KYUNGPOOK Math. J. 56(2016), 979-991
http://dx.doi.org/10.5666/KMJ.2016.56.3.979
pISSN 1225-6951 eISSN 0454-8124

$\mathcal{Z}$ Tensor on N(k)-Quasi-Einstein Manifolds

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Abstract. The object of the present paper is to study N(k)-quasi-Einstein manifolds. We study an N(k)-quasi-Einstein manifold satisfying the curvature conditions $R(\xi, X) \cdot \mathcal{Z} = 0$, $\mathcal{Z}(X, \xi) \cdot R = 0$, and $P(\xi, X) \cdot \mathcal{Z} = 0$, where R, P and $\mathcal{Z}$ denote the Riemannian curvature tensor, the projective curvature tensor and $\mathcal{Z}$ tensor respectively. Next we prove that the curvature condition $C \cdot \mathcal{Z} = 0$ holds in an N(k)-quasi-Einstein manifold, where C is the conformal curvature tensor. We also study $\mathcal{Z}$-recurrent N(k)-quasi-Einstein manifolds. Finally, we construct an example of an N(k)-quasi-Einstein manifold and mention some physical examples.

1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = dimM \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n}g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to Besse ([3], p. 432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a

* Corresponding Author.
Received June 19, 2015; accepted July 6, 2016.
2010 Mathematics Subject Classification: 53C25.
Key words and phrases: k-nullity distribution, quasi-Einstein manifolds, N(k)-quasi-Einstein manifolds, $\mathcal{Z}$ tensor, projective curvature tensor, conformal curvature tensor.

natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([3], p. 432-433). For instance, every Einstein manifold belongs to the class of Riemannian or semi-Riemannian manifolds (M^n, g) realizing the following relation :

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1.2}$$

where a, b are smooth functions and η is a non-zero 1-form such that

$$g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = \eta(\xi) = 1 \tag{1.3}$$

for all vector fields X.

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold ([7]) if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (1.2). We shall call η the associated 1-form and the unit vector field ξ is called the generator of the manifold.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. Several authors have studied Einstein's field equations. For example, in [17], Naschie turned the tables on the theory of elementary particles and showed the expectation number of elementary particles of the standard model using Einstein's unified field equation. He also discussed possible connections between Gödel's classical solution of Einstein's field equations and E-infinity in [16]. Also quasi-Einstein manifolds have some importance in the general theory of relativity. For instance, the Robertson-Walker spacetimes are quasi Einstein manifolds. Further, quasi Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [10]. Perfect fluid spacetimes in n-dimensions subjected to the restriction $\nabla_m C^m_{jkl} = 0$, where C is the Weyl conformal curvature tensor, recently investigated in [27] by Mantica, Molinari and De (see also [28]).

The study of quasi Einstein manifolds was continued by Chaki ([8]), Guha ([18]), De and Ghosh ([11, 12]), Bejan ([1]), Debnath and Konar ([14]) and many others. The notion of quasi-Einstein manifolds have been generalized by several authors in several ways such as generalized Einstein manifolds ([2]), generalized quasi-Einstein manifolds ([4, 13]), mixed generalized quasi-Einstein manifolds[5] and many others. In recent papers Özgür studied super quasi-Einstein manifolds ([32]) and generalized quasi-Einstein manifolds ([33]). Also Nagaraja ([29]) studied N(k)-mixed quasi-Einstein manifolds.

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ of a Riemannian manifold M ([36]) is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being some smooth function. In a similar way we can define the k-nullity distribution $N(k)$ of a Lorentzian manifold. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then M is said to be an $N(k)$-quasi Einstein manifold ([30]). In fact k is not arbitrary as the following:

Lemma 1 ([30]). *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1} \quad . \tag{1.4}$$

