \text{for } i &= 1,\ H_1H_1' = -k_{n-2}H_1H_2\ , \\ \text{for } i &= 2,\ H_2H_2^{'} = k_{n-2}H_1H_2 - k_{n-3}H_2H_3\ , \\ \text{for } i &= 3,\ H_3H_3^{'} = k_{n-3}H_2H_3 - k_{n-4}H_3H_4\ , \\ &\vdots \\ \text{for } i &= n-6,\ H_{n-6}H_{n-6}^{'} = k_6H_{n-7}H_{n-6} - k_5H_{n-6}H_{n-5}\ , \\ \text{for } i &= n-5,\ H_{n-5}H_{n-5}^{'} = k_5H_{n-6}H_{n-5} - k_4H_{n-5}H_{n-4}\ , \\ \text{for } i &= n-4,\ H_{n-4}H_{n-4}^{'} = k_4H_{n-5}H_{n-4} - k_3H_{n-4}H_{n-3}\ , \\ \text{for } i &= n-3,\ H_{n-3}H_{n-3}^{'} = k_3H_{n-4}H_{n-3} - k_2H_{n-3}H_{n-2}\ , \end{aligned}$$

and we have $H_{n-2}^{'} = H_{n-1}$, $H_{n-1}^{\prime} = -H_n - k_1 H_{n-2}$.

So, we show that by using an algebric calculus

$$H_1 H_1^{\prime} + H_2 H_2^{\prime} + ... + H_{n-4} H_{n-4}^{\prime} + H_{n-3} H_{n-3}^{\prime} +$$

$$H_{n-1} H_{n-1}^{\prime} + H_n H_{n-2}^{\prime} + H_{n-2} H_n^{\prime} = 0.$$

Consequently, the expression $2H_{n-2}H_n + H_{n-1}^2 + \sum_{i=1}^{n-3} H_i^2$ is a nonzero constant.

(4) $\Rightarrow$ (1) : Let $2H_{n-2}H_n + H_{n-1}^2 + \sum_{i=1}^{n-3} H_i^2$ be a nonzero constant.We obtain that

$$H_n H_{n-2}^{\prime} + H_{n-2} H_n^{\prime} + H_{n-1} H_{n-1}^{\prime} + H_1 H_1^{\prime} +$$
$$H_2 H_2^{\prime} + ... + H_{n-4} H_{n-4}^{\prime} + H_{n-3} H_{n-3}^{\prime} = 0.$$

From eq. (3.1), we have

$$H_n H_{n-1} + H_{n-2} H_n^{\prime} + H_{n-1} \{-H_n - k_1 H_{n-2}\} + H_1 \{-k_{n-2} H_2\}$$

$$+H_2 \{-k_{n-3} H_3 + k_{n-2} H_1\} + H_3 \{-k_{n-4} H_4 + k_{n-3} H_2\} +$$

$$+ \ldots + H_{n-4} \{-k_3 H_{n-3} - k_4 H_{n-5}\} + H_{n-3} \{-k_2 H_{n-2} + k_3 H_{n-4}\} = 0.$$

So, an algebric calculus show that

$$H_{n-2} \left\{ H_n^{'} - k_1 H_{n-1} - k_2 H_{n-3} \right\} = 0$$

and since $H_{n-2} \neq 0$, we obtain

$$H_n^{'} - k_1 H_{n-1} - k_2 H_{n-3} = 0.$$

From Definition (3.7),

$$D = H_n L + H_{n-2} N + H_{n-1} W_1 + H_{n-3} W_2 + ... + H_2 W_{n-3} + H_1 W_{n-2} + W_n.$$

And, differentiating of the Darboux vector field D, we obtain

$$D^{\prime}(s) = \left\{ H_n^{'} - k_1 H_{n-1} - k_2 H_{n-3} \right\} L(s) = 0.$$

Consequently, D is the constant vector field. From Corollary (3.6), we have

$$X(s) = D(s) f(\varphi)$$

Since X and D are the constant vector fields for all s, then $f(\varphi)$ is the constant function. Thus, the curve α is the null W_n-slant helix. □

Remark 3.9. Considering the above theorem we can easily obtain following corollary.

Corollary 3.10. *The curve α is the null W_n-slant helix with the harmonic curvature functions $\{H_1, H_2, ..., H_n\}$ in M_1^{n+2}. Then we have the following classification for the Darboux vector field D*

(i) D is spacelike vector field if the $H_{n-2}H_n$ is a positive,
(ii) D is timelike vector field if the $H_{n-2}H_n < 0$ and unequality $|H_{n-2}H_n| > H_{n-1}^2 + \sum_{i=1}^{n-3} H_i^2$ is satisfied,
(iii) D is lightlike vector field if the $H_{n-2}H_n < 0$ and equality $|H_{n-2}H_n| = H_{n-1}^2 + \sum_{i=1}^{n-3} H_i^2$ is satisfied.

