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On Semi-cubically Hyponormal Weighted Shifts with First Two Equal Weights

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Abstract. It is known that a semi-cubically hyponormal weighted shift need not satisfy the flatness property, in which equality of two weights forces all or almost all weights to be equal. So it is a natural question to describe all semi-cubically hyponormal weighted shifts W_α with first two weights equal. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ be a backward 3-step extension of a recursively generated weight sequence with $1 < x < u < v < w$ and let W_α be the associated weighted shift. In this paper we characterize completely the semi-cubical hyponormal W_α satisfying the additional assumption of the positive determinant coefficient property, which result is parallel to results for quadratic hyponormality.

1. Introduction and Notation

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For A, $B \in \mathcal{L}(\mathcal{H})$, we set $[A, B] := AB - BA$. A k-tuple $\mathbf{T} = (T_1, ..., T_k)$ of operators on $\mathcal{H}$ is called *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^k$ is positive on the direct sum of $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$

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(k copies). Also an operator T is said to be (*strongly*) *k-hyponormal* for each positive integer k if $(I, T, ..., T^k)$ is hyponormal. The Bram-Halmos criterion shows that an operator T is subnormal if and only if T is k-hyponormal for all $k \geq 1$ ([2], [16]). An operator T is *polynomially hyponormal* if for every polynomial p, $p(T)$ is hyponormal, and T is *weakly k-hyponormal* if for every polynomial p of degree k or less, $p(T)$ is hyponormal ([5],[10],[11]). In particular, weak 2-hyponormality (or weak 3-hyponormality) is referred to as *quadratically hyponormal* (or *cubically hyponormal*, respectively). For a positive integer k, an operator $T \in \mathcal{L}(\mathcal{H})$ is called *semi-weakly k-hyponormal* if $T + sT^k$ is hyponormal for all $s \in \mathbb{C}$ ([12]). It is obvious that a weakly k-hyponormal operator is semi-weakly k-hyponormal. In particular, weak 2-hyponormality is equivalent to semi-weak 2-hyponormality.

It is well known that k-hyponormality implies weak k-hyponormality for each positive integer k. The following results provide a bridge between subnormal and hyponormal operators: subnormal $\Rightarrow$ polynomially hyponormal $\Rightarrow \cdots \Rightarrow$ weakly 3-hyponormal $\Rightarrow$ weakly 2-hyponormal $\Rightarrow$ hyponormal. However, one does not yet have concrete examples about the converse implications for $n \geq 3$; see [9], [17] and [18] for weak 2- and weak 3-hyponormalities.

J. Stampfli ([21]) proved that a subnormal weighted shift with two equal weights $\alpha_n = \alpha_{n+1}$ for some nonnegative n has the property that $\alpha_1 = \alpha_2 = \cdots$, which is known as the "flatness property." Stampfli's result has been used to attempt the construction of nonsubnormal polynomially hyponormal weighted shifts (cf. [1],[3],[4],[7],[12],[15],[17]). In [3], Choi proved that if a weighted shift W_α is polynomially hyponormal with the first two weights equal, then W_α has the flatness property. In [4], Curto obtained a quadratically hyponormal weighted shift with first two weights equal but not satisfying flatness. Also in [17], the authors showed that a weighted shift W_α with weights $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{n+1}{n+2}}$ $(n \geq 2)$ is not cubically hyponormal. And in [19], it was shown that if a weighted shift W_α is cubically hyponormal with first two weights equal, then W_α has flatness. However, in [12], it was proved that there exists a semi-cubically hyponormal weighted shift W_α with $\alpha_0 = \alpha_1 < \alpha_2$ which is not 2-hyponormal. Hence the following problem arises naturally as the analog to the question for quadratically hyponormal weighted shifts.

Problem 1.1. Describe all semi-cubically hyponormal weighted shifts W_α with first two weights equal.