Now, it is immediate to note that in an n-dimensional N(k)-quasi-Einstein manifold ([30])

$$R(X,Y)\xi = \frac{a+b}{n-1}[\eta(Y)X - \eta(X)Y], \tag{1.5}$$

which is equivalent to

$$R(X,\xi)Y = \frac{a+b}{n-1}[\eta(Y)X - g(X,Y)\xi] = -R(\xi,X)Y. \tag{1.6}$$

From (1.5) we get

$$R(\xi,X)\xi = \frac{a+b}{n-1}[\eta(X)\xi - X]. \tag{1.7}$$

In [30] it was shown that an n-dimensional conformally flat quasi Einstein manifold is an $N(\frac{a+b}{n-1})$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(\frac{a+b}{2})$-quasi Einstein manifold. Also in [31] Özgür, cited some physical examples of $N(k)$-quasi Einstein manifolds. Recently, Taleshian and Hosseinzadeh ([20, 35]), De, De and Gazi ([9]) studied some curvature conditions on N(k)-quasi-Einstein manifold. All these motivated us to study such a manifold.

The conformal curvature tensor play an important role in differential geometry and also in general theory of relativity. The Weyl conformal curvature tensor C of a Riemannian manifold (M^n, g) $(n > 3)$ is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY \\ &\qquad +S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.8}$$

where r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$. If the dimension $n = 3$, then the conformal curvature tensor vanishes identically. The conformal curvature tensor have been studied by several authors in several ways such as ([15, 19, 21, 22]) and many others.
The projective curvature tensor P in a Riemannian manifold (M^n, g) is defined by ([37])

$$P(X,Y)W = R(X,Y)W - \frac{1}{n-1}[S(Y,W)X - S(X,W)Y], \tag{1.9}$$

In 2012, Mantica and Molinari ([26]) defined a generalized (0,2) symmetric $\mathcal{Z}$ tensor given by

$$\mathcal{Z}(X, Y) = S(X, Y) + \phi g(X, Y), \tag{1.10}$$

where ϕ is an arbitrary scalar function. In Refs. [23], [26] and [24] various properties of the $\mathcal{Z}$ tensor were pointed out.
The derivation conditions $R(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot S = 0$ have been studied in [36], where R and S denote the curvature and Ricci tensor respectively. In [30], the derivation conditions $\bar{C}(\xi, X) \cdot R = 0$ and $\bar{C}(\xi, X) \cdot \bar{C} = 0$ on N(k)-quasi-Einstein manifold were studied, where $\bar{C}$ is the concircular curvature tensor. Moreover in [30], for an N(k)-quasi-Einstein manifold it was proved that $k = \frac{a+b}{n-1}$. *Özgür* in [31] studied the condition $R \cdot P = 0$, $P \cdot S = 0$ and $P \cdot P = 0$ for an N(k)-quasi-Einstein manifolds, where P denotes the projective curvature tensor and some physical examples of N(k)-quasi-Einstein manifolds are given. Again, in 2008, *Özgür* and Sular ([34]) studied N(k)-quasi-Einstein manifold satisfying $R \cdot C = 0$ and $R \cdot \tilde{C} = 0$, where C and $\tilde{C}$ represent the conformal curvature tensor and the quasi-conformal curvature tensor, respectively. Recently, Yildiz, De and Cetinkaya ([38]) studied quasi-conformally recurrent N(k)-quasi-Einstein manifold. This paper is a continuation of the previous studies.

The paper is organized as follows:
After preliminaries in Section 3, we study an N(k)-quasi-Einstein manifold satisfying the curvature condition $R(\xi, X) \cdot \mathcal{Z} = 0$. In the next two sections, we study an N(k)-quasi-Einstein manifold satisfying $\mathcal{Z}(X, \xi) \cdot R = 0$, and $P(\xi, X) \cdot \mathcal{Z} = 0$ respectively. Section 6 deals with the nature of the curvature condition $C \cdot \mathcal{Z} = 0$ in an N(k)-quasi-Einstein manifold. We study $\mathcal{Z}$-recurrent N(k)-quasi-Einstein manifolds in Section 7. Finally, we give some examples of N(k)-quasi-Einstein manifolds.

