

## $\epsilon$ -Compatible Maps and Common Fixed Point Theorems

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ABSTRACT. The aim of this paper is to introduce the notion of  $\epsilon$ -compatible maps and obtain some common fixed point theorems. Also, our results generalize some well known fixed point theorems.

### 1. Introduction and Preliminaries

Stephen Banach in 1922, formulated a classical theorem, in nonlinear functional analysis, which became known as the Banach contraction principle. This theorem states that if a self-mapping  $f$  of a complete metric space  $(X, d)$  satisfies the condition

$$d(fx, fy) \leq kd(x, y), 0 \leq k < 1$$

for each  $x, y \in X$ , then  $f$  has a unique fixed point, that is, there exists a unique  $z$  in  $X$  such that  $f(z) = z$ .

This classical theorem is used to determine the existence and uniqueness of a fixed point for a contraction map and became an excellent tool to compute the fixed point through iteration process. This theorem also gave birth to many other notable fixed point theorems such as Edelstein(1962), Kannan(1968), Ciric(1974), Nudler(1969) and, etc. The fixed theorems which give the existence of fixed point are widely used in physics to determine the steady state temperature distribution, to analyse neutron transport, in biology to analyse epidemiological parameters, in economical analysis, and also in computer sciences.

Every fixed point theorem of a self-mapping  $f$  of a metric space  $(X, d)$  can also be considered as a common fixed theorem of  $f$  and the identity mapping on  $X$ . Goebel in 1968 obtained a coincidence point theorem by replacing the identity map in Banach contraction principle with a function  $g$  of  $X$ , in such a way that

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**Theorem 1.1.** *Let  $A$  be an arbitrary set and  $X$  be a metric space with the metric  $d$ . Suppose that  $f, g$  are two mappings defined on the set  $A$  with the values in  $X$ . If  $f(A) \subseteq g(A)$ ,  $g(A)$  is a complete subspace of  $X$  and for all  $x, y \in A$*

$$d(fx, fy) \leq kd(gx, gy), 0 \leq k < 1,$$

*then  $f$  and  $g$  have a coincidence point, that is, there exists  $z \in A$  such that  $f(z) = g(z)$ .*

The property of common fixed point for contractive type mappings necessarily implies the commutativity conditions, a condition on the ranges of the mappings. The theory of common fixed point is equally interesting as that of a mapping. In 1976, Jungck [1] initiated the study of existence of common fixed point under commutativity of mappings and it was the direct generalization of Banach contraction mapping principle for a pair of mappings. According to his theorem,

**Theorem 1.2([1]).** *Let  $(X, d)$  be a complete metric space and let  $f$  and  $g$  be commuting self-maps of  $X$  satisfying the conditions:*

- (i)  $fX \subseteq gX$
- (ii)  $d(fx, fy) \leq kd(gx, gy)$ , for all  $x, y \in X$  and some  $0 \leq k < 1$ .

*If  $g$  is continuous, then  $f$  and  $g$  have a unique common fixed point.*

Sessa [2] established a fixed point theorem for non-commuting pair of mappings by introducing weakly commutativity. In 1986, Jungck [3] generalized the weak commutativity by introducing the compatibility of mappings. Since then, many studies of common fixed point of self-mappings, satisfying contractive type conditions were initiated regarding the compatibility of mappings. In addition, many generalized notions of compatible mappings were established and the results were obtained for non-commuting mappings, see [4]-[12].

Now, we recall the relevant concepts to common fixed point theorems.

**Definition 1.1([3]).** Self-maps  $f$  and  $g$  of a metric space  $(X, d)$  are compatible if  $\lim_n d(fgx_n, gfx_n) = 0$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_n fx_n = \lim_n gx_n = t$  for some  $t \in X$ .

**Definition 1.2([7]).** Two self-maps  $f, g$  of a non-empty set  $X$  are weakly compatible if  $fgx = gfx$  whenever  $fx = gx$ ,  $x \in X$ .

**Definition 1.3([8]).** Two self-maps  $f, g$  of a non-empty set  $X$  are occasionally weakly compatible if there exists an element  $x \in X$  such that  $fx = gx$  implies  $fgx = gfx$ .

**Definition 1.4([9]).** Two self-maps  $f$  and  $g$  of a metric space  $(X, d)$  are sub-compatible if there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_n fx_n = \lim_n gx_n = t$ , for some  $t \in X$  and  $\lim_n d(fgx_n, gfx_n) = 0$ .