*Proof.*Let α be a null W_n-slant helix in M_1^{n+2}. So, we can calculate from Definition (3.7),

$$g(D, D) = 2H_{n-2}H_n + H_{n-1}^2 + \sum_{i=1}^{n-3} H_i^2.$$

Having obtained the following conditions are satisfied

$$\begin{aligned} H_{n-2}H_n &< 0 \text{ and } |H_{n-2}H_n| = H_{n-1}^2 + \sum_{i=1}^{n-3} H_i^2, \\ H_{n-2}H_n &\geq 0, \\ H_{n-2}H_n &< 0 \text{ and } |H_{n-2}H_n| > H_{n-1}^2 + \sum_{i=1}^{n-3} H_i^2, \end{aligned}$$

then we say that D is a lightlike, spacelike and timelike vector, respectively. As a result, we can see that the causal character of the Darboux vector field depends on the harmonic curvature functions of the curve α. □

In [11], defined a generalized helix in $n-$dimensional space (n *odd*) as a curve satisfying that the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, ...$ are constants. It is also proven that a curve is a generalized helix if and only if there exists a fixed direction which makes constant angle with all the vectors of the Frenet frame. Clearly, $ccr-$curves are a subset of generalized helices in the sense of [10]. Via similar idea, if we can consider null $ccr-$curves which has constant curvature ratios $\frac{k_{i+1}}{k_i}$, where k_i are Cartan curvature functions of the null curve α.

Theorem 3.11. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{2m+2}$ be a null $ccr-$curve in M_1^{2m+2} given by the Cartan frame $\{L, N, W_1, ..., W_{2m}\}$ and the harmonic curvature functions $\{H_1, H_2, ..., H_{2m}\}$. The Harmonic curvature functions of the curve α are given by*

$$\begin{cases} H_{2i-1} = \prod_{k=1}^{i} \frac{k_{2m+1-(2k-1)}}{k_{2m+1-2k}} & , \quad 1 \leq i \leq m-1, \\ H_{2m-1} = 0 & , \\ H_{2i} = 0 & , \quad 1 \leq i \leq m. \end{cases}$$

Proof. We apply the induction method for the proof.
For the case of $i = 1$:
From eq. (3.1), we can write

$$\begin{aligned} H_2 &= -\frac{1}{k_{2m-2}} \{H_1' - k_{2m-1} H_0\} \\ &= -\frac{1}{k_{2m-2}} \left(\frac{k_{2m}}{k_{2m-1}}\right)'. \end{aligned}$$

Since the curve α is a $ccr-$curve then $\frac{k_{2m}}{k_{2m-1}}$ is a constant. So, the Harmonic curvature function H_2 is a vanish. Then, eq. (3.1) gives

$$H_3 = \frac{k_{2m}}{k_{2m-1}} \frac{k_{2m-2}}{k_{2m-3}}.$$

Let us assume that Theorem (3.11) is true for the case $i = p$ then, we have

$$H_{2p} = 0$$

and

$$H_{2p-1} = \frac{k_{2m}}{k_{2m-1}} \frac{k_{2m-2}}{k_{2m-3}} \frac{k_{2m-4}}{k_{2m-5}} \cdots \frac{k_{2m-(2p-4)}}{k_{2m-(2p-3)}} \frac{k_{2m-(2p-2)}}{k_{2m-(2p-1)}}.$$

From Definition (3.2), we obtain

$$H_{2p+1} = -\frac{1}{k_{2m-2p-1}} \left\{H_{2p}' - k_{2m-2p} H_{2p-1}\right\}.$$

Then the last equation reduce to

$$H_{2p+1} = \frac{k_{2m}}{k_{2m-1}} \frac{k_{2m-2}}{k_{2m-3}} \frac{k_{2m-4}}{k_{2m-5}} \cdots \frac{k_{2m-(2p-2)}}{k_{2m-(2p-1)}} \frac{k_{2m-2p}}{k_{2m-(2p+1)}}.$$

Also, from eq. (3.1) we obtain

$$\begin{aligned} H_{2m-1} &= H_{2m-2}' \\ &= 0. \end{aligned}$$

□

Corollary 3.12. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_1^{2m+2}$ be a null $W_{2m}-$slant helix in M_1^{2m+2}. If the ratios $\frac{k_4}{k_3}, \frac{k_6}{k_5}, ..., \frac{k_{2m-2}}{k_{2m-3}}, \frac{k_{2m}}{k_{2m-1}}$ are constants, then the axis of a null $W_{2m}-$slant helix α is*

$$D = H_{2m-3}W_2 + H_{2m-5}W_4 + ... + H_1W_{2m-2} + W_{2m}.$$

Proof. According to Definition (3.7) for $n = 2m$, we have

$$D = H_nL + H_{n-2}N + H_{n-1}W_1 + H_{n-3}W_2 + ... + H_2W_{n-3} + H_1W_{n-2} + W_n$$

and from Theorem (3.11), we get

$$D = H_{2m-3}W_2 + H_{2m-5}W_4 + ... + H_1W_{2m-2} + W_{2m}.$$

□

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