2. Preliminaries

From (1.2) and (1.3) it follows that

$$r = an + b \quad \textit{and} \quad QX = aX + b\eta(X)\xi,$$

$$S(X, \xi) = k(n-1)\eta(X),$$

where r is the scalar curvature and Q is the Ricci operator.
In an n-dimensional N(k)-quasi-Einstein manifold M, the generalized $\mathcal{Z}$ tensor takes the form

$$\mathcal{Z}(X, Y) = (a + \phi)g(X, Y) + b\eta(X)\eta(Y), \tag{2.1}$$

and scalar $\mathcal{Z}$ takes the form

$$\mathcal{Z} = (a + \phi)n + b. \tag{2.2}$$

Also,

$$\mathcal{Z}(\xi, Y) = (a + b + \phi)\eta(Y). \tag{2.3}$$

Also the projective curvature tensor P in an n-dimensional N(k)-quasi-Einstein manifold satisfies the following relations:

$$P(X, Y)\xi = 0, \tag{2.4}$$

$$P(\xi, X)Y = \frac{b}{n-1}[g(X, Y)\xi - \eta(X)\eta(Y)\xi], \tag{2.5}$$

$$\eta(P(X, Y)U) = \frac{b}{n-1}[g(Y, U)\eta(X) - g(X, U)\eta(Y)], \tag{2.6}$$

for all vector fields X, Y, U on M.
Again in an n-dimensional N(k)-quasi-Einstein manifold M, the conformal curvature tensor C satisfies

$$C(X, Y)Z = -\frac{b}{n-2}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \tag{2.7}$$

$$C(X, Y)\xi = -\frac{b}{n-2}[\eta(Y)X - \eta(X)Y], \tag{2.8}$$

$$\eta(C(X, Y)Z) = 0, \tag{2.9}$$

$$\eta(C(X, Y)\xi) = 0, \tag{2.10}$$

$$C(\xi, Y)Z = -\frac{b}{n-2}[\eta(Y)\eta(Z)\xi - \eta(Z)Y], \tag{2.11}$$

for all vector fields X, Y, Z on M.

3. N(k)-Quasi-Einstein Manifold Satisfying $R(\xi, X) \cdot \mathcal{Z} = 0$

Let us suppose that the manifold (M^n, g) be an N(k)-quasi-Einstein manifold. Then the condition $R(\xi, X) \cdot \mathcal{Z} = 0$ gives

$$\mathcal{Z}(R(\xi, X)Y, \xi) + \mathcal{Z}(Y, R(\xi, X)\xi) = 0. \tag{3.1}$$

In view of (1.6) and (2.1) we get

$$\mathcal{Z}(R(\xi, X)Y, \xi) = k(a + b + \phi)[g(X, Y) - \eta(X)\eta(Y)]. \tag{3.2}$$

Also, in view of (1.7) and (2.1) we obtain

$$\mathcal{Z}(Y, R(\xi, X)\xi) = k(a+\phi)[\eta(X)\eta(Y) - g(X,Y)]. \tag{3.3}$$

Thus equations (3.1), (3.2) and (3.3) together give

$$bk[g(X,Y) - \eta(X)\eta(Y)] = 0. \tag{3.4}$$

Since in a quasi-Einstein manifold $b \neq 0$, therefore, either $k = 0$ or, $g(X,Y) = \eta(X)\eta(Y)$. In the second case, from (1.2) it follows that the manifold becomes an Einstein manifold. This is a contradiction. Thus we have $k = 0$, that is, $a + b = 0$. Conversely, if $k = 0$, then in view of (1.6) and (1.7) the manifold satisfies $R(\xi, X) \cdot \mathcal{Z} = 0$.

Thus we can state the following:

Theorem 3.1. *An N(k)-quasi-Einstein manifold* (M^n, g) *satisfies the condition* $R(\xi, X) \cdot \mathcal{Z} = 0$ *if and only if* $a + b = 0$.