**Definition 1.5([11]).** Two self-maps  $f$  and  $g$  of a metric space  $(X, d)$  are conditionally compatible if whenever the set of sequences  $\{y_n\}$  satisfying  $\lim_n fy_n = \lim_n gy_n$

is non-empty, there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_n fx_n = \lim_n gx_n = t$ , for some  $t \in X$  and  $\lim_n d(fgx_n, gfx_n) = 0$ .

**Definition 1.6**([13]). Two self-mappings  $A$  and  $S$  of a metric space  $(X, d)$  are called reciprocally continuous if  $\lim_{n \rightarrow \infty} ASx_n = At$  and  $\lim_{n \rightarrow \infty} SAx_n = St$  whenever  $\{x_n\}$  is a sequence such that  $\lim_{n \rightarrow \infty} Sx_n = t$  and  $\lim_{n \rightarrow \infty} Ax_n = t$  for some  $t$  in  $X$ .

Now, we give the definition of  $\epsilon$ -compatible mappings.

**Definition 1.7.** Two self-maps  $f$  and  $g$  of a metric space  $(X, d)$  are said to be  $\epsilon$ -compatible if for every  $\epsilon > 0$ , there exists an element  $x \in X$  such that  $d(fx, gx) < \epsilon$  implies  $d(fgx, gfx) < \phi(\epsilon)$ , where  $\phi : [0, \infty) \rightarrow [0, \infty)$  is continuous at 0.

**Example 1.1.** Let  $X = \mathbb{R}$  be a metric space with the usual metric  $d(x, y) = |x - y|$ . Define  $f, g : X \rightarrow X$  by

$$fx = \begin{cases} x^2, & x \neq 0 \\ 1, & x = 0. \end{cases} \quad gx = \begin{cases} 2x, & x \neq 0 \\ 2, & x = 0. \end{cases}$$

For  $x \neq 0$ ,  $|fx - gx| \rightarrow 0$  and  $|fgx - gfx| \rightarrow 0$  as  $x \rightarrow 0$ . That is, the pair  $f$  and  $g$  is  $\epsilon$ -compatible. For  $x = 2$ ,  $fx = gx = 4$ , but  $|fgx - gfx| \neq 0$ . Hence the pair  $f$  and  $g$  are not compatible and also not occasionally weakly compatible.

**Remark 1.1.** The concept of  $\epsilon$ -compatible is independent of that of compatible.

Consider the two mappings  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $fx = x + 1$  and  $gx = x + 2$ . The pair  $f$  and  $g$  are compatible but not  $\epsilon$ -compatible.

**Example 1.2.** Consider the two self-mappings on  $\mathbb{R}$  such that

$$fx = \begin{cases} \frac{1}{x} + 2x, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases} \quad \text{and} \quad gx = 2x$$

Here, there is no sequence  $\{x_n\}$  and  $t$  in  $\mathbb{R}$  such that  $fx_n, gx_n$  converges to  $t$ . But,  $|fx - gx| = \frac{1}{x} \rightarrow 0$  as  $x \rightarrow \infty$ . Then  $fgx = \frac{1}{2x} + 4x$  and  $gfx = \frac{2}{x} + 4x$  gives  $|fgx - gfx| \rightarrow 0$  as  $x \rightarrow \infty$ . Hence,  $f$  and  $g$  are  $\epsilon$ -compatible but not subcompatible.

**Lemma 1.1.** *Let  $f$  be a self-map on a metric space  $(X, d)$ . If  $f$  has a fixed point in  $X$ , then, there exists a mapping  $g$  on  $X$  such that  $f$  and  $g$  are  $\epsilon$ -compatible.*

*Proof.* Let  $a$  be the fixed point of  $f$ . Define a map  $g$  on  $X$  such that  $g(x) = a$ , for all  $x \in X$ . For any  $\epsilon > 0$ ,  $d(fa, ga) < \epsilon$  implies  $d(fga, gfa) < \epsilon$ . Thus,  $f, g$  are  $\epsilon$ -compatible.  $\square$

**Lemma 1.2.** *Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in a metric space  $(X, d)$  such that  $d(x_n, y_n) \rightarrow 0$  as  $n \rightarrow \infty$ . If  $\{x_n\}$  is a Cauchy sequence, then  $\{y_n\}$  so is.*