In [12], Do-Exner-Jung-Li characterized the semi-cubical hyponormality of the weighted shift $W_{\alpha(x)}$ with positive determinant coefficients (p.d.c. – definition reviewed below), where $\alpha(x) : \sqrt{x}, \sqrt{x}, \sqrt{\frac{k+1}{k+2}}$ $(k \geq 2)$ is a weight sequence with Bergman tail. In this paper we describe the semi-cubical hyponormality of the weighted shifts having the p.d.c. property but with recursive tails. More precisely, for a three step backward extended weight sequence $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$, where $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ is the Stampfli (recursively generated) subnormal completion of u, v, w (cf. [21]), we characterize completely the semi-cubical hyponormality of W_α with p.d.c. Note that, by the nature of a recursive tail, the one and two step backward extensions are special cases of the three step backward extension (see Remark 3.5).

For the reader's convenience, we recall the Stampfli subnormal completion (cf. [6],[21]). For given numbers $\alpha_0, \alpha_1, \alpha_2$ with $0 < \alpha_0 < \alpha_1 < \alpha_2$, define

$$\alpha_n^2 = \Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2} \quad \text{for all } n \geq 3, \tag{1.1}$$

where $\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)}{\alpha_1^2-\alpha_0^2}$ and $\Psi_1 = \frac{\alpha_1^2(\alpha_2^2-\alpha_0^2)}{\alpha_1^2-\alpha_0^2}$. Then we may obtain a weight sequence $\{\alpha_n\}_{n=0}^\infty$ generated recursively by (1.1), which is usually denoted by $(\alpha_0, \alpha_1, \alpha_2)^\wedge$ (for example, see [21]); the associated shift is subnormal. It follows from [6] that

$$\alpha_n \nearrow L := \frac{1}{\sqrt{2}}\left(\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0}\right)^{1/2} \quad \text{as } \; n \to \infty.$$

The organization of this paper is as follows. In Section 2 we recall some terminology concerning semi-cubically hyponormal weighted shifts. In Section 3 we characterize the semi-cubic hyponormality of weighted shifts W_α with p.d.c., where $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$, and then consider a related example.

Throughout this paper, $\mathbb{R}_+$, $\mathbb{N}$, and $\mathbb{N}_0$ are the sets of nonnegative real numbers, positive integers, and nonnegative integers, respectively.

2. Preliminaries

We recall some standard terminology for semi-cubically hyponormal weighted shifts (cf. [12]). Let $\ell^2(\mathbb{N}_0)$ be the space of square summable sequences in $\mathbb{C}$ and let $\{e_i\}_{i=0}^\infty$ be an orthonormal basis of $\ell^2(\mathbb{N}_0)$. For a weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$ in $\mathbb{R}_+$, the associated weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$ is *semi-cubically hyponormal* if

$$D(s) := [(W_\alpha + sW_\alpha^3)^*, W_\alpha + sW_\alpha^3] \geq 0, \quad s \in \mathbb{C}. \tag{2.1}$$

In fact, the condition in (2.1) is equivalent to a simpler one, as in the following proposition whose proof comes from [8, Prop. 1].

Proposition 2.1. *Let W_α be a weighted shift with a weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$ in $\mathbb{R}_+$. Then W_α is semi-weakly n-hyponormal if and only if $W_\alpha + tW_\alpha^n$ is hyponormal for all $t \geq 0$.*

Proof. It is sufficient to show the necessity. For any $s \in \mathbb{C}$, we may take nonnegative real numbers t and θ such that $s = te^{i(n-1)\theta}$. Recall that there exists a unitary operator U such that $UW_\alpha U^* = e^{-i\theta}W_\alpha$. Then

$$U(W_\alpha + sW_\alpha^n)U^* = UW_\alpha U^* + s(UW_\alpha U^*)^n = e^{-i\theta}(W_\alpha + tW_\alpha^n),$$

so the inequality for all $t \geq 0$ suffices to yield (2.1). $\square$

By Proposition 2.1, W_α is semi-cubically hyponormal if and only if $D(s) \geq 0$ for all $s \in \mathbb{R}_+$. Observe that

$$D(s) = \begin{pmatrix} q_0 & 0 & z_0 & 0 & \cdots \\ 0 & q_1 & 0 & z_1 & \ddots \\ z_0 & 0 & q_2 & \ddots & \ddots \\ 0 & z_1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad s \in \mathbb{R}_+,$$

