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A New Type of Helix Constructed by Plane Curves

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Abstract. In this paper, we give an algorithm to construct a space curve in Euclidean 3-space $\mathbb{E}^3$ from a plane curve which is called PDP-helix of order d. The notion of the PDP-helices is a generalization of a general helix and a slant helix in $\mathbb{E}^3$. It is naturally shown that the PDP-helix of order 1 and order 2 are the same as the general helix and the slant helix, respectively. We give a characterization of the PDP-helix of order d. Moreover, we study some geometric properties of that of order 3.

1. Introduction

The geometry of curves is an important and exciting branch of mathematics. For centuries, in fact, it has played a significant role in the development of applied and natural sciences (see [3]).

In differential geometry, a *helix* in Euclidean 3-space $\mathbb{E}^3$ is the simplest space curve which is defined by a Frenet curve in $\mathbb{E}^3$ with non-zero constant curvature and non-zero constant torsion. It is well-known that DNA has the double helix shape in biology.

Sometimes, many researchers attempt to generalize the fundamental notions in their works because of the mathematical importance and the possibility of various applications. The helix was first generalized in 1802 by M. A. Lancret who defined a *general helix* by the property that its tangent vector field makes a constant angle with a fixed direction. In 1850 ([1]), J. Bertrand gave another generalization which is called a *Bertrand curve*. Recently, S. Izumiya and N. Takeuchi introduced the notion of a *slant helix* in [5] that contains the general helix. It is well-known that helices, Bertrand curves and slant helices are respectively characterized by $\frac{\tau}{\kappa} = c$, $a\kappa + b\tau = 1$ and $\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' = c$, where κ and τ are respectively their curvature and their torsion, and c, $a \neq 0$ and b are constants.

These generalizations have led to a number of studies in the field of science as well as mathematics for a long time. Especially, they are still applied in biology,

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physics, material engineering and computer aided design.

In [2], recently, author and Y. H. Kim introduced the notions of a principal-donor curve and a principal-directional curve in $\mathbb{E}^3$. By using them, they proved that a general helix and a slant helix are a principal-donor curve and a second principal-donor curve of a plane curve, respectively.

In this paper, motivated by above, we define a *PDP-helix of order d* in $\mathbb{E}^3$ as a generalization of helices. It is naturally shown that a PDP-helix of order 1 and order 2 are the same as a general helix or a slant helix, respectively. Moreover, we give a characterization of the PDP-helix of order d and study some geometric properties of that of order 3.

2. Preliminaries

A Frenet curve $\gamma := \gamma(s)$ in Euclidean 3-space $\mathbb{E}^3$ defined on an open interval I is a unit speed regular curve satisfying $\gamma''(s) \neq 0$ for all $s \in I$. Then, the Frenet frame along the curve γ is defined by a field of an orthonormal frame $\{\mathbf{T} = \gamma', \mathbf{N} = \gamma''/||\gamma''||, \mathbf{B} = \mathbf{T} \times \mathbf{N}\}$ and it satisfies the followging Frenet-Serret equations:

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}, \tag{2.1}$$

where $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are called the *tangent vector*, the *principal normal vector* and the *binormal vector* of γ, respectively. Two functions κ and τ in (2.1) are respectively called the *curvature* and the *torsion* of γ. At each point of the curve γ, the planes spanned by $\{\mathbf{T}, \mathbf{N}\}$, $\{\mathbf{T}, \mathbf{B}\}$ and $\{\mathbf{N}, \mathbf{B}\}$ are called the *osculating plane*, the *rectifying plane* and the *normal plane*, respectively.

In [2], author and Y. H. Kim defined the principal-donor curve and the principal-directional curve and they gave the relationship between them as follows:

Definition 2.1([2]). Let $\gamma := \gamma(s)$ be a Frenet curve $\gamma := \gamma(s)$ in $\mathbb{E}^3$ with Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature κ and torsion τ. Then, an integral curve of $-\cos\left(\int \tau(s)ds\right) \mathbf{N}(s) + \sin\left(\int \tau(s)ds\right) \mathbf{B}(s)$ is called a *principal-donor curve* of γ. Similarly, an integral curve of $\mathbf{N}(s)$ is called a *principal-directional curve* of γ.