4. N(k)-Quasi-Einstein Manifold Satisfying $\mathcal{Z}(X, \xi) \cdot R = 0$

In this section we consider an n-dimensional N(k)-quasi-Einstein manifold (M^n, g) satisfying the condition

$$(\mathcal{Z}(X,\xi) \cdot R)(U,V)W = 0. \tag{4.1}$$

Now we have

$$(\mathcal{Z}(X,\xi) \cdot R)(U,V)W = ((X \wedge_{\mathcal{Z}} \xi) \cdot R)(U,V)W,$$

where the endomorphism $(X \wedge_{\mathcal{Z}} U)V$ is defined by

$$(X \wedge_{\mathcal{Z}} U)V = \mathcal{Z}(U,V)X - \mathcal{Z}(X,V)U.$$

Then

$$\begin{aligned} (\mathcal{Z}(X,\xi) \cdot R)(U,V)W = {} & ((X \wedge_{\mathcal{Z}} \xi)R)(U,V)W - R((X \wedge_{\mathcal{Z}} \xi)U, V)W \\ & - R(U, (X \wedge_{\mathcal{Z}} \xi)V)W - R(U,V)(X \wedge_{\mathcal{Z}} \xi)W. \end{aligned} \tag{4.2}$$

Then from (4.1) and (4.2) we have

$$\begin{aligned} & \mathcal{Z}(\xi, R(U,V)W)X - \mathcal{Z}(X, R(U,V)W)\xi - \mathcal{Z}(\xi, U)R(X,V)W \\ & + \mathcal{Z}(X,U)R(\xi,V)W - \mathcal{Z}(\xi,V)R(U,X)W + \mathcal{Z}(X,V)R(U,\xi)W \\ & - \mathcal{Z}(\xi,W)R(U,V)X + \mathcal{Z}(X,W)R(U,V)\xi = 0. \end{aligned} \tag{4.3}$$

Using (1.5), (1.6) and (2.1) in (4.3) and then taking inner product with ξ, we obtain

$$bk[g(X,U)\eta(V)\eta(W) - g(X,V)\eta(U)\eta(W)] = 0. \tag{4.4}$$

Putting $V = \xi$ in (4.4) we get

$$bk\eta(W)[g(X,U) - \eta(X)\eta(U)] = 0. \tag{4.5}$$

Since in a quasi-Einstein manifold $b \neq 0$, the 1-form η is non-zero and $g(X,U) - \eta(X)\eta(U) \neq 0$, from equation (4.5) it follows that $k = 0$. Again, if we take $k = 0$, then the converse is trivial.

Thus we can state the following:

Theorem 4.1. *An n-dimensional N(k)-quasi-Einstein manifold (M^n, g) satisfies the condition $\mathcal{Z}(X,\xi) \cdot R = 0$ if and only if $a + b = 0$.*

Therefore, by the Theorems 3.1 and 4.1 we can state the following corollary:

Corollary 4.1. *Let (M^n, g) be an n-dimensional N(k)-quasi-Einstein manifold. Then the following statements are equivalent:*

(i) $R(\xi, X) \cdot \mathcal{Z} = 0$,

(ii) $\mathcal{Z}(X,\xi) \cdot R = 0$,

(iii) $a + b = 0$,

for every vector field X on (M^n, g).

5. N(k)-Quasi-Einstein Manifold Satisfying $P(\xi, X) \cdot \mathcal{Z} = 0$

In this section we consider an n-dimensional N(k)-quasi-Einstein manifold (M^n, g) satisfying the condition

$$P(\xi, X) \cdot \mathcal{Z} = 0. \tag{5.1}$$

From the condition $P(\xi, X) \cdot \mathcal{Z} = 0$, we get

$$\mathcal{Z}(P(\xi, X)Y, U) + \mathcal{Z}(Y, P(\xi, X)U) = 0, \tag{5.2}$$

which in view of (2.5) gives

$$\begin{aligned}\frac{b}{n-1}[g(X,Y)Z(\xi,U) - \eta(X)\eta(Y)Z(\xi,U) + g(X,U)Z(Y,\xi)\\ -\eta(X)\eta(Y)Z(Y,\xi)] = 0.\end{aligned} \tag{5.3}$$

Since $b \neq 0$, using (2.3) we obtain

$$(a + b + \phi)[g(X,Y)\eta(U) + g(X,U)\eta(Y) - 2\eta(X)\eta(Y)\eta(U)] = 0, \tag{5.4}$$

which gives $a + b + \phi = 0$.