where for all $k \in \mathbb{N}_0$,

$$\begin{aligned} &q_k := u_k + v_k s^2, \quad z_k := \sqrt{w_k} s, \quad u_k := \alpha_k^2 - \alpha_{k-1}^2, \\ &v_k := \alpha_k^2 \alpha_{k+1}^2 \alpha_{k+2}^2 - \alpha_{k-3}^2 \alpha_{k-2}^2 \alpha_{k-1}^2, \; w_k := \alpha_k^2 \alpha_{k+1}^2 (\alpha_{k+2}^2 - \alpha_{k-1}^2)^2, \end{aligned}$$

with $\alpha_{-3} = \alpha_{-2} = \alpha_{-1} = 0$. Consider two submatrices

$$D^{(1)}(s) = \begin{pmatrix} q_0 & z_0 & 0 & & \\ z_0 & q_2 & z_2 & 0 & \\ 0 & z_2 & q_4 & z_4 & \ddots \\ & 0 & z_4 & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix} \text{ and } D^{(2)}(s) = \begin{pmatrix} q_1 & z_1 & 0 & & \\ z_1 & q_3 & z_3 & 0 & \\ 0 & z_3 & q_5 & z_5 & \ddots \\ & 0 & z_5 & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

and observe that $D(s) = D^{(1)}(s) \oplus D^{(2)}(s)$, $s \in \mathbb{R}_+$. Define

(2.2)
$$D_n^{(1)}(t) = \begin{pmatrix} q_0 & z_0 & 0 & & \\ z_0 & q_2 & z_2 & 0 & \\ 0 & z_2 & q_4 & z_4 & \ddots \\ & 0 & z_4 & \ddots & \ddots \\ & & \ddots & \ddots & q_{2n} \end{pmatrix}, \quad D_n^{(2)}(t) = \begin{pmatrix} q_1 & z_1 & 0 & & \\ z_1 & q_3 & z_3 & 0 & \\ 0 & z_3 & q_5 & z_5 & \ddots \\ & 0 & z_5 & \ddots & \ddots \\ & & \ddots & \ddots & q_{2n+1} \end{pmatrix},$$

where $t = s^2$. Then W_α is semi-cubically hyponormal if and only if $D_n^{(j)}(t) \geq 0$ for all $n \geq 0$, $j = 1, 2$.

To detect the positivity of $D_n^{(j)}(t)$ in (2.2), we consider a matrix with the form below:

$$M_n(t) = \begin{pmatrix} \check{q}_0 & \check{r}_0 & 0 & & & \\ \check{r}_0 & \check{q}_1 & \check{r}_1 & 0 & & \\ 0 & \check{r}_1 & \check{q}_2 & \check{r}_2 & \ddots & \\ & 0 & \check{r}_2 & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \check{q}_{n-1} & \check{r}_{n-1} \\ & & & 0 & \check{r}_{n-1} & \check{q}_n \end{pmatrix},$$

where $\check{q}_k := \check{u}_k + \check{v}_k t, \check{r}_k := \sqrt{\check{w}_k t}\ (k \geq 0)$, and $\check{u}_k \geq 0, \check{v}_k \geq 0, \check{w}_k \geq 0, t \geq 0$. (We take the approach in [6] for what follows.) Then

$$d_n(t) := \det M_n(t) = \sum_{i=0}^{n+1} c(n,i)\, t^i, \tag{2.3}$$

and it follows from [6] that

$$\begin{aligned} c(0,0) &= \check{u}_0, \quad c(0,1) = \check{v}_0, \\ c(1,0) &= \check{u}_0\check{u}_1, \ c(1,1) = \check{u}_1\check{v}_0 + \check{u}_0\check{v}_1 - \check{w}_0, \ c(1,2) = \check{v}_1\check{v}_0, \\ c(n,i) &= \check{u}_n c(n-1,i) + \check{v}_n c(n-1,i-1) - \check{w}_{n-1} c(n-2,i-1), \\ c(n,n+1) &= \check{v}_0\check{v}_1\cdots\check{v}_n, \quad n \geq 2, \ 0 \leq i \leq n, \end{aligned}$$