*Proof.* Since  $\{x_n\}$  is Cauchy and  $\lim_n d(x_n, y_n) = 0$ , then, we have that given  $\frac{\epsilon}{3} > 0$ , there exists  $N \in \mathbb{N}$  such that

$$d(x_n, x_m) < \frac{\epsilon}{3} \quad \text{and} \quad d(x_n, y_n) < \frac{\epsilon}{3}, \quad \forall m, n \geq N.$$

Now, for all  $m, n \geq N$ ,

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

$$\text{i.e., } d(y_n, y_m) < \epsilon,$$

Thus,  $\{y_n\}$  is Cauchy sequence.  $\square$

In this paper, we prove some common fixed point theorems using  $\epsilon$ -compatible maps. Our first result is proved in the setting of compact metric space and it follows.

## 2. Main Results

**Theorem 2.1.** *Let  $f$  and  $g$  be a  $\epsilon$ -compatible maps of a compact metric space  $(X, d)$ . If  $f$  is continuous and satisfies the condition that  $d(gx, gy) < d(fx, fy), x \neq y$ , then  $f$  and  $g$  have a unique common fixed point.*

*Proof.* Since  $f$  and  $g$  are  $\epsilon$ -compatible, for every  $n \in \mathbb{N}$ , there exists an element  $x_n$  in  $X$  such that  $d(fx_n, gx_n) < \frac{1}{n} \Rightarrow d(fgx_n, gfx_n) < \phi(\frac{1}{n})$ . By the compactness of  $X$ ,  $\{x_n\}$  has a subsequence,  $\{x_{n_k}\}$  converges to an element in  $X$ , let it be  $p$ . Since  $f$  is continuous,  $fx_{n_k} \rightarrow fp$  as  $n_k \rightarrow \infty$ . Now,  $d(gx_{n_k}, gp) < d(fx_{n_k}, fp)$  implies  $\{gx_{n_k}\}$  converges to  $gp$ . Since  $d(fx_n, gx_n) \rightarrow 0$ , we have that  $fp = gp = q$ . The continuity of  $f$  implies that  $fgx_{n_k}, f^2x_{n_k} \rightarrow fq$  as  $n_k \rightarrow \infty$ . But  $d(gfx_{n_k}, gq) < d(f^2x_{n_k}, fq)$  implies that  $\lim_{n_k \rightarrow \infty} gfx_{n_k} = gq$ . By  $\epsilon$ -compatibility of  $f$  and  $g$ , we have that  $fq = gq = q_1$ . Next, we prove the unique point of coincidence of  $f$  and  $g$ . Suppose that  $q$  and  $q_1$  are two distinct elements in  $X$  such that  $q = fp = gp$  and  $q_1 = fq = gq$ . Now,  $d(gp, gq) < d(fp, fq)$ , i.e.,  $d(q, q_1) < d(q, q_1)$  contradicts to our assumption. Therefore,  $f$  and  $g$  have a unique point of coincidence. By the unique point of coincidence, we have  $q = q_1$ . Thus,  $fp = gp = fgp = gfp$  i.e.,  $fq = gq = q$ . Hence,  $f$  and  $g$  have a unique common fixed point.  $\square$

**Theorem 2.2.** *Let  $f$  be a continuous self-mapping of a complete metric space  $(X, d)$  and  $g : X \rightarrow X$  be a mapping such that*

$$(3.1) \quad d(gx, gy) \leq \alpha d(fx, fy), \quad \alpha \in (0, 1)$$

*for all  $x, y \in X$ . Then  $f$  and  $g$  are  $\epsilon$ -compatible pair if, and only if  $f$  and  $g$  have a unique common fixed point.*

*Proof.* To prove the necessary part, we assume that  $f$  and  $g$  have a unique common fixed point,  $p$  (say), in  $X$ .

Given  $\epsilon > 0$ ,  $p \in X$  such that  $d(fp, gp) < \epsilon \Rightarrow d(fgp, gfp) < \epsilon$ . That is,  $f$  and  $g$  are  $\epsilon$ -compatible.