Theorem 2.2([2]). *Let γ be a Frenet curve in $\mathbb{E}^3$ with curvature κ and torsion τ and $\bar{\gamma}$ a principal-directional curve of γ. Then the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of $\bar{\gamma}$ are respectively given by*

$$\bar{\kappa} = \sqrt{\kappa^2 + \tau^2} \quad \text{and} \quad \bar{\tau} = \frac{\kappa^2}{\kappa^2 + \tau^2} \left(\frac{\tau}{\kappa}\right)'. \tag{2.2}$$

Theorem 2.3([2]). *Let $\bar{\gamma}$ be a Frenet curve in $\mathbb{E}^3$ with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ and γ a principal-donor curve of $\bar{\gamma}$. Then the curvature κ and the torsion τ of γ*

are respectively given by

$$\kappa = \bar{\kappa} \left| \cos\left(\int \bar{\tau} ds \right) \right| \quad \textit{and} \quad \tau = \bar{\kappa} \sin\left(\int \bar{\tau} ds \right). \tag{2.3}$$

Proposition 2.4. *With the same notations as above, it follows that*

$$\int \bar{\tau} ds = \sin^{-1}\left(\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right). \tag{2.4}$$

Proof. If follows from (2.2) and (2.3). □

The following remark is useful to calculate the principal-donor curve of a given Frenet curve.

Remark 2.5. The equation (2.4) implies that

$$\sin\left(\int \bar{\tau} ds \right) = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \quad \text{and} \quad \cos\left(\int \bar{\tau} ds \right) = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}. \tag{2.5}$$

Remark 2.6. If a Frenet curve γ is a principal-donor curve of $\bar{\gamma}$, $\bar{\gamma}$ is a principal-directional curve of γ, and vise versa.

Remark 2.7. In physics, a Frenet curve in Euclidean 3-space is considered as a path traced by an article with the unit constant speed in space. Then, the tangent vector and the principal normal vector can be interpreted as the velocity and the unit direction of acceleration of the article, respectively. The principal-directional curve can be considered as a path of another article whose velocity is a unit direction of acceleration of the given article.

Throughout this paper, we will only consider a unit speed regular curve in $\mathbb{E}^3$ which do not include part of straight lines unless otherwise stated.

3. The PDP-helix of order d

In this section, we study PDP-helices in Euclidean 3-space $\mathbb{E}^3$. Firstly, we define a PDP-helix of order d.

Definition 3.1. A Frenet curve γ_1 in $\mathbb{E}^3$ is called a *PDP-helix of order 1* if γ_1 is a principal-donor curve of a plane curve γ_0 but not planar. For an integer $d \geq 2$, inductively, a Frenet curve γ_d is called a *PDP-helix of order d* if γ is a principal-donor curve of a PDP-helix γ_{d-1} of order $d-1$. In addition, the plane curve γ_0 which generates the PDP-helix γ_d is called a *base curve* of γ_d. Equivalently, we say that γ_d is generated by γ_0.

Sometime, we will call a PDP-helix γ_d of order d by a *d'th principal-donor curve* of a plane curve γ_0. Equivalently, we call γ_0 by a *d'th principal-directional curve* of γ_d.

In Definition 3.1, 'PDP' means 'Princinal-Donor curve of a Plane curve'. From (2.2) and (2.3), we have

Proposition 3.2. *A Frenet curve γ is a PDP-helix of order 1 if and only if γ is a general helix. Moreover, γ is a PDP-helix of order 2 if and only if γ is a slant helix.*

Proposition 3.2 says that the notion of PDP-helices is a generalization of a general helix and a slant helix.

In [4], Ç. Camcı, K. İlarslan, L. Kula and H. H. Hacısalihoğlu studied a general helix in a n-dimensional Euclidean space $\mathbb{E}^n$. They gave a characterization of general helices in $\mathbb{E}^n$ in terms of *harmonic curvatures* $\{H_1, H_2, \cdots, H_{n-2}\}$. If γ is a Frenet curve in $\mathbb{E}^n$ with the Frenet frame $\{V_1, V_2, \cdots, V_n\}$ and the curvature functions $\{k_1, k_2, \cdots, k_{n-1}\}$, the harmonic curvatures of γ are defined by

$$H_i = \begin{cases} 0, & i = 0 \\ \frac{k_1}{k_2}, & i = 1 \\ \{V_1[H_{i-1}] + H_{i-2}k_i\}, & i = 2, 3, \cdots, n-2. \end{cases}$$