with $c(-n,-i) := 0$ for all $n, i \in \mathbb{N}$. Suppose that $\check{u}_n\check{v}_{n+1} = \check{w}_n$ $(n \geq 2)$; this yields that

$$c(n,i) = \begin{cases} \check{v}_n \cdots \check{v}_2 c(1,2), & \text{if } i = n+1, \\ \check{u}_n c(n-1,n) + \check{v}_n \cdots \check{v}_3 \rho, & \text{if } i = n, \\ \check{u}_n c(n-1,n-1) + \check{v}_n \cdots \check{v}_3 \tau, & \text{if } i = n-1, \\ \check{u}_n c(n-1,i), & \text{if } 0 \leq i \leq n-2, \end{cases} \tag{2.4}$$

for all $n \geq 3$, where

$$\rho := \check{v}_2 c(1,1) - \check{w}_1 c(0,1) \text{ and } \tau := \check{v}_2 c(1,0) - \check{w}_1 c(0,0). \tag{2.5}$$

To detect the positivity of $D_n^{(j)}(t)$, $j = 1, 2$, we consider

$$d_n^{(j)}(t) := \det D_n^{(j)}(t) = \sum_{i=0}^{n+1} c_j(n,i) t^i,$$

for $j = 1, 2$. Note that if $c_j(n, n+1) > 0$ and $c_j(n,i) \geq 0$ for all $n \geq 0$ with $0 \leq i \leq n$, then every matrix $D_n^{(j)}(t)$ is obviously positive for all $n \geq 0$ and $t > 0$. Recall that W_α has *positive determinant coefficients (p.d.c.)* (and is therefore semi-cubically hyponormal) if all coefficients in $d_n^{(j)}(t)$ are nonnegative and the $c_j(n, n+1)$ are strictly positive for all $n \in \mathbb{Z}_+$ and $j = 1, 2$ (cf. [12, Def. 2.2]).

The following is the crucial lemma which can be obtained from the proof of Theorem 4.3 in [6], and which we will apply in succession to $D_n^{(1)}(t)$ and $D_n^{(2)}(t)$.

Lemma 2.2. *Under the notation above, we suppose that* $\breve{u}_n\breve{v}_{n+1} = \breve{w}_n$ $(n \geq 2)$. *Then the determinant of* $M_n(t)$ *in* (2.3) *has non-negative coefficients* $c(n,i)$ *for all* n *and* i *if and only if the following conditions hold:*

(i) $c(1,1), c(2,1), c(2,2), c(3,2)$ *are all positive,*

(ii) $\Gamma_n := \breve{v}_2\breve{v}_1\breve{v}_0 + \frac{\breve{v}_n}{\breve{u}_n}\rho \geq 0$, *for all* $n \geq 3$,

(iii) $\Omega_n := \breve{v}_2\breve{v}_1\breve{v}_0 + \frac{\breve{v}_{n-1}}{\breve{u}_{n-1}}\rho + \frac{\breve{v}_{n-1}}{\breve{u}_{n-1}}\frac{\breve{v}_n}{\breve{u}_n}\tau \geq 0$, *for all* $n \geq 4$.

3. A Special Case of Semi-cubical Hyponormality with p.d.c.

Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. To determine when W_α is a semi-cubically hyponormal weighted shift with p.d.c., we will use Lemma 2.2 for $j = 1, 2$ considering conditions (i), (ii), and (iii) in Lemma 2.2. We will denote the coefficients of the determinants $c_1(n,i)$ and $c_2(n,i)$ with the obvious meaning, and distinguish other quantities between $j = 1$ and $j = 2$ with superscripts (for example, $\rho^{(1)}$ and $\rho^{(2)}$). Note first that it follows from Lemma 3.1 of [20] that we have both $u_n^{(1)}v_{n+1}^{(1)} = w_n^{(1)}$ and $u_n^{(2)}v_{n+1}^{(2)} = w_n^{(2)}$ for $n \geq 2$.