Now, we prove the sufficient part of our theorem. Assume that  $f$  and  $g$  are  $\epsilon$ -compatible. Since  $f$  and  $g$  are  $\epsilon$ -compatible pair, for every  $n \in \mathbb{N}$ , there exists an element  $x_n$  in  $X$  such that  $d(fx_n, gx_n) < \frac{1}{n^2} \Rightarrow d(fgx_n, gfx_n) < \phi(\frac{1}{n^2})$ . Now, for  $x = x_n$  and  $y = x_n + 1$ , we have

$$(3.2) \quad d(gx_n, gx_{n+1}) \leq \alpha d(fx_n, fx_{n+1}).$$

By the triangle inequality  $d(fx_n, fx_{n+1}) \leq d(fx_n, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, fx_{n+1})$ , (3.2) gives,

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \alpha(d(fx_n, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, fx_{n+1})) \\ &< \alpha\left(\frac{1}{n^2} + \frac{1}{(n+1)^2}\right) + \alpha d(gx_n, gx_{n+1}) \\ \Rightarrow (1 - \alpha)d(gx_n, gx_{n+1}) &< \alpha\left(\frac{1}{n^2} + \frac{1}{(n+1)^2}\right) \\ \text{i.e., } d(gx_n, gx_{n+1}) &< \left(\frac{\alpha}{(1 - \alpha)}\right)\left(\frac{1}{n^2} + \frac{1}{(n+1)^2}\right). \end{aligned}$$

For  $m \leq n$ , we have

$$\begin{aligned} d(gx_m, gx_n) &\leq d(gx_m, gx_{m+1}) + d(gx_{m+1}, gx_{m+2}) + \dots + d(gx_{n-1}, gx_n) \\ &\leq \left(\frac{\alpha}{(1 - \alpha)}\right)\left[\frac{1}{m^2} + \frac{2}{(m+1)^2} + \frac{2}{(m+2)^2} + \dots + \frac{2}{(n-1)^2} + \frac{1}{n^2}\right]. \end{aligned}$$

By Cauchy criteria for the convergence of a series, for every  $\epsilon > 0$ , there exists  $N$ , such that

$$d(gx_n, gx_m) < \epsilon, \quad \forall m, n \geq N.$$

i.e.,  $\{gx_n\}$  is Cauchy sequence. By the completeness of  $X$ , there exists  $u$  in  $X$  such that  $gx_n \rightarrow u$  as  $n \rightarrow \infty$ . Since  $d(fx_n, gx_n) \rightarrow 0$ ,  $fx_n$  converges to  $u$ . Since  $f$  is continuous,  $f^2x_n, fgx_n \rightarrow fu$ . Now, (3.1) gives

$$d(gfx_n, gu) \leq \alpha d(f^2x_n, fu).$$

As  $n \rightarrow \infty$ , we have that  $gfx_n \rightarrow gu$ . Since  $d(fgx_n, gfx_n) \rightarrow 0$ , we have  $fu = gu$ . Now by the hypothesis,

$$d(gx_n, gu) \leq \alpha d(fx_n, fu).$$

Now we get that  $d(u, gu) \leq \alpha d(u, fu) = \alpha d(u, gu)$  as  $n$  tends to  $\infty$ . Then  $(1 - \alpha)d(u, gu) \leq 0 \Rightarrow fu = gu = u$ . Thus, Clearly, by (3.1),  $f$  and  $g$  have a unique common fixed point.  $\square$

**Example 2.1.** Let  $X = [4, 10]$  be a metric space with usual metric  $d(x, y) = |x - y|$ . Define continuous functions  $f, g : X \rightarrow X$  such that

$$f(x) = \begin{cases} \frac{1}{10}(-10x + 139), & x \in [4, 7] \\ \frac{1}{30}(31x - 10), & x \in (7, 10] \end{cases} \quad g(x) = \begin{cases} \frac{1}{50}(-3x + 507), & x \in [4, 7] \\ \frac{1}{5}(2x + 30), & x \in (7, 10]. \end{cases}$$

At  $x = 4$ , the pair  $\{f, g\}$  is not compatible, for  $f(4) = g(4)$  but  $(fg)(4) \neq (gf)(4)$ . Further, the pair  $\{f, g\}$  is  $\epsilon$ -compatible and also satisfies the sufficient requirements of Theorem 2.1 with  $\alpha = \frac{12}{31}$ . Thus  $f$  and  $g$  have a unique fixed point, i.e.,  $f(10) = g(10) = 10$ .