Indeed, they proved that *a Frenet in $\mathbb{E}^n$ is a general helix with the Frenet frame $\{V_1, V_2, \cdots, V_n\}$ and the Harmonic curvatures $\{H_1, H_2, \cdots, H_{n-2}\}$ if and only if it satisfies that $V_1[H_{n-2}] + k_{n-1}H_{n-3} = 0$.*

Motivated by above characterization, we give a characterization of a PDP-helix in $\mathbb{E}^3$. Before that, we define the *n'th PDP-ratio* and the *n'th PDP-torsion* of a Frenet curve γ in $\mathbb{E}^3$ by

$$\mu_n = \begin{cases} \frac{\tau}{\kappa}, & n = 0 \\ -\frac{1}{\tau_{n-1}} \left(\frac{1}{\sqrt{1+(\mu_{n-1})^2}} \right)', & n = 1, 2, 3, \cdots \end{cases} \tag{3.1}$$

and

$$\tau_n = \begin{cases} \tau, & n = 0 \\ \frac{(\mu_{n-1})'}{1+(\mu_{n-1})^2}, & n = 1, 2, 3, \cdots, \end{cases} \tag{3.2}$$

where κ and τ is the curvature and the torsion of γ, respectively.

Then, we can give a characterization of a PDP-helix of order d.

Theorem 3.3. *Let γ be a Frenet curve with the curvature κ and the torsion τ. Then, γ is a PDP-helix of order d ($d \geq 1$) if and only if its $(d-1)$'th PDP-ratio μ_{d-1} is constant.*

Proof. Two equations of (2.2) in Theorem 2.2 are intrinsic information for the principal-directional curve of a given Frenet curve in $\mathbb{E}^3$. Also, they are respectively equivalent to

$$\bar{\kappa} = \kappa\sqrt{1+\mu^2} \quad \text{and} \quad \bar{\tau} = \frac{\mu'}{1+\mu^2}, \tag{3.3}$$

where $\mu := \frac{\tau}{\kappa}$. Then, (3.3) leads to

$$\bar{\mu} := \frac{\bar{\tau}}{\bar{\kappa}} = -\frac{1}{\tau}\left(\frac{1}{\sqrt{1+\mu^2}}\right)'. \tag{3.4}$$

It follows from (3.4) that the PDP-ratio μ_n in (3.1) is the ratio for an n'th principal-directional curve of γ. From the definition, the curve γ is a PDP-helix of order d if and only if its d'th principal-directional curve is a plane curve, equivalently, its $(d-1)$'th principal-directional curve is a general helix, that is, μ_{d-1} is constant. □

Example 3.4. A general helix γ in $\mathbb{E}^3$ is a PDP-helix of order 1. In fact, it satisfies that $\mu_0 = \frac{\tau}{\kappa}$ is constant.

Example 3.5. A slant helix γ in $\mathbb{E}^3$ is a PDP-helix of order 2. By the simple calculation, we get that $\mu_0 = \frac{\tau}{\kappa}$, $\tau_0 = \tau$, from which,
$\mu_1 = -\frac{1}{\tau_0}\left(\frac{1}{\sqrt{1+(\mu_0)^2}}\right)' = -\frac{1}{\tau}\left(\frac{1}{\sqrt{1+(\tau/\kappa)^2}}\right)' = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' = c$. The last equality coincides with the characterization of slant helices.

Example 3.6. Let $\gamma(s) = (x(s), y(s), z(s))$ be a Frenet curve in $\mathbb{E}^3$, where

$$x(s) = \int_0^s \frac{t}{1+t^2}\left\{\frac{t^2}{\sqrt{1+t^2}}\sin(\frac{\sqrt{3}}{\sqrt{1+t^2}}) - \sqrt{3}\cos(\frac{\sqrt{3}}{\sqrt{1+t^2}})\right\}dt,$$

$$y(s) = \int_0^s \frac{-1}{2(1+t^2)^{\frac{3}{2}}}\left\{\sqrt{3}(t^2-1)\sqrt{1+t^2}\cos(\frac{\sqrt{3}}{\sqrt{1+t^2}}) + (1+3t^2)\sin(\frac{\sqrt{3}}{\sqrt{1+t^2}})\right\}dt,$$

$$z(s) = -\frac{1}{2}\int_0^s \left\{\cos(\frac{\sqrt{3}}{\sqrt{1+t^2}}) + \frac{\sqrt{3}}{\sqrt{1+t^2}}\sin(\frac{\sqrt{3}}{\sqrt{1+t^2}})\right\}dt.$$