3.1. The p.d.c. condition for $D_n^{(1)}$.

The following $(n+1) \times (n+1)$ matrix is the expression of $D_n^{(1)}$:

$$D_n^{(1)} = \begin{pmatrix} u_0 + v_0 t & \sqrt{w_0 t} & 0 & 0 & \cdots & 0 \\ \sqrt{w_0 t} & u_2 + v_2 t & \sqrt{w_2 t} & 0 & \cdots & 0 \\ 0 & \sqrt{w_2 t} & u_4 + v_4 t & \sqrt{w_4 t} & \cdots & 0 \\ 0 & 0 & \sqrt{w_4 t} & u_6 + v_6 t & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \sqrt{w_{2n-2} t} \\ 0 & 0 & 0 & \cdots & \sqrt{w_{2n-2} t} & u_{2n} + v_{2n} t \end{pmatrix}.$$

Since $c_1(1,1) = (uv-1)x > 0$ and positivity conditions for the coefficients $c_1(2,1)$, $c_1(2,2)$, $c_1(3,2)$ can be obtained by some direct computations, we will concentrate our consideration on (ii) and (iii) in Lemma 2.2.

Set $\eta_n = \frac{v_n}{u_n}$. Then we may prove that $\eta_n \nearrow U$; see [20, Lemma 3.3], where

$$U = \frac{(\Psi_1^2 + \Psi_0)^2}{2\Psi_0^2}\left(2\Psi_0 + \Psi_1^2 + \Psi_1\sqrt{4\Psi_0 + \Psi_1^2}\right),$$

where the values Ψ_0 and Ψ_1 are those associated with the Stampfli completion.

Lemma 3.1. *With the notation above, we have that* $\Gamma_n^{(1)} := v_4v_2v_0 + \eta_{2n}\rho^{(1)} \geq 0$ *for all* $n \geq 3$, *if and only if one of the following conditions holds:*

(i) $\rho^{(1)} \geq 0$,

(ii) $\rho^{(1)} < 0$ *and* $v_4 v_2 v_0 + U\rho^{(1)} \geq 0$,

where $\rho^{(1)} = v_4 c_1(1,1) - w_2 c_1(0,1)$ *is as in* (2.5).

Proof. We consider first $\Gamma_n^{(1)} = v_4 v_2 v_0 + \eta_{2n}\rho^{(1)}$, where $\rho^{(1)}$ is as in (2.5). In this case, to check the positivity of $\Gamma_n^{(1)}$, we define

$$\Delta\Gamma_n^{(1)} = \Gamma_{n+1}^{(1)} - \Gamma_n^{(1)}.$$

Then $\Delta\Gamma_n^{(1)} = (\eta_{n+2} - \eta_n)\rho^{(1)}$. If $\rho^{(1)} \geq 0$, since $\{\eta_n\}$ is increasing, $\Delta\Gamma_n^{(1)}$ is positive, i.e., $\Gamma_n^{(1)}$ is increasing in n, so to detect the positivity of $\Gamma_n^{(1)}$ for $n \geq 3$, it is enough to consider only the positivity of $\Gamma_3^{(1)}$. On the other hand, if $\rho^{(1)} < 0$, then $\Delta\Gamma_n^{(1)}$ is negative. So $\Gamma_n^{(1)}$ is decreasing in n, and to detect the positivity of $\Gamma_n^{(1)}$ for $n \geq 3$, it is enough to examine the positivity of the limit of $\Gamma_n^{(1)}$. Since $\eta_n \nearrow U$, we can obtain

$$\lim_{n\to\infty} \Gamma_n^{(1)} = v_4 v_2 v_0 + U\rho^{(1)}.$$

Hence the proof is complete. □

Lemma 3.2. *With above notation, we have that* $\Omega_n^{(1)} := v_4 v_2 v_0 + \eta_{2n-2}\rho^{(1)} + \eta_{2n-2}\eta_{2n}\tau^{(1)} \geq 0$ *for* $n \geq 4$ *if and only if one of the following conditions holds:*

(i) $s \geq 0$ *and* $v_4 v_2 v_0 + \eta_6\rho^{(1)} + \eta_6\eta_8\tau^{(1)} \geq 0$,