**Corollary 2.1.** Let  $f$  and  $g$  be reciprocal continuous mappings of a complete metric space  $(X, d)$  such that

$$d(gx, gy) \leq \alpha d(fx, fy), \quad \alpha \in (0, 1),$$

for all  $x, y \in X$ . If  $f$  and  $g$  are  $\epsilon$ -compatible pair, then,  $f$  and  $g$  have a unique common fixed point.

*Proof.* By the reciprocal continuity of  $f, g$  and  $\epsilon$ -compatibility, it gives straight away that  $fu = gu$ . The remaining part of the proof follow Theorem 2.2.  $\square$

Let  $\mathbb{R}_+$  be set of all non-negative real numbers. Consider a function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  which is non-decreasing and  $\psi(t) < t$  for  $t > 0$ .

**Theorem 2.3.** Let  $f, g$  be  $\epsilon$ -compatible mappings on a complete metric space  $(X, d)$  satisfying

$$(3.3) \quad d(gx, gy) \leq \psi(m(x, y)), \quad \forall x, y \in X$$

where  $m(x, y) = \max\{d(fx, gx), d(fy, gy)\}$ . If  $f$  is continuous on  $X$ , then there exists  $u$  in  $X$  such that  $fu = gu = u$  and such a point is unique.

*Proof.* The  $\epsilon$ -compatibility of  $f$  and  $g$  implies that there exists a sequence  $\{x_n\} \in X$  such that

$$d(fx_n, gx_n) < \frac{1}{n^2} \quad \text{and} \quad d(fgx_n, gfx_n) < \phi\left(\frac{1}{n^2}\right).$$

The inequality (3.3) gives,

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \psi(m(x_n, x_{n+1})), \\ m(x_n, x_{n+1}) &= \max\{d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1})\}. \end{aligned}$$

Since  $\psi$  is non-decreasing, we have  $\psi(m(x_n, x_{n+1})) \leq \psi\left(\frac{1}{n^2}\right)$ , then

$$d(gx_n, gx_{n+1}) \leq \psi(d(m(x_n, x_{n+1}))) \leq \psi\left(\frac{1}{n^2}\right) < \frac{1}{n^2}.$$

For  $m \leq n$ , we have

$$\begin{aligned} d(gx_m, gx_n) &\leq d(gx_m, gx_{m+1}) + d(gx_{m+1}, gx_{m+2}) + \cdots + d(gx_{n-1}, gx_n) \\ &< \frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{n^2}. \end{aligned}$$

By Cauchy criteria for the series  $\sum \frac{1}{n^2}$  and  $s_n = \sum_1^n \frac{1}{k^2}$ , for every  $\epsilon > 0$ , there exists  $N$  such that  $|s_n - s_{n+p}| < \epsilon$ , for all  $n \geq N$  and  $p \in \mathbb{N}$ . Therefore,  $\{gx_n\}$  is Cauchy sequence and by Lemma 1.2,  $\{fx_n\}$  is also Cauchy sequence. Since  $X$  is complete, there exists an element  $u \in X$  such that  $fx_n, gx_n$  converges to  $u$ . The continuity of  $f$  and  $\epsilon$ -compatibility of  $f$  and  $g$  implies  $fgx_n, gfx_n$  converges to  $fu$ . Suppose that  $fu \neq gu$ . By our hypothesis, we have

$$\begin{aligned} d(gfx_n, gu) &\leq \psi(m(fx_n, u)), \\ m(fx_n, u) &= \max\{d(ffx_n, gfx_n), d(fu, gu)\}. \end{aligned}$$

As  $n \rightarrow \infty$ , we have  $d(fu, gu) \leq \psi(d(fu, gu)) < d(fu, gu)$  which contradicts. Then,  $u$  is the coincidence point of  $f$  and  $g$ , i.e.,  $fu = gu$ . Now, we shall show that  $fu = gu = u$ . Our hypothesis gives,

$$\begin{aligned} d(gx_n, gu) &\leq \psi(m(x_n, u)) \\ &= \psi(\max\{d(fx_n, gx_n), d(fu, gu)\}). \end{aligned}$$