Then, the curvature κ and the torsion τ of γ are calculated by $\frac{\sqrt{3}}{(1+s^2)^{3/2}}\cos(\frac{\sqrt{3}}{\sqrt{1+s^2}})$ and $\frac{\sqrt{3}}{(1+s^2)^{3/2}}\sin(\frac{\sqrt{3}}{\sqrt{1+s^2}})$, respectively. From (3.1) and (3.2), we get that $\mu_0 = \cos(\frac{\sqrt{3}}{\sqrt{1+s^2}})$, $\tau_0 = \tau$, $\mu_1 = -s$, $\tau_1 = -\frac{\sqrt{3}s}{(1+s^2)^{3/2}}$ and $\mu_2 = \frac{1}{\sqrt{3}}$. Therefore, the curve γ is a PDP-helix of order 3. Moreover, the curvature of its base curve equals to $\frac{2}{1+s^2}$ and hence the base curve is expressed by $\left(2\tan^{-1}s - s, \ln(1+s^2)\right)$. See Fig.3.1.

Remark 3.7. For any positive integer d greater than 3, it is hard to study some geometry properties of a PDP-helix of order d because of the difficulty of integral. Nevertheless, it is meaningful to study the PDP-helices. Because it is important

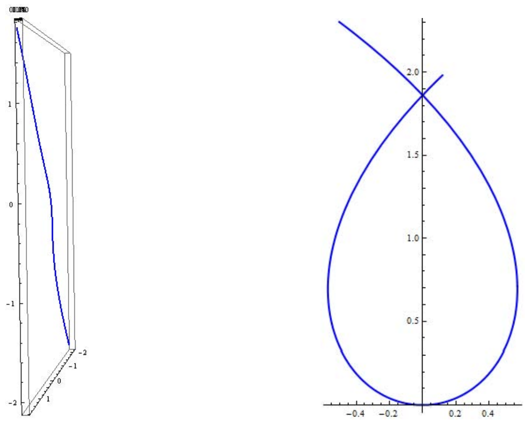

Figure 3.1: A PDP-helix of order 3 and its base curve in Example 3.6.

to find the various new examples of curves in applied sciences. If it need to draw some PDP-helices of high order, we can do by the computation using the PDP-helix algorithm.

4. The PDP-helix of order 3

In differential geometry, it is well-known that the position vector of a unit speed plane curve in $\mathbb{E}^2$ with the curvature κ is given by (see [6])

$$\left(\int_0^s \cos\left(\int_0^\sigma \kappa dt\right) d\sigma, \int_0^s \sin\left(\int_0^\sigma \kappa dt\right) d\sigma\right).$$

But, unfortunately, nothing is known about the the case of space curves in $\mathbb{E}^3$ with the curvature κ and the torsion τ so far.

Recently, author and Y. H. Kim gave the position vectors of a general helix and a slant helix described by their curvature and torsion as follows:

Theorem A([2]). *A general helix and a slant helix in $\mathbb{E}^3$ with the curvature κ and the torsion τ are respectively locally expressed by*

$$\frac{1}{A}\left(\int_0^s \sin\left(|A|\int_0^\sigma \kappa dt\right) d\sigma, -\int_0^s \cos\left(|A|\int_0^\sigma \kappa dt\right) d\sigma, cs\right)$$

and

$$\left(\int_0^s \frac{1}{\sqrt{\kappa^2+\tau^2}}\left\{|\tau|\cos\left(|A|\int_0^\sigma \sqrt{\kappa^2+\tau^2}dt\right)+\frac{c\kappa}{A}\sin\left(|A|\int_0^\sigma \sqrt{\kappa^2+\tau^2}dt\right)\right\}d\sigma,\right.$$
$$\int_0^s \frac{1}{\sqrt{\kappa^2+\tau^2}}\left\{|\tau|\sin\left(|A|\int_0^\sigma \sqrt{\kappa^2+\tau^2}dt\right)-\frac{c\kappa}{A}\cos\left(|A|\int_0^\sigma \sqrt{\kappa^2+\tau^2}dt\right)\right\}d\sigma,$$
$$\left.-\frac{1}{A}\int_0^s \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}dt\right),$$

where $A=\pm\sqrt{1+c^2}$.