(ii) $s < 0$ *and* $v_4 v_2 v_0 + U\rho^{(1)} + U^2\tau^{(1)} \geq 0$,

where $\tau^{(1)} = v_4 c_1(1,0) - w_2 c_1(0,0)$ *as in* (2.5) *and*

$$s = \rho^{(1)} + \frac{\eta_8\eta_{10} - \eta_6\eta_8}{\eta_8 - \eta_6}\tau^{(1)}.$$

Proof. To consider the positivity of $\Omega_n^{(1)}$, we first define $\Delta\Omega_n^{(1)} = \Omega_{n+1}^{(1)} - \Omega_n^{(1)}$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned} \Delta\Omega_n^{(1)} &= (\eta_{2n} - \eta_{2n-2})\,\rho^{(1)} + (\eta_{2n}\eta_{2n+2} - \eta_{2n-2}\eta_{2n})\,\tau^{(1)} \\ &= (\eta_{2n} - \eta_{2n-2})\left(\rho^{(1)} + \frac{\eta_{2n}\eta_{2n+2} - \eta_{2n-2}\eta_{2n}}{\eta_{2n} - \eta_{2n-2}}\tau^{(1)}\right). \end{aligned} \tag{3.1}$$

For brevity, we set

$$s_n(x,u,v,w) := \rho^{(1)} + \frac{\eta_{2n}\eta_{2n+2} - \eta_{2n-2}\eta_{2n}}{\eta_{2n} - \eta_{2n-2}}\tau^{(1)}.$$

We will claim that $s_n(x,u,v,w)$ is constant in $n \geq 4$, i.e., s_n is independent of n for $n \geq 4$. The original idea for the proof comes from [14, Lemma 4.3]; see more recently [13]. Define

$$Q_n := \frac{\eta_n\eta_{n+2} - \eta_{n-2}\eta_n}{\eta_n - \eta_{n-2}}. \tag{3.2}$$

Observe that with $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}$,

$$\eta_n(\alpha) = \eta_{n-3}\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right), \quad n \geq 6. \tag{3.3}$$

Then it follows from (3.1) and (3.2) that

$$Q_n(\alpha) = Q_{n-3}\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right), n \geq 8.$$

A direct computation shows that

$$Q_5\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right) = Q_6\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right);$$

in fact, we can confirm that its value is

$$\frac{v(2u^2w - u^2v + 2uv^2 - 4uvw + vw^2)(uv^2 + u^2w - 3uvw + vw^2)^2}{u^2(v-u)^4(w-v)^2}.$$

Put $p = \alpha_6^2$, where α_6 is the 6th term of α. Then it is well known that

$$(\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge} = \sqrt{u}, (\sqrt{v}, \sqrt{w}, \sqrt{p})^{\wedge},$$

which implies that

$$Q_n\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right) = Q_{n-1}\left((\sqrt{v}, \sqrt{w}, \sqrt{p})^{\wedge}\right), n \geq 5.$$

If we mimic the proof of [14, Lemma 4.3], we get

$$Q_7\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right) = Q_6\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right).$$

Repeating this argument, we obtain

$$Q_n\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right) = Q_5\left((\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}\right), \text{ for all } n \geq 5.$$

Hence Q_n is constant in n, $n \geq 8$, and so $s_n(x, u, v, w)$ is constant in n for $n \geq 4$.

Now, we set $s = s(x, u, v, w) := s_4(x, u, v, w)$. By (3.1), obviously

$$\Delta\Omega_n^{(1)} = (\eta_{2n} - \eta_{2n-2})\, s, \quad n \geq 4.$$

Repeating the method in the proof of Lemma 3.1, we obtain that $\Omega_n^{(1)} \geq 0$ for $n \geq 4$ if and only if either (i) $s \geq 0$ and $\Omega_4^{(1)} = v_4v_2v_0 + \eta_6\rho^{(1)} + \eta_6\eta_8\tau^{(1)} \geq 0$ or (ii) $s < 0$ and

$$\lim_{n\to\infty} \Omega_n^{(1)} = v_4v_2v_0 + U\rho^{(1)} + U^2\tau^{(1)} \geq 0.$$