Letting  $n \rightarrow \infty$ , we have  $d(u, gu) \leq \psi(0) = 0$ . Thus  $u$  is the fixed point of  $f$  and  $g$ . If there exists  $v \in X$  such that  $fv = gv = v$ , then

$$\begin{aligned} d(gu, gv) &\leq \psi(m(u, v)) \\ &= \psi(\max\{d(fu, gu), d(fv, gv)\}) = 0. \end{aligned}$$

i.e.,  $u = v$ . Hence,  $f$  and  $g$  have a unique common fixed point. □

**Corollary 2.2.** *Let  $f$  and  $g$  be  $\epsilon$ -compatible mappings on a complete metric space  $(X, d)$  satisfying the condition, for all  $x, y \in X$ ,*

$$d(gx, gy) \leq \alpha \max\{d(fx, gx), d(fy, gy)\}, \quad \alpha \in (0, 1).$$

*If  $f$  is continuous on  $X$ , then  $f$  and  $g$  have a unique common fixed point.*

*Proof.* Consider the function  $\psi(t) = \alpha t$  in Theorem 2.3 and prove Corollary 2.2. □

**Theorem 2.4.** *Let  $(X, d)$  be a complete metric space and  $f, g$  be  $\epsilon$ -compatible mappings on  $X$ . If the maps  $f$  and  $g$  satisfy*

$$(3.4) \quad d(gx, gy) \leq ad(fx, fy) + bd(fx, gx) + cd(fy, gy), \quad \forall x, y \in X$$

*where  $a, b$  and  $c$  are in  $\mathbb{R}_+$  such that  $a, c \in [0, 1)$ .*

*Proof.* The  $\epsilon$ -compatibility of  $f$  and  $g$  ensures the existence of a sequence  $\{x_n\}$  in

$X$  such that  $d(fx_n, gx_n) < \frac{1}{n^2}$  implies  $d(fgx_n, gfx_n) < \phi(\frac{1}{n^2})$ . Now, (3.4) gives,

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq ad(fx_n, fx_{n+1}) + bd(fx_n, gx_n) + cd(fx_{n+1}, gx_{n+1}) \\ &\leq a[d(fx_n, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, fx_{n+1})] \\ &\quad + bd(fx_n, gx_n) + cd(fx_{n+1}, gx_{n+1}), \\ (1-a)d(gx_n, gx_{n+1}) &\leq (a+b)d(fx_n, gx_n) + (a+c)d(fx_{n+1}, gx_{n+1}), \\ d(gx_n, gx_{n+1}) &\leq \left(\frac{a+b}{1-a}\right)d(fx_n, gx_n) + \left(\frac{a+c}{1-a}\right)d(fx_{n+1}, gx_{n+1}) \\ &< \left(\frac{a+b}{1-a}\right)\frac{1}{n^2} + \left(\frac{a+c}{1-a}\right)\frac{1}{(n+1)^2}, \\ \text{i.e., } d(gx_n, gx_{n+1}) &< \left(\frac{2a+b+c}{1-a}\right)\frac{1}{n^2}. \end{aligned}$$

For  $m \leq n$ , we have

$$\begin{aligned} d(gx_m, gx_n) &\leq \sum_m^{n-1} d(gx_i, gx_{i+1}) \\ &< \left(\frac{2a+b+c}{1-a}\right) \sum_m^{n-1} \frac{1}{k^2}. \end{aligned}$$

Since the sequence  $s_n = \sum_1^n \frac{1}{k^2}$  is Cauchy,  $\{gx_n\}$  is also Cauchy. Then, by Lemma 1.2,  $\{fx_n\}$  is Cauchy sequence. Now, by hypothesis of  $X$ , there exists  $u \in X$  such that  $\lim fx_n = \lim gx_n = u$ . The continuity of  $f$  gives that  $\lim fgx_n = \lim f^2x_n = fu$  and  $d(fgx_n, gfx_n) \rightarrow 0$  implies  $\{gfx_n\}$  converges to  $fu$ . Suppose that  $fu \neq gu$ , by (3.4), we have

$$\begin{aligned} d(gfx_n, gu) &\leq ad(ffx_n, fu) + bd(ffx_n, gfx_n) + cd(fu, gu) \\ \text{As } n \rightarrow \infty, d(fu, gu) &\leq cd(fu, gu) \\ \text{i.e., } d(fu, gu) &< d(fu, gu) \end{aligned}$$

contradicts. Then, we have shown that  $fu = gu$ . Our hypothesis gives,

$$\begin{aligned} d(gx_n, gu) &\leq ad(fx_n, fu) + bd(fx_n, gx_n) + cd(fu, gu) \\ \text{As } n \rightarrow \infty, d(u, gu) &\leq ad(u, fu) \\ (1-a)d(u, gu) &\leq 0 \end{aligned}$$