By the similar method, we can give the position vector of a PDP-helix of order 3 in $\mathbb{E}^3$ with the curvature κ and the torsion τ.

The last expression of a slant helix leads to its Frenet frame $\{T,N,B\}$ as follows:

$$\begin{aligned}
T(s)=&\Big(\tfrac{1}{\sqrt{\kappa^2+\tau^2}}\left\{|\tau|\cos\left(|A|\textstyle\int_0^s\sqrt{\kappa^2+\tau^2}dt\right)+\tfrac{c\kappa}{A}\sin\left(|A|\textstyle\int_0^s\sqrt{\kappa^2+\tau^2}dt\right)\right\},\\
&\tfrac{1}{\sqrt{\kappa^2+\tau^2}}\left\{|\tau|\sin\left(|A|\textstyle\int_0^\sigma\sqrt{\kappa^2+\tau^2}dt\right)-\tfrac{c\kappa}{A}\cos\left(|A|\textstyle\int_0^\sigma\sqrt{\kappa^2+\tau^2}dt\right)\right\},\\
&-\tfrac{1}{A}\tfrac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big),\\
N(s)=&\tfrac{1}{A}\left(\sin(|A|\textstyle\int_0^s\sqrt{\kappa^2+\tau^2}dt,-\cos(|A|\textstyle\int_0^s\sqrt{\kappa^2+\tau^2}dt),c\right),\\
B(s)=&T(s)\times N(s)\\
=&\Big(\tfrac{1}{\sqrt{\kappa^2+\tau^2}}\left\{\tfrac{c|\tau|}{A}\sin\left(|A|\textstyle\int_0^\sigma\sqrt{\kappa^2+\tau^2}dt\right)-\kappa\cos\left(|A|\textstyle\int_0^\sigma\sqrt{\kappa^2+\tau^2}dt\right)\right\},\\
&-\tfrac{1}{\sqrt{\kappa^2+\tau^2}}\left\{\tfrac{c|\tau|}{A}\cos\left(|A|\textstyle\int_0^\sigma\sqrt{\kappa^2+\tau^2}dt\right)+\kappa\sin\left(|A|\textstyle\int_0^\sigma\sqrt{\kappa^2+\tau^2}dt\right)\right\},\\
&-\tfrac{1}{A}\tfrac{|\tau|}{\sqrt{\kappa^2+\tau^2}}\Big).
\end{aligned}$$

Since a PDP-helix $\hat{\gamma}$ of order 3 is a principal-donor curve of a slant helix, we have the tangent vector of $\hat{\gamma}$ by using (2.2) and (2.5) as follows:

$$\begin{aligned}
&\hat{\gamma}'(s)\\
&=-\cos(\textstyle\int\tau ds)N(s)+\sin(\textstyle\int\tau dt)B(s)\\
&=\Big(-\tfrac{\hat{\tau}}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}\sin(|A|\textstyle\int_0^s\hat{K}(t)dt)\\
&\qquad+\tfrac{\hat{\kappa}}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}\tfrac{1}{\hat{K}(t)}\left\{\tfrac{c|\tau|}{A}\sin\left(|A|\textstyle\int_0^s\hat{K}(t)dt\right)-\kappa\cos\left(|A|\textstyle\int_0^s\hat{K}(t)dt\right)\right\},\\
&\quad\tfrac{\hat{\tau}}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}\cos(|A|\textstyle\int\hat{K}(t)dt)\\
&\qquad-\tfrac{\hat{\kappa}}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}\tfrac{1}{\hat{K}(t)}\left\{\tfrac{c|\tau|}{A}\cos\left(|A|\textstyle\int_0^s\hat{K}(t)dt\right)+\kappa\sin\left(|A|\textstyle\int_0^s\hat{K}(t)dt\right)\right\},\\
&\quad-\tfrac{c\hat{\tau}}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}-\tfrac{\hat{\kappa}}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}\tfrac{|\tau|}{\hat{K}(s)}\Big),
\end{aligned}$$

where $\hat{K}(s)=\sqrt{\kappa^2+\tau^2}=\sqrt{\hat{\kappa}^2(1+\hat{\mu}^2)+\frac{(\hat{\mu}')^2}{(1+\hat{\mu}^2)^2}}$ and $\hat{\mu}=\frac{\hat{\tau}}{\hat{\kappa}}$. Thus, we have