Thus we can state the following:

Theorem 5.1. *Let M be an n-dimensional N(k)-quasi-Einstein manifold. Then M satifies the condition $P(\xi, X) \cdot \mathcal{Z} = 0$ if and only if $a + b + \phi = 0$.*

6. The Nature of the Curvature Condition $C \cdot \mathcal{Z} = 0$ in an N(k)-Quasi-Einstein Manifold

In this section we consider an n-dimensional N(k)-quasi-Einstein manifold (M^n, g) satisfying the condition

$$(C(X,Y)\cdot \mathcal{Z})(U,V) = -\mathcal{Z}(C(X,Y)U,V) - \mathcal{Z}(U, C(X,Y)V). \tag{6.1}$$

Using (1.10) and (1.2) in (6.1) we get

$$\begin{aligned}(C(X,Y)\cdot \mathcal{Z})(U,V) &= -ag(C(X,Y)U,V) - b\eta(C(X,Y)U)\eta(V)\\ &\quad -ag(U,C(X,Y)V) - b\eta(U)\eta(C(X,Y)V)\\ &\quad -\phi g(C(X,Y)U,V) - \phi g(U,C(X,Y)V),\end{aligned} \tag{6.2}$$

from which we obtain

$$(C(X,Y)\cdot \mathcal{Z})(U,V) = -b[\eta(C(X,Y)U)\eta(V) + \eta(C(X,Y)V)\eta(U)]. \tag{6.3}$$

Using (2.9) in (6.3) we get

$$(C(X,Y)\cdot \mathcal{Z})(U,V) = 0.$$

Hence we can state the following:

Theorem 6.1. *In an n-dimensional N(k)-quasi-Einstein manifold (M^n, g), the relation $C \cdot \mathcal{Z} = 0$ holds for all vector fields X, Y, U, V on (M^n, g).*

7. $\mathcal{Z}$-Recurrent N(k)-Quasi-Einstein Manifolds

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) is said to be a $\mathcal{Z}$-recurrent manifold ([25]) if its $\mathcal{Z}$ tensor satisfies the condition

$$(\nabla_X \mathcal{Z})(U,V) = \eta(X)\mathcal{Z}(U,V), \tag{7.1}$$

where η is a non-zero 1-form.
We have

$$(\nabla_X \mathcal{Z})(U,V) = X\mathcal{Z}(U,V) - \mathcal{Z}(\nabla_X U, V) - \mathcal{Z}(U, \nabla_X V). \tag{7.2}$$

Using (7.1) and (7.2) we get

$$X\mathcal{Z}(U,V) - \mathcal{Z}(\nabla_X U, V) - \mathcal{Z}(U, \nabla_X V) = \eta(X)\mathcal{Z}(U,V). \tag{7.3}$$

Putting $U = V = \xi$ in (7.3) we obtain

$$(a + b + \phi)\eta(X) = X(a + b + \phi).$$

Thus we can state the following:

Theorem 7.1. *If (M^n, g) is a $\mathcal{Z}$-recurrent N(k)-qusi-Einstein manifold, then*

$$(a+b+\phi)\eta(X) = X(a+b+\phi),$$

for all $X \in TM$.