gives  $gu = u$ . Thus, we have shown that  $fu = gu = u$ . If  $v$  is the fixed point of  $f$  and  $g$ , we have, by hypothesis,

$$\begin{aligned} d(gu, gv) &\leq ad(fu, fv) + bd(fu, gu) + cd(fv, gv) \\ d(u, v) &\leq ad(u, v) \\ (1-a)d(u, v) &\leq 0 \end{aligned}$$

implies  $u = v$ . Hence,  $f$  and  $g$  have a unique common fixed point. □

**Corollary 2.3.** *Let  $f$  and  $g$  be  $\epsilon$ -compatible mappings on a complete metric space  $(X, d)$  satisfying the condition, for all  $x, y \in X$ ,*

$$d(gx, gy) \leq \alpha[d(fx, gx) + d(fy, gy)], \quad \alpha \in (0, 1).$$

*If  $f$  is continuous on  $X$ , then  $f$  and  $g$  have a unique common fixed point.*

*Proof.* With  $a = 0$  in Theorem 2.4, we can prove the Corollary easily. □

**Theorem 2.5.** *Let  $f$  and  $g$  be  $\epsilon$ -compatible mappings on a complete metric space  $(X, d)$  satisfying the condition, for all  $x, y \in X$ ,*

$$(3.5) \quad d(gx, gy) \leq \alpha m(x, y)$$

*where,  $m(x, y) = \max\{d(fx, fy), d(fx, gx), d(fx, gy), d(fy, gx), d(fy, gy)\}$  and  $\alpha \in (0, 1)$ . If  $f$  is continuous on  $X$ , then  $f$  and  $g$  have a unique common fixed point.*

*Proof.* The  $\epsilon$ -compatibility of  $f$  and  $g$  implies that there exists a sequence  $\{x_n\}$  in  $X$  such that for given  $\frac{1}{n^2} > 0$ ,  $d(fx_n, gx_n) < \frac{1}{n^2}$  implies  $d(fgx_n, gfx_n) < \phi(\frac{1}{n^2})$ . Then, (3.5) gives,

$$(3.6) \quad \begin{aligned} d(gx_n, gx_{n+1}) \leq & \alpha \max\{d(fx_n, fx_{n+1}), d(fx_n, gx_n), d(fx_n, gx_{n+1}), \\ & d(fx_{n+1}, gx_n), d(fx_{n+1}, gx_{n+1})\} \end{aligned}$$

Now, we examine the possibilities to be occurred, by cases.

(i)  $m(x_n, x_{n+1}) = d(fx_n, fx_{n+1})$ , by (3.5), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) & \leq \alpha[d(fx_n, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, fx_{n+1})] \\ (1 - \alpha)d(gx_n, gx_{n+1}) & < \alpha \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right] \\ d(gx_n, gx_{n+1}) & < \frac{\alpha}{1 - \alpha} \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right] \end{aligned}$$

(ii)  $m(x_n, x_{n+1}) = d(fx_n, gx_n)$   
i.e.,  $d(gx_n, gx_{n+1}) < \alpha[\frac{1}{n^2}]$

(iii)  $m(x_n, x_{n+1}) = d(fx_n, gx_{n+1})$

$$\begin{aligned} d(gx_n, gx_{n+1}) & \leq \alpha[d(fx_n, gx_n) + d(gx_n, gx_{n+1})] \\ d(gx_n, gx_{n+1}) & < \frac{\alpha}{1 - \alpha} \left[ \frac{1}{n^2} \right] \end{aligned}$$

(iv)  $m(x_n, x_{n+1}) = d(fx_{n+1}, gx_n)$

$$\begin{aligned} d(gx_n, gx_{n+1}) & \leq \alpha[d(fx_{n+1}, gx_{n+1}) + d(gx_{n+1}, gx_n)] \\ d(gx_n, gx_{n+1}) & < \frac{\alpha}{1 - \alpha} \left[ \frac{1}{(n+1)^2} \right] \end{aligned}$$

$$(v) \quad m(x_n, x_{n+1}) = d(fx_{n+1}, gx_{n+1}) \\ \text{i.e., } d(gx_n, gx_{n+1}) < \frac{\alpha}{1-\alpha} \left[ \frac{1}{(n+1)^2} \right]$$