Theorem 4.1. *A PDP-helix $\hat{\gamma}$ of order 3 in $\mathbb{E}^3$ with the curvature $\hat{\kappa}$ and the torsion $\hat{\tau}$ is locally expressed by*

$$\begin{aligned}\hat{\gamma}(s) = \int_0^s \Bigg(&\frac{1}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}\left[-\hat{\tau}\sin(|A|\int_0^s \hat{K}(t)dt)\right.\\ &\left.+\frac{\hat{\kappa}}{\hat{K}(t)}\left\{\frac{c|\tau|}{A}\sin\left(|A|\int_0^s \hat{K}(t)dt\right) - \kappa\cos\left(|A|\int_0^s \hat{K}(t)dt\right)\right\}\right],\\ &\frac{1}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}\left[\hat{\tau}\cos(|A|\int \hat{K}(t)dt)\right.\\ &\left.-\frac{\hat{\kappa}}{\hat{K}(t)}\left\{\frac{c|\tau|}{A}\cos\left(|A|\int_0^s \hat{K}(t)dt\right) + \kappa\sin\left(|A|\int_0^s \hat{K}(t)dt\right)\right\}\right],\\ &-\frac{1}{A\sqrt{\hat{\kappa}^2+\hat{\tau}^2}}\left[c\hat{\tau}+\frac{\hat{\kappa}|\tau|}{\hat{K}(s)}\right]\Bigg)d\sigma,\end{aligned}$$

where $\hat{K}(s) = \sqrt{\kappa^2+\tau^2} = \sqrt{\hat{\kappa}^2(1+\hat{\mu}^2)+\frac{(\hat{\mu}')^2}{(1+\hat{\mu}^2)^2}}$ *and* $\hat{\mu} = \frac{\hat{\tau}}{\hat{\kappa}}$.

Up to isometry, a plane curve in $\mathbb{E}^2$ is completely determined by a real valued function called by the *curvature*. Also, a PDP-helix of order d in $\mathbb{E}^3$ is determined by a plane curve $\mathbb{E}^2$. Consequently, a PDP-helix of order d in $\mathbb{E}^3$ is completely determined by a real-valued curvature function up to isometry. In [2], author and Y. H. Kim gave an expression of slant helix generated by a plane curve $(x(s), y(s))$ with the curvature $\kappa_0 = \sqrt{x''(s)+y''(s)}$.

Theorem B([2]). *Let γ be a slant helix generated by a base curve $(x(s), y(s))$. Then, γ is expressed by*

$$\begin{aligned}&\left(-\int_0^s \cos\left(\int_0^\sigma b\kappa_0 dt\right)x'(\sigma) + b\sin\left(\int_0^\sigma b\kappa_0 dt\right)y'(\sigma)d\sigma,\right.\\ &-\int_0^s \cos\left(\int_0^\sigma b\kappa_0 dt\right)y'(\sigma) - b\sin\left(\int_0^\sigma b\kappa_0 dt\right)x'(\sigma)d\sigma,\\ &\left.a\sin\left(\int_0^s b\kappa_0 d\sigma\right)\right).\end{aligned} \tag{4.1}$$

Similarly, we give a local expression of a PDP-helix of order 3 in terms of the components of its base curve. This is very useful to find many examples of a PDP-helix of order 3.

Theorem 4.2. *A PDP-helix of order* 3 *generated by a base curve* $(x(s), y(s))$ *with the curvature* $\kappa_0 = \sqrt{x''(s)^2 + y''(s)^2}$ *is locally expressed by*

$$
\begin{aligned}
(4.2)\qquad \Big(&-\int_0^s \{a\cos(A_2(t))\,y'(t) - \sin(A_2(t))\sin(A_1(t))\,x'(t) \\
&\qquad\qquad + \sin(A_2(t))\cos(A_1(t))\,y'(t)\}\,dt, \\
&\int_0^s \{a\cos(A_2(t))\,x'(t) + \sin(A_2(t))\sin(A_1(t))\,y'(t) \\
&\qquad\qquad + \sin(A_2(t))\cos(A_1(t))\,x'(t)\}\,dt, \\
&\int_0^s \{-b\cos(A_2(t)) + a\sin(A_2(t))\cos(A_1(t))\}\,dt\Big),
\end{aligned}
$$

where $A_1(t) = b\int_0^t \kappa_0(u)du$ *and* $A_2(t) = a\int_0^t \kappa_0 \sin(A_1(u))\,du$ *for constants* a *and* b *with* $a^2 + b^2 = 1$.