Hence the proof is complete. □

3.2. The p.d.c. condition for $D_n^{(2)}$.

The following $(n+1)\times(n+1)$ matrix is $D_n^{(2)}$:

$$D_n^{(2)} = \begin{pmatrix} u_1+v_1t & \sqrt{w_1t} & 0 & 0 & \cdots & 0 \\ \sqrt{w_1t} & u_3+v_3t & \sqrt{w_3t} & 0 & \cdots & 0 \\ 0 & \sqrt{w_3t} & u_5+v_5t & \sqrt{w_5t} & \cdots & 0 \\ 0 & 0 & \sqrt{w_5t} & u_7+v_7t & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \sqrt{w_{2n-1}t} \\ 0 & 0 & 0 & \cdots & \sqrt{w_{2n-1}t} & u_{2n+1}+v_{2n+1}t \end{pmatrix}.$$

As in the case of $D_n^{(1)}$, conditions for the positivity of the coefficients $c_2(1,1)$, $c_2(2,1)$, $c_2(2,2)$, $c_2(3,2)$ can be obtained by some direct computations. One may also compute that

$$(3.4)\quad c_2(2,1) = (w-v)c_2(1,1) \text{ and } c_2(3,2) = \frac{u^2(w-u)^3}{w(v-u)(uv-2uw+w^2)}c_2(2,2).$$

It is straightforward to verify that $\frac{u^2(w-u)^3}{w(v-u)(uv-2uw+w^2)}$ is positive, and it results from the above that $c_2(2,1)$ and $c_2(1,1)$ have the same sign, as do $c_2(3,2)$ and $c_2(2,2)$.

Lemma 3.3. *With the above notation, the following conditions are equivalent:*

(i) $\Gamma_n^{(2)} := v_5v_3v_1 + \eta_{2n+1}\rho^{(2)} \geq 0$ *for all* $n \geq 3$,

(ii) $\Omega_n^{(2)} = v_5v_3v_1 + \eta_{2n-1}\rho^{(2)} \geq 0$ *for all* $n \geq 4$,

(iii) *one of the following conditions holds:*

(iii-a) $\rho^{(2)} \geq 0$,

(iii-b) $\rho^{(2)} < 0$ *and* $v_5v_3v_1 + U\rho^{(2)} \geq 0$,

where $\rho^{(2)} = v_5c_2(1,1) - w_3c_2(0,1)$ *is as in* (2.5).

Proof. (i) ⇔ (iii). Since the proof is exactly that of Lemma 3.1, we omit it here.
(i) ⇔ (ii). Recall that $\Omega_n^{(2)} := v_5v_3v_1 + \eta_{2n-1}\rho^{(2)} + \eta_{2n-1}\eta_{2n+1}\tau^{(2)}$. Since

$$\tau^{(2)} = v_5c_2(1,0) - w_3c_2(0,0) = u_1(v_5u_3 - w_3) = 0,$$

$\Omega_n^{(2)} = \Gamma_{n-1}^{(2)}$ for all $n \geq 4$. Hence the proof is complete. □

3.3. The main theorem.

We now give the main theorem of this paper. Combining results in Subsections 3.1 and 3.2, we obtain the following theorem with the above notation.

Theorem 3.4. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ *with* $1 < x < u < v < w$. *Then* W_α *is a semi-cubically hyponormal weighted shift with p.d.c. if and only if the following conditions hold:*

(i) $c_1(2,1), c_1(2,2), c_1(3,2), c_2(1,1), c_2(2,2)$ *are nonnegative,*

(ii) *one of the following conditions holds:*
 (ii-a) $\rho^{(1)} \geq 0$,
 (ii-b) $\rho^{(1)} < 0$ *and* $v_4 v_2 v_0 + U\rho^{(1)} \geq 0$,

(iii) *one of the following conditions holds:*
 (iii-a) $s \geq 0$ *and* $v_4 v_2 v_0 + \eta_6 \rho^{(1)} + \eta_6 \eta_8 \tau^{(1)} \geq 0$,
 (iii-b) $s < 0$ *and* $v_4 v_2 v_0 + U\rho^{(1)} + U^2 \tau^{(1)} \geq 0$,

(iv) *one of the following conditions holds:*
 (iv-a) $\rho^{(2)} \geq 0$,
 (iv-b) $\rho^{(2)} < 0$ *and* $v_5 v_3 v_1 + U\rho^{(2)} \geq 0$.