From all the possibilities, it can be concluded that for  $n \in N$ ,  $d(gx_n, gx_{n+1}) < k \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right]$ , where  $\delta = \max\{\alpha, \frac{\alpha}{1-\alpha}\}$ . For  $m \leq n$ , we have

$$d(gx_m, gx_n) \leq \sum_m^{n-1} d(gx_i, gx_{i+1}) \\ < \delta \sum_m^{n-1} \frac{2}{k^2}.$$

Since the sequence  $s_n = \sum_1^n \frac{1}{k^2}$  is Cauchy,  $\{gx_n\}$  is Cauchy and  $\{fx_n\}$  is also Cauchy. On account of complete metric space  $(X, d)$ , there exists  $z \in X$  such that  $fx_n, gx_n \rightarrow z$  as  $n \rightarrow \infty$ . Continuity of  $f$  and the  $\epsilon$ -compatibility of mappings together implies that  $fgx_n, gfx_n$  converges to  $fz$ . Now by hypothesis,

$$d(gfx_n, gz) \leq \alpha m(fx_n, x_n) \\ m(fx_n, x_n) = \max\{d(ffx_n, fz), d(ffx_n, gfx_n), d(ffx_n, gz), \\ d(fz, gfx_n), d(fz, gz)\}, \\ \lim_{n \rightarrow \infty} m(fx_n, x_n) = d(fz, gz),$$

Then, we have

$$\text{i.e., } d(fz, gz) = \lim_n d(gfx_n, gz) \leq \alpha d(fz, gz) \\ (1 - \alpha)d(fz, gz) \leq 0$$

Since  $\alpha < 1$ , we have  $fz = gz$ . Then,

$$d(gx_n, gz) \leq \alpha m(x_n, z) \\ m(x_n, z) = \max\{d(fx_n, fz), d(fx_n, gx_n), d(fx_n, gz), \\ d(fz, gx_n), d(fz, gz)\} \\ \lim_{n \rightarrow \infty} m(x_n, z) = d(z, gz) \\ \text{i.e., } (1 - \alpha)d(z, gz) \leq d(z, gz)$$

gives  $z = gz$ . Thus,  $z$  is the common fixed point of  $f$  and  $g$ . Explicitly, the inequality (3.5) implies the uniqueness of coincidence point and hence,  $f$  and  $g$  have a unique common fixed point.  $\square$

**Remark 2.1.** In the above theorems, we can obtain the same conclusion with reciprocal continuity of  $f$  and  $g$  and  $\epsilon$ -compatibility keeping the contractive condition.

## References

- [1] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly., **83**(1976), 261–263.
- [2] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. Soc., **32**(1982), 149–153.
- [3] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **9**(4)(1986), 771–779
- [4] G. Jungck, P. P. Murthy and Y. J. Cho, *Compatible mappings of type(A) and common fixed points*, Math. Japonica, **38**(1993), 381–390.
- [5] H. K. Patha, Y. J. Cho, S. M. Kang and B. S. Lee, *Fixed point theorems for compatible mappings of type(P) and Applications to dynamic programming*, Le Mathematicne, L(1995), 15–33.
- [6] G. Jungck and H. K. Pathak, *Fixed points via "Biased maps"*, Proc. Amer. Math. Soc., **123**(1995), 2049–2060.
- [7] G. Jungck, *Common fixed points for noncontinuous non-self maps on non-metric spaces*, Far East J. Math. Sci., **4**(1996), 199–215.
- [8] M. A. Al-Thagafi and N. Shahzad, *Generalized I-non-expansive self-maps and invariant approximations* Acta Math. Sin., **24**(2008), 867–876.
- [9] H. Bouhadjera and C. Godet-Thobie, *Common fixed point theorems for pair of sub-compatible maps* math. FA arXiv:0906.3159v1, (2009).
- [10] V. Pant and R. P. Pant, *Common fixed points of conditionally commuting maps*, Fixed Point Theory, **1**(2012), 113–118.
- [11] R. P. Pant and R. K. Bisht, *Occasionally weakly compatible mappings and fixed points*, Bull. Belg. Math. Soc. Simon Stevin, **19**(2012), 655–661.
- [12] R. Bisht and N. Shahzad, *Faintly compatible mappings and common fixed point*, Fixed Point Theory Appl. 2013, **2013**:156.
- [13] R. P. Pant, *A common fixed point under a new condition*, J. Pure Appl. Math., **30**(2)(1999), 147–152.