Proof. The proof is similar to that of theorem 4.1. □

Remark 4.3. In case of $b = 0$, (4.2) is the same as the local expression of slant helices. That is, the notion of PDP-helices of order 3 is a generalization of slant helices.

A helix in $\mathbb{E}^3$ is often called by a *circular helix*. In fact, a helix in $\mathbb{E}^3$ is a PDP-helix of order 1 generated by a circle. Thus, we call a PDP-helix generated by a circle a *circular PDP-helix*. The following is a characterization of circular PDP-helices of order 3.

Example 4.4. (Circular PDP-helix of order 3**)**
A circular PDP-helix of order 3 can be constructed from a circle with the radius r, i.e., $(r\cos(\frac{s}{r}), r\sin(\frac{s}{r}))$. The circle has the curvature $\kappa_0 = \frac{1}{r}$. Thus, from (4.2) in theorem 4.2 we get the expression of a circular PDP-helix of order 3 as follows:

$$
\begin{aligned}
\int \Big(&-\tfrac{1}{\sqrt{1+c^2}}\cos(\tfrac{s}{r})\cos\big(\tfrac{1}{c}\cos(\tfrac{cs}{r\sqrt{(1+c^2)}})\big) + \sin\big(\tfrac{1}{c}\cos(\tfrac{cs}{r\sqrt{(1+c^2)}})\big)\times \\
&\qquad \Big\{\tfrac{c}{\sqrt{1+c^2}}\cos(\tfrac{s}{r})\cos\big(\tfrac{cs}{r\sqrt{(1+c^2)}}\big) + \sin(\tfrac{s}{r})\sin\big(\tfrac{cs}{r\sqrt{(1+c^2)}}\big)\Big\}, \\
&-\tfrac{1}{\sqrt{1+c^2}}\sin(\tfrac{s}{r})\cos\big(\tfrac{1}{c}\cos(\tfrac{cs}{r\sqrt{(1+c^2)}})\big) + \sin\big(\tfrac{1}{c}\cos(\tfrac{cs}{r\sqrt{(1+c^2)}})\big)\times \\
&\qquad \Big\{\tfrac{c}{\sqrt{1+c^2}}\sin(\tfrac{s}{r})\cos\big(\tfrac{cs}{r\sqrt{(1+c^2)}}\big) - \cos(\tfrac{s}{r})\sin\big(\tfrac{cs}{r\sqrt{(1+c^2)}}\big)\Big\}, \\
&\tfrac{1}{\sqrt{1+c^2}}\Big\{c\cos\big(\tfrac{1}{c}\cos(\tfrac{cs}{r\sqrt{(1+c^2)}})\big) + \cos(\tfrac{cs}{r\sqrt{(1+c^2)}})\sin\big(\tfrac{1}{c}\cos(\tfrac{cs}{r\sqrt{(1+c^2)}})\big)\Big\}\Big)\,ds.
\end{aligned}
$$

Moreover, it has the curvature $\kappa(s) = \frac{1}{r\sqrt{1+c^2}}\cos(\frac{s}{r})\cos\big(\frac{1}{c}\cos(\frac{cs}{\sqrt{r(1+c^2)}})\big)$, the torsiion $\tau(s) = \frac{1}{r\sqrt{1+c^2}}\cos(\frac{s}{r})\sin\big(\frac{1}{c}\cos(\frac{cs}{\sqrt{r(1+c^2)}})\big)$ and the ratio $\mu(s) = \tan\big(\frac{1}{c}\cos(\frac{cs}{\sqrt{r(1+c^2)}})\big)$. (See Fig 4.1.)

Example 4.5. Consider a function $f(s) = \frac{1}{1+s^2}$. Then, the function $f(s)$ determines completely a plane curve $(\sqrt{1+s^2}, \sinh^{-1} s)$. As a base curve, this plane