A $\mathcal{Z}$-recurrent manifold is $\mathcal{Z}$-symmetric if and only if the 1-form η is zero. Thus we have the following corollaries:

Corollary 7.2. *If (M^n, g) is a $\mathcal{Z}$-symmetric N(k)-quasi-Einstein manifold, then $a+b+\phi$ is constant.*

Corollary 7.3. *If (M^n, g) is a $\mathcal{Z}$-recurrent N(k)-quasi-Einstein manifold and if $a+b+\phi$ is constant, then either $a+b+\phi = 0$ or (M^n, g) reduces to a $\mathcal{Z}$-symmetric N(k)-quasi-Einstein manifold.*

8. Example of $N(k)$-Quasi Einstein Manifolds

Let us consider a Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{8.1}$$

where $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant, $(i, j = 1, 2, 3, 4)$. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{q}{1+2q}, \;\; \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{q}{1+2q},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{q}{1+2q},$$

$$R_{2332} = R_{2442} = R_{3443} = \frac{q^2}{1+2q}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors are:

$$R_{11} = \frac{3q}{(1+2q)^2}, \;\; R_{22} = R_{33} = R_{44} = \frac{q}{1+2q}. \tag{8.2}$$

It can be easily shown that the scalar curvature r of the resulting space $(\mathbb{R}^4, g)$ is $r = \frac{6q(1+q)}{(1+2q)^3}$, which is non-vanishing and non-constant. Now we shall show that this $(\mathbb{R}^4, g)$ is an $N(k)$-quasi-Einstein manifold.

To show that the manifold under consideration is an $N(k)$-quasi-Einstein manifold, let us choose the scalar functions a, b and the 1-form η as follows:

$$a = \frac{q}{(1+2q)^2}, \qquad b = \frac{2q(1-q)}{(1+2q)^3}, \tag{8.3}$$

$$\eta_i(x) = \begin{cases} \sqrt{1+2q} & \text{for i=1} \\ 0 & \text{otherwise,} \end{cases} \tag{8.4}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.2) reduces to the equations

$$R_{11} = ag_{11} + b\eta_1\eta_1, \tag{8.5}$$

$$R_{22} = ag_{22} + b\eta_2\eta_2 \tag{8.6}$$

$$R_{33} = ag_{33} + b\eta_3\eta_3, \tag{8.7}$$

and

$$R_{44} = ag_{44} + b\eta_4\eta_4, \tag{8.8}$$

since, for the other cases (1.2) holds trivially. By (8.3) and (8.4) we get

$$\begin{aligned} \text{R.H.S. of (8.5)} &= ag_{11} + b\eta_1\eta_1 \\ &= \frac{q}{(1+2q)^2}(1+2q) + \frac{2q(1-q)}{(1+2q)^3}(1+2q) \\ &= \frac{3q}{(1+2q)^2} = R_{11} \\ &= \text{L.H.S. of (8.5).} \end{aligned}$$

By similar argument it can be shown that (8.6), (8.7) and (8.8) are also true. So, $(\mathbb{R}^4, g)$ is an $N(\frac{q}{(1+2q)^3})$-quasi-Einstein manifold.

9. Physical Examples of N(k)-Quasi-Einstein Manifolds

Example 9.1. This example is concerned with example of an N(k)-quasi-Einstein manifold in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentzian manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity.

An n-dimemsional Riemannian or semi-Riemannian manifold (M^n, g), $(n > 3)$ is said to be pseudo Ricci symmetric manifold[6] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y), \tag{9.1}$$

where A is a non-zero 1-form defined by

$$g(X,U) = A(X),$$

for all X. Such a manifold is denoted by $(PRS)_n$.
Here we consider a perfect fluid $(PRS)_4$ spacetime with non-zero scalar curvature and having the basic vector field U as the timelike vector field of the fluid, that is, $g(U,U) = -1$. In a recent paper[9] De et al. prove the following:

A perfect fluid pseudo Ricci symmetric spacetime with non-zero scalar curvature is an N($\frac{2r}{9}$)-quasi-Einstein manifold.

Example 9.2([31]). A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's equation without cosmological constant is an N($\frac{k(3\sigma+p)}{6}$)-quasi-Einstein manifold.

Example 9.3([31]). A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's equation with cosmological constant is an N($\frac{\lambda}{3} + \frac{k(3\sigma+p)}{6}$)-quasi-Einstein manifold, where k is the gravitational constant, σ is the energy density and p is the isotropic pressure of the fluid.

Acknowledgement. The authors are thankful to the reviewer for careful reading of the manuscript and his/her thoughtful comments for the improvement of the paper.

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