Remark 3.5. According to the construction of Stampfli's completion from three values u, v and w, Theorem 3.4 can yield a characterization of semi-cubical hyponormality of a backward 2-step weighted shift W_α with Stampfli's completion tail, where $\alpha : 1, 1, (\sqrt{x}, \sqrt{u}, \sqrt{v})^\wedge$ with $1 < x < u < v$, because if we choose $w = \frac{u(v^2+ux-2vx)}{v(u-x)}$ for our backward 3-step extension, then $\alpha' : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ produces the same weighted shift W_α. Of course, the case of 1-step extension can be considered similarly; this case was studied in [20, Th. 3.7].

Remark 3.6. Using some technical computations, we believe it is possible to characterize the semi-cubical hyponormality of backward n-step extended weighted shifts W_α with Stampfli's completion tail. We presume their proofs will be intricate.

In what follows, we consider an example related to Theorem 3.4.

Example 3.7. Let $\alpha : 1, 1, \sqrt{x}, \left(\sqrt{\frac{111}{100}}, \sqrt{\frac{112}{100}}, \sqrt{\frac{113}{100}}\right)^\wedge$, where x is a real variable with $1 < x < \frac{111}{100}$. We obtain a range of x for semi-cubical hyponormality with p.d.c. of W_α.

(i) $c_1(2,1) \geq 0 \Leftrightarrow 69375x^2 - 158688x + 90160 \leq 0$,
 $c_1(2,2) \geq 0 \Leftrightarrow x \leq \frac{195776}{152625}$,
 $c_1(3,2) \geq 0 \Leftrightarrow 1149751875x^2 - 2603155192x + 1462768720 \leq 0$,
 $c_2(1,1) \geq 0 \Leftrightarrow 122 - 111x \geq 0$,
 $c_2(2,2) \geq 0 \Leftrightarrow 4648721 - 4082025x \geq 0$.

Thus we get (i) holds if and only if $\frac{4(19836-\sqrt{2538771})}{69375} < x < \frac{122}{111}$.

(ii) Since $1 < x < \frac{111}{100}$, we can check without difficulty that $\rho^{(1)} > 0$; i.e., (ii) always holds for $1 < x < \frac{111}{100}$.

(iii) One computes that

$$s \geq 0 \Leftrightarrow \varphi(x) = 1800279654375x^2 - 4093200707344x + 2314100115040 \leq 0;$$

a computation shows that this yields also $v_4 v_2 v_0 + \eta_6 \rho^{(1)} + \eta_6 \eta_8 \tau^{(1)} \geq 0$. Hence (iii-a) holds $\Leftrightarrow \delta \leq x < \frac{111}{100}$, where $\delta \approx 1.053$ is the smallest root of $\varphi(x) = 0$. And

$$v_4 v_2 v_0 + U\rho^{(1)} + U^2 \tau^{(1)} \geq 0 \Leftrightarrow \phi(x) = a_3 x^3 + a_2 x^2 - a_1 x + a_0 \leq 0,$$

where a_j are positive real numbers. We can check easily that $\phi(x)$ has roots $\delta_3 < 0 < \delta_2 < \delta_1$, with $\delta_2 \approx 1.036$. Assembling these computations gives

$$s < 0 \text{ and } v_4v_2v_0 + U\rho^{(1)} + U^2\tau^{(1)} \geq 0 \Leftrightarrow \delta_2 \leq x < \delta.$$

Therefore we get (iii) holds if and only if $\delta_2 \leq x < \frac{111}{100}$.

(iv) holds for $1 < x < \frac{111}{100}$, because further computation shows $v_5v_3v_1 + \eta_7\rho^{(2)} \geq 0$ and $v_5v_3v_1 + U\rho^{(2)} \geq 0$ for $1 < x < \frac{111}{100}$.

Combining the analyses of (i)-(iv) above, we get that W_α is semi-cubically hyponormal with p.d.c. if and only if $\frac{4(19836-\sqrt{2538771})}{69375} < x < \frac{122}{111}$.

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