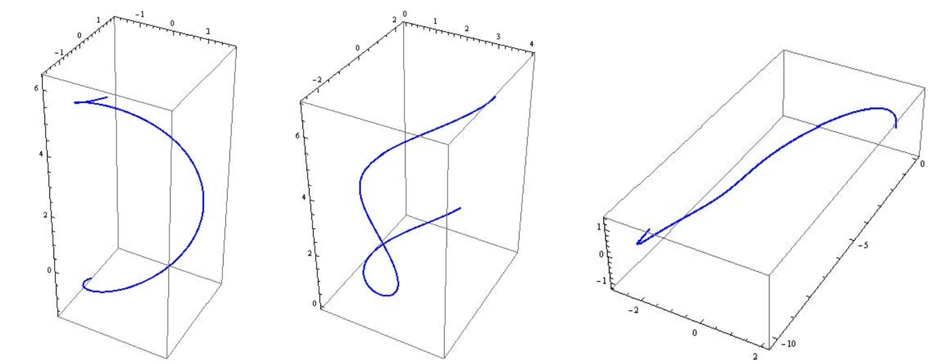

Figure 4.1: Circular PDP-helices of order 1, 2 and 3 generated by a circle.

curve also construct the PDP-helices of order 1, 2 and 3. Set $c = \frac{1}{\sqrt{3}}$ in (4.2) of theorem 4.2. Then, we get the figures of PDP-helices of order 1, 2, and 3 by the PDP-helix algorithm and the computation. See Fig 4.2.

A general helix in $\mathbb{E}^3$ has an exciting property that its tangent vector makes a constant angle with a fixed direction. Moreover, this property characterizes general helices. Similarly, the principal-normal vector of a slant helix in $\mathbb{E}^3$ has the same property. In point of view of the PDP-helix, the fixed directions are all the z-axis.

Here, we find the similar geometric property of PDP-helices of order 3.

Theorem 4.6. *A Frenet curve γ in $\mathbb{E}^3$ with the curvature κ and the torsion τ is a PDP-helix of order 3 if and only if the vector $-\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B$ makes a constant with the fixed direction, where $\{T, N, B\}$ be the Frenet frame of γ.*

Proof. Let $\overline{\gamma}$ be a principal-directional curve of γ and $\{\overline{T}, \overline{N}, \overline{B}\}$ the Frenet frame of $\overline{\gamma}$. Then, the relationship between $\{T, N, B\}$ and $\{\overline{T}, \overline{N}, \overline{B}\}$ is given by

$$\begin{cases} T = -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\overline{N} + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\overline{B}, \\ N = \overline{T}, \\ B = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\overline{N} + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\overline{B}, \end{cases}$$

from which, $\overline{N} = -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B$. We know that γ is a PDP-helix of order 3 if and only if $\overline{\gamma}$ is a slant helix. Also, since $\overline{\gamma}$ is a slant helix if and only if $\overline{N}$ makes a constant angle with the fixed direction, our assertion is proved. $\square$

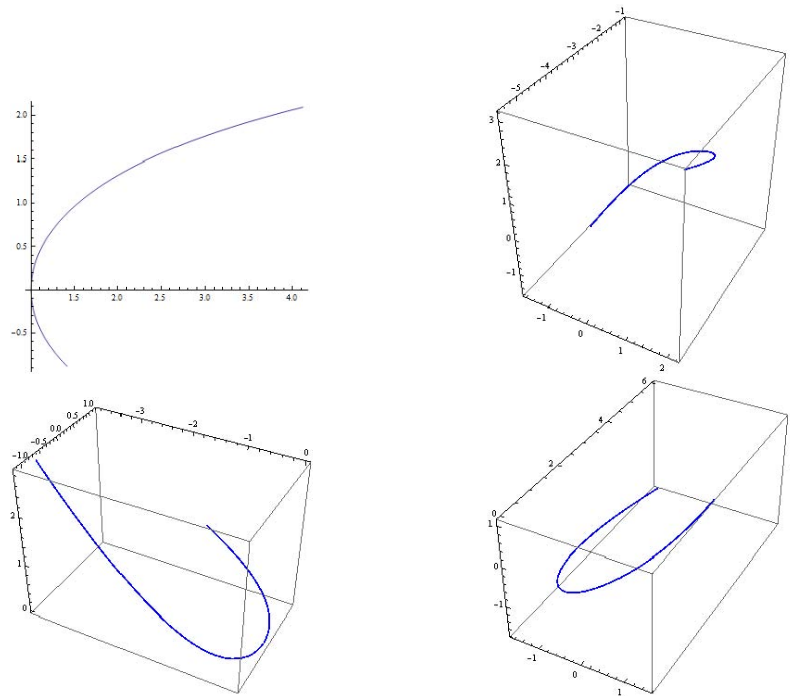

Figure 4.2: A plane curve and the PDP-helices of order 1, 2 and 3 in Example 4.